Control of PDE Systems

Lecture 1 (Meetings 1 & 2)

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Nonlinear Models

In this course we will deal with nonlinear dynamical systems that are model by a set of coupled first-order ordinary differential equations (ODE),

$$
\begin{array}{rcl}
\dot{x}_1 & = & f_1(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\
\dot{x}_2 & = & f_2(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\
\vdots & = & \vdots \\
\dot{x}_n & = & f_n(t, x_1, \ldots, x_n, u_1, \ldots, u_p)
\end{array} \tag{1}
$$

where $x_1, \ldots, x_n$ denote the n states, $u_1, \ldots, u_p$ denote the p inputs, t denotes time and $\dot{x}_i$ denotes the time derivative of the state x_i.

Nonlinear Models

After defining

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}, \qquad f(t,x,u) = \begin{bmatrix} f_1(t,x,u) \\ f_2(t,x,u) \\ \vdots \\ f_n(t,x,u) \end{bmatrix}$$

we can rewrite the *state equation* (1) as

$$\dot{x} = f(t,x,u) \tag{2}$$

which may be associated with the *output equation*

$$y = h(t,x,u) \tag{3}$$

where y denotes the q-dimensional output.

Nonlinear Models

- Nonlinear Control: Design control law

$$u = \gamma(t, x)$$

for

$$\dot{x} = f(t, x, u)$$

- Nonlinear Analysis: We study the dynamics of the unforced system

$$\dot{x} = f(t, x)$$

where u has been either forced to zero or replaced by the control law $\gamma(t, x)$.

$\dot{x} = f(t, x)$	nonautonomous or time-varying
$\dot{x} = f(x)$	autonomous or time-invariant

Nonlinear Models

A point $x = x^*$ in the state space is said to be an equilibrium point if it has the property that whenever the state of the system starts at x^*, it will remain at x^* for all future time. For the autonomous system

$$\dot{x} = f(x) \tag{4}$$

the equilibrium points are the real roots of the equation

$$0 = f(x) \tag{5}$$

Equilibrium points can be isolated or there can be a continuum of points.

Autonomous Systems

Consider the autonomous system

$$\dot{x} = f(x) \tag{6}$$

where $f : D \to R^n$ is a *locally Lipschitz* map from a domain $D \subset R^n$ into R^n. Suppose $\bar{x} = 0 \in D$ is an equilibrium point of (6).

Our goal is to characterize and study stability of the equilibrium $\bar{x} = 0$ (no loss of generality).

Lipschitz Condition

Lemma: Let $f : [a, b] \times D \to \mathcal{R}^m$ be continuous for some domain $D \subset \mathcal{R}^n$. Suppose that $[\partial f / \partial x]$ exists and is continuous on $[a, b] \times D$. If, for a convex subset $W \subset D$, there exists a constant $L \geq 0$ such that $\left\| \frac{\partial f}{\partial x} \right\| \leq L$ on $[a, b] \times W$, then f satisfies the Lipschitz condition

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|$$

for all $t \in [a, b]$, $x \in W$ and $y \in W$.

- The lemma indeed shows how a Lipschitz constant can be computed using knowledge of $[\partial f / \partial x]$

$$\mathcal{C}^0 \text{ (Continuity)} \Leftarrow \text{Lipschitz} \Leftarrow \mathcal{C}^1 \text{ (Continuously Differentiability)}$$

Examples:

- Lipschitz but not $\mathcal{C}^1$: $f(x) = |x|$, $f(x) = sat(x)$.
- $\mathcal{C}^0$ but not Lipschitz: $f(x) = \sqrt{x}$.

Stability

Definition [K] 4.1: The equilibrium point $x = 0$ of (6) is

- *stable* if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \forall t \geq 0$$

- *unstable* if not stable
- *asymptotically stable* if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} x(t) = 0$$

The $\epsilon - \delta$ requirement for stability takes a challenge-answer form.

Stability

"Stability is a property of the equilibrium, not of the system"

Stability of the equilibrium is equivalent to stability of the system only when there exists only one equilibrium (e.g., linear systems). In this case stability $\equiv$ global stability.

The equilibrium point $x = 0$ of (6) is

- *attractive* if there is $\delta > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

  Example: Attractive but unstable
- *asymptotically stable (a.s.)* if it is stable and attractive.
- *globally asymptotically stable (g.a.s.)* if a.s. $\forall x(0) \in \mathcal{R}^n$.

