ME 433 – STATE SPACE CONTROL

Lecture 11

Final-time-free

1. Minimum-time Problems

One special class of final-time-free problems is defined by a performance index

$$J(t_0) = \int_{t_0}^{T} 1 dt$$

which arises when we are interested in minimizing the time T-t_0 required to make a function of the final state $\psi(x(T),T)$ zero given some initial state $x(t_0)$. The Hamiltonian is in this case

$$H(t,x,u) = 1 + \lambda^T(t) f(t,x,u)$$

Generalized boundary condition

$$\left(\phi_x + \psi_x^T \nu - \lambda\right)^T dx\big|_T + \left(\phi_t + \psi_t^T \nu + H\right)^T dt\big|_T = 0$$

Final-time-free

Case 1: The final state $x(T)$ is required to be fixed at a given value r_T. Then $dx(T)=0$. In this case,

$$\psi\big(T, x(T)\big) = x(T) - r_T = 0$$

is independent of T, and since $\phi(x(T),T)=0$ in the minimum-time problem, the boundary condition requires

$$H(T) = 0$$

if the Hamiltonian is not an explicit function of time, we must have

$$H(t) = 0 \qquad \forall t \in [t_0, T]$$

Case 2: Both $x(T)$ and T are free, but they are independent. The boundary condition demands that

$$\left(\phi_x + \psi_x^T \nu - \lambda\right)^T dx\big|_T = 0 \qquad \left(\phi_t + \psi_t^T \nu + H\right)^T dt\big|_T = 0$$

Final-time-free

Case 3: Both $x(T)$ and T are free, but they are dependent. An examples is when the final state $x(T)$ is required to be on a specified moving point $p(t)$ at time T, but $x(T)$ and T are otherwise free. Then

$$x(T) = p(T)$$

$$dx(T) = \frac{dp(T)}{dT} dT$$

The boundary condition becomes (note that there is no $\psi(T)$, or $\psi(T)$ is identical to zero)

$$\left(\phi_x - \lambda\right)^T \frac{dp(t)}{dt} dt\bigg|_T + \left(\phi_t + H\right)^T dt\big|_T = 0$$

Since dT is non-zero, this requires

$$\left(\phi_x(T) - \lambda(T)\right)^T \frac{dp(T)}{dT} + \left(\phi_t(T) + H(T)\right)^T = 0$$

Constrained Input Problems

2. Pontryagin's Minimum Principle

Let the plant

$$\dot{x} = f(t,x,u), \qquad t_0 < t < T$$

Have an associated cost index of

$$J(t_0) = \phi(T,x(T)) + \int_{t_0}^{T} L(t,x(t),u(t))dt$$

Where the final state must satisfy

$$\psi(T,x(T)) = 0$$

and $x(t_0)$ is given.

Constrained Input Problems

If the control is unconstrained, the optimal control problem has been already solved, where the condition for optimality was

$$\frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} = 0, \qquad H(t,x,u) = L(t,x,u) + \lambda^T(t) f(t,x,u)$$

Now suppose the control $u(t)$ is constrained to line in an *admissible region*. It was shown by Pontryagin that the unconstrained solution still holds but the stationary condition must be replaced by the more general condition (where * denotes optimal quantities)

$$H(x^*,u^*,\lambda^*,t) \le H(x^*,u^*+\delta u,\lambda^*,t) \qquad \text{all admissible } \delta u$$

or equivalently

$$H(x^*,u^*,\lambda^*,t) \le H(x^*,u,\lambda^*,t) \qquad \text{all admissible } u$$

This is Pontryagin's Minimum Principle: "The Hamiltonian must be minimized over all admissible u for optimal values of the state and costate"

Constrained Input Problems

System Properties | SUMMARY | Controller Properties

System Model

$$x(t) = f(t,x,u)$$

Performance Index

$$J(t_0) = \phi(T,x(T)) + \int_{t_0}^{T} L(t,x(t),u(t))dt$$

Final-state Constraint

$$\psi(T,x(T)) = 0$$

Hamiltonian

$$H(t,x,u,\lambda) = L(t,x,u) + \lambda(t)f(t,x,u)$$

State Equation

$$\dot{x} = \frac{\partial H}{\partial \lambda} = f(t,x,u), \qquad t \geq t_0$$

Costate Equation

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x}, \qquad t \leq T$$

Stationary Condition

$$H(x^*,u^*,\lambda^*,t) \leq H(x^*,u,\lambda^*,t)$$

Boundary Condition

$$x(t_0) \text{ given}$$

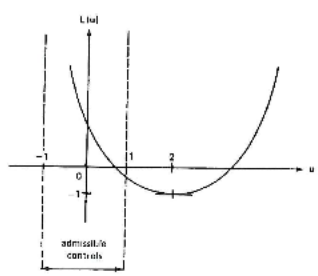
$$\left(\phi_x + \psi_x^T \nu - \lambda^T\right)^T dx\big|_T + \left(\phi_t + \psi_t^T \nu + H\right)^T dt\big|_T = 0$$