Derivative along the trajectory

Definition: Let $V : D \to R$ be a *continuously differentiable* function defined in a domain $D \in R^n$ that contains the origin. The derivative of V along the trajectory (solution) of (6), denoted by $\dot{V}(x)$ is given by

$$\begin{aligned}
\dot{V}(x) &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i \\
&= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \\
&= [\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ldots, \frac{\partial V}{\partial x_n}][f_1(x), f_2(x), \ldots, f_n(x)]^T \\
&= \frac{\partial V}{\partial x} f(x)
\end{aligned}$$

Lyapunov Stability Theorem

Theorem [K] 4.1: Let $x = 0$ be an equilibrium for (6) and $D \in R^n$ be a domain containing $x = 0$. Let $V : D \to R$ be a continuously differentiable function, such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\} \tag{7}$$

$$\dot{V}(x) \leq 0 \text{ in } D \tag{8}$$

Then, $x = 0$ is *stable*. Moreover, if

$$\dot{V}(x) < 0 \text{ in } D - \{0\} \tag{9}$$

then $x = 0$ is *asymptotically stable*.

Lyapunov Stability Theorem

- Lyapunov function candidate

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\}$$

- Lyapunov function

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\}$$
$$\dot{V}(x) \leq 0 \text{ in } D$$

- Lyapunov surface (level surface, level set)

$$\{x | V(x) = c\}$$

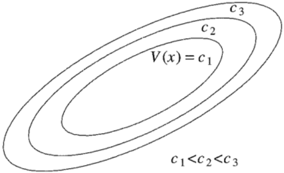

Figure 1: Level surfaces of a Lyapunov function.

Lyapunov Stability Theorem

- Positive definite

$$V(0) = 0, V(x) > 0, \forall x \neq 0$$

- Positive semidefinite

$$V(0) = 0, V(x) \geq 0, \forall x \neq 0$$

$V(x)$ is negative (semi)definite if $-V(x)$ is positive (semi)definite

- Lyapunov Theorem
V pdf + $\dot{V}$ nsdf $\rightarrow$ stable
V pdf + $\dot{V}$ ndf $\rightarrow$ asymptotically stable

Lyapunov Stability Theorem

Example [K] 4.4: Consider the pendulum equation with friction

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\left(\frac{g}{l}\right)\sin x_1 - \left(\frac{k}{m}\right)x_2\end{aligned}$$

- $V_1(x) = a(1-\cos(x_1)) + (1/2)x_2^2 \Rightarrow$ Stable.
- $V_2(x) = a(1-\cos(x_1)) + (1/2)x^T P x \Rightarrow$ Asympt. Stable.

Conclusion:

- Lyapunov's stability conditions are *only sufficient*.
- $V_1(x)$ good enough to prove a.s. via LaSalle's theorem.
- Backward approach $\rightarrow$ *Variable Gradient Method*.

Region of Attraction

When the origin $x = 0$ is asymptotically stable, we are often interested in determining how far from the origin the trajectory can be and still converge to the origin as $t \to \infty$. This gives rise to the definition of *region of attraction* (also called *region of asymptotically stability*, domain of attraction, or basin).

Definition: Let $\phi(t, x)$ be the solution of (6) that starts at initial state x at time $t = 0$. The, the region of attraction is defined as the set of all points x such that $\lim_{t\to\infty} \phi(t, x) = 0$

Question: Under what conditions will the region of attraction be the whole space R^n? In other words, for any initial state x, under what conditions the trajectory $\phi(t, x)$ approaches the origin as $t \to \infty$, no matter how large $\|x\|$ is. If an a.s. equilibrium point at the origin has this property, it is said to be *globally asymptotically stable* (g.a.s.).

Global Lyapunov Stability Theorem

Theorem [K] 4.2: Let $x = 0$ be an equilibrium for (6). Let $V : R^n \to R$ be a continuously differentiable function, such that

$$V(0) = 0 \text{ and } V(x) > 0, \ \forall x \neq 0 \tag{10}$$

$$\|x\| \to \infty \Rightarrow V(x) \to \infty \tag{11}$$

$$\dot{V}(x) < 0, \ \forall x \neq 0 \tag{12}$$

Then, $x = 0$ is *globally asymptotically stable* and is the *unique* equilibrium point.

NOTE: It is not enough to satisfy Theorem 4.1 for $D = R^n$!!!

Chetaev's Instability Theorem

Theorem [K] 4.3: Let $x = 0$ be an equilibrium for (6). Let $V : D \to R$ be a continuously differentiable function, such that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 with arbitrarily small $\|x_0\|$. Define a set

$$U = \{x \in B_r | V(x) > 0\}$$

where

$$B_r = \{x \in R^n | \|x\| \leq r\}.$$

Suppose that $\dot{V}(x) > 0$ in U. Then $x = 0$ is *unstable*.

Crucial Condition: $\dot{V}$ must be positive in the entire set where $V > 0$.

Chetaev's Instability Theorem

Example [K] 4.7: Consider the second order system

$$\begin{aligned} \dot{x}_1 &= x_1 + g_1(x) \\ \dot{x}_2 &= -x_2 + g_2(x) \end{aligned}$$

where $g_1()$ and $g_2()$ are locally Lipschitz functions that satisfy the inequalities

$$|g_1(x)| \leq k\|x\|^2, \qquad |g_2(x)| \leq k\|x\|^2$$

Use the Lyapunov function candidate $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$ and Chetaev's theorem to show that the origin is unstable.

Chetaev's Instability Theorem

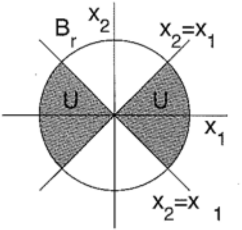

Figure 2: The set U for $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$.

Invariance Principle

Example [K] 4.4: Consider the pendulum equation with friction

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\left(\frac{g}{l}\right)\sin x_1 - \left(\frac{k}{m}\right)x_2\end{aligned}$$

We consider the Lyapunov function candidate

$$V(x) = \left(\frac{g}{l}\right)(1-\cos x_1) + \frac{x_2^2}{2} \Rightarrow \dot{V}(x) = -\left(\frac{k}{m}\right)x_2^2 \leq 0$$

The energy Lyapunov function fails to satisfy the asymptotic stability condition of Theorem 4.1.
But can $\dot{V}(x) = 0$ be maintained at $x \neq 0$?

Invariance Principle

Idea: If we can find a Lyapunov function in a domain containing the origin whose derivative along the trajectories of the system is *negative semidefinite*, and if we can establish that no trajectory can stay identicaly at points where $\dot{V}(x) = 0$ except at the origin, then the origin is asymptotically stable (LaSalle's Invariance Principle).

Definition: A set M is *(positively) invariant* w.r.t. $\dot{x} = f(x)$ if $x(0) \in M \Rightarrow x(t) \in M$ for all $t \in R$ ($t \in R_+$).

Invariance Principle

Theorem [K] 4.4 (LaSalle's Theorem): Let $\Omega \subset D$ be a compact set that is positively invariant w.r.t. $\dot{x} = f(x)$. Let $V : D \to R$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then, every solution starting in Ω approaches M as $t \to \infty$.

Note: Unlike Lyapunov's theorem, LaSalle's theorem does NOT require the function $V(x)$ to be positive definite.

Note: When we are interested in showing that $x(t) \to 0$ as $t \to \infty$, we need to establish that the largest invariant set M in E is the origin, i.e., $M \equiv 0$.

Invariance Principle

Corollary [K] 4.1 (4.2): Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$. Let $V : D(R^n) \to R$ be a continuously differentiable (radially unbounded, positive definite) function on a domain D containing the origin (on R^n), such that $\dot{V}(x) \leq 0$ in D (in R^n). Let $S = \{x \in D(R^n) | \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution. Then, the origin is *(globally) asymptotically stable*.

Invariance Principle

LaSalle's theorem

- relaxes the negative definiteness requirement for $\dot{V}(x)$ of Lyapunov's theorem
- does not require $V(x)$ to be positive definite
- gives an estimate of the region of attraction Ω which is not necessarily a level set of $V(x)$
- applies not only to equilibrium points but also to equilibrium sets

Example:

$$\begin{aligned} \dot{x} &= -|x|x + (1-|x|)xy \\ \dot{y} &= -\frac{1}{8}(1-|x|)x^2 \end{aligned}$$