Constrained Input Problems

Example: Optimization with Constraints

$$L = \frac{1}{2}u^2 - 2u + 1$$

$$|u| \leq 1$$



Constrained Input Problems

2.1 Constrained Minimum-Time Problem (Bang-Bang Control)

Let the plant

$$\dot{x} = Ax + Bu$$

have an associated cost index of

$$J(t_0) = \int_{t_0}^{T} 1 dt$$

with T free. Where the final state must satisfy

$$\psi(T, x(T)) = 0$$

and $x(t_0)$ is given. Suppose the control is required to satisfy

$$|u(t)| \leq 1, \quad \forall t \in [t_0, T]$$

Constrained Input Problems

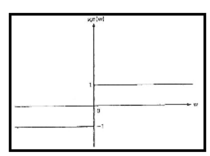
The Hamiltonian is in this case

$$H(t,x,u) = 1 + \lambda^T(t)\left(Ax(t) + Bu(t)\right)$$

Pontryagin's principle

$$1 + \left(\lambda^*\right)^T\left(Ax^* + Bu^*\right) \leq 1 + \left(\lambda^*\right)^T\left(Ax^* + Bu\right)$$

$$\left(\lambda^*\right)^T Bu^* \leq \left(\lambda^*\right)^T Bu$$

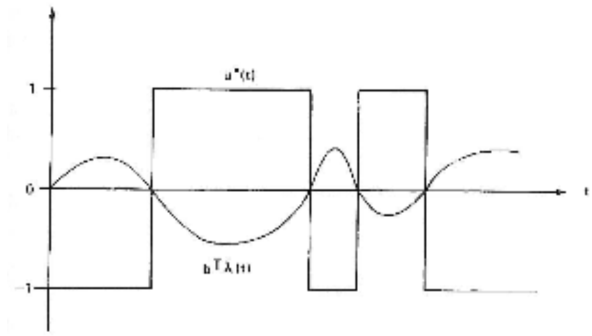


Then, we can show that

$$u^*(t) = -\text{sgn}\left(\left(\lambda^*\right)^T B\right)$$

$$sign(w) = \begin{cases} 1 & w > 0 \\ \text{indeterminate} & w = 0 \\ -1 & w < 0 \end{cases}$$

Constrained Input Problems



$$u^*(t) = -\mathrm{sgn}\left(\left(\lambda^*\right)^T B\right)$$

Constrained Input Problems

Example: Bang-Bang Control of Systems Obeying Newton's Law

Let the plant obey Newton's laws so that

$$\dot{y} = v,$$
$$\dot{v} = u$$

The goal is to minimize the associated cost index (time T)

$$J(t_0) = \int_0^T 1dt$$

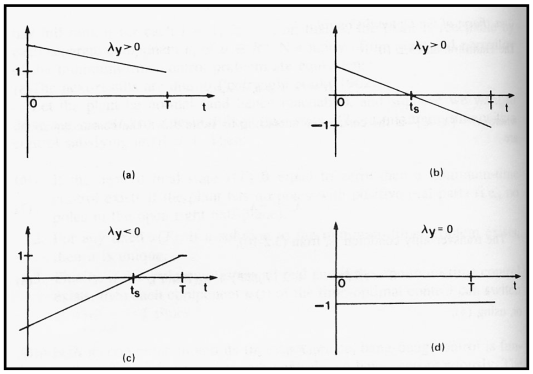
with T free. Where the final state must satisfy

$$\psi(T, x(T)) = \begin{bmatrix} y(T) & v(T) \end{bmatrix}^T = 0$$

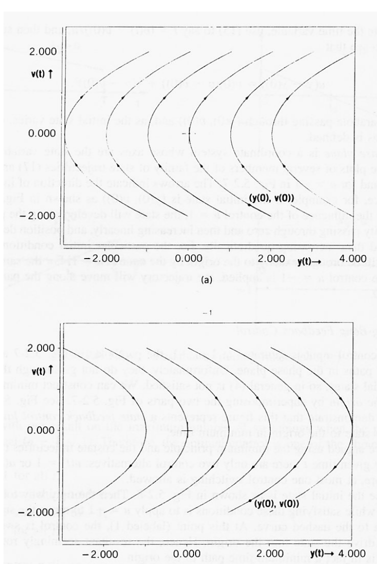
and $x(0)=[y(0) \quad v(0)]^T$ is given. Suppose the control is required to satisfy

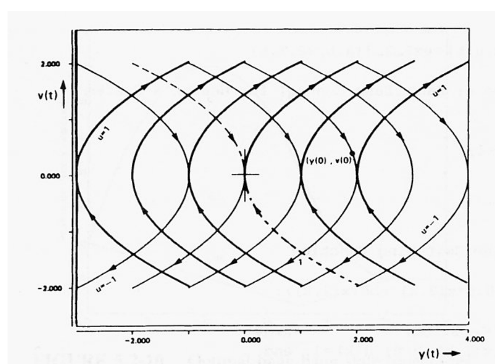
$$|u(t)| \le 1, \quad \forall t \in [t_0, T]$$

Constrained Input Problems



Constrained Input Problems





Constrained Input Problems

2.2 Constrained Minimum-Fuel Problem (Bang-Off-Bang Control)

Let the plant

$$\dot{x} = Ax + Bu$$

have an associated cost index of

$$J(t_0) = \int_{t_0}^{T} \sum_{i=1}^{m} c_i |u_i(t)| dt$$

with T either free or fixed. Where the final state must satisfy

$$\psi(T, x(T)) = 0$$

and $x(t_0)$ is given. Suppose the control is required to satisfy

$$|u(t)| \leq 1, \quad \forall t \in [t_0, T]$$

Constrained Input Problems

The Hamiltonian is in this case

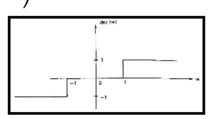
$$H(t,x,u) = C^T |u(t)| + \lambda^T(t)\left(Ax(t) + Bu(t)\right)$$

$$C^T = \begin{bmatrix} c_1 & c_2 & \cdots & c_m \end{bmatrix}, |u| = \begin{bmatrix} |u_1| & |u_2| & \cdots & |u_m| \end{bmatrix}^T$$

Pontryagin's principle

$$C^T |u^*| + (\lambda^*)^T (Ax^* + Bu^*) \leq C^T |u| + (\lambda^*)^T (Ax^* + Bu)$$

$$C^T |u^*| + (\lambda^*)^T Bu^* \leq C^T |u| + (\lambda^*)^T Bu$$



Then, we can show that

$$u_i^*(t) = -dez\left(\frac{b_i^T \lambda^*(t)}{c_i}\right), \quad i = 1, \ldots, m$$

$$dez(w) = \begin{cases} -1 & w < -1 \\ \text{between } -1 \text{ and } 0 & w = -1 \\ 0 & -1 < w < 1 \\ \text{between } 0 \text{ and } 1 & w = 1 \\ 1 & w > 1 \end{cases}$$

Constrained Input Problems

2.3 Constrained Minimum-Energy Problem

Let the plant

$$\dot{x} = Ax + Bu$$

have an associated cost index of

$$J(t_0) = \frac{1}{2}\int_{t_0}^{T} u^T(t)Ru(t)dt, \qquad R > 0$$

with T either free or fixed. Where the final state must satisfy

$$\psi(T, x(T)) = 0$$

and $x(t_0)$ is given. Suppose the control is required to satisfy

$$|u(t)| \le 1, \qquad \forall t \in [t_0, T]$$

Constrained Input Problems

The Hamiltonian is in this case

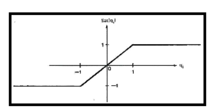
$$H(t,x,u) = \frac{1}{2}u^T(t)Ru(t) + \lambda^T(t)\big(Ax(t) + Bu(t)\big)$$

Pontryagin's principle

$$\frac{1}{2}(u^*)^T Ru^* + (\lambda^*)^T(Ax^* + Bu^*) \le \frac{1}{2}u^T Ru + (\lambda^*)^T(Ax^* + Bu)$$

$$\frac{1}{2}(u^*)^T Ru^* + (\lambda^*)^T Bu^* \le \frac{1}{2}u^T Ru + (\lambda^*)^T Bu$$

Then, we can show that



$$u_i^*(t) = -sat\left(\left[R^{-1}B^T\lambda(t)\right]_i\right), \qquad i = 1,\ldots,m$$

$$sat(w) = \begin{cases} 1 & w > 1 \\ w & |w| < 1 \\ -1 & w < -1 \end{cases}$$