ME 433 – STATE SPACE CONTROL

Lecture 5

State Feedback

Problem Definition: "A system is said to be controllable if and only if it is possible, by means of the input, to transfer the system from any initial state $x(0)$ to any other state $x(t)$ in a finite time $t \geq 0$."

Theorem: "A system is controllable if and only if the matrix

$$\overline{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$ Controllability Matrix

is full-rank."

State Feedback

We consider the linear, time-invariant system

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

and we look for a state gain K such that

$$u = -Kx$$

In this case we have the closed-loop system

$$\dot{x} = Ax - BKx = (A - BK)x$$

We should note that we can modify the dynamics (eigenvalues) of the system by state feedback. If the system is controllable, it is always possible to find a state gain K to set the eigenvalues of the closed-loop system at arbitrary values.

State Feedback

Given the desired characteristic equation $\alpha(s)$, we can compute the closed-loop characteristic equation.

$$a_k(s) = \det(sI - A + BK)$$

By equating coefficients of identical power of $a_k(s)$ and $\alpha(s)$, we can obtain n algebraic equations for the coefficients of K.

Example: Desired eigenvalues: -1, -3.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Note: This method becomes rather cumbersome when n is large.

State Transformation

We consider the linear, time-invariant system

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

We define the state transformation

$$x(t) = Tz(t) \Leftrightarrow T^{-1}x(t) = z(t)$$

Then we can write

$$T\dot{z} = ATz + Bu \Rightarrow \dot{z} = T^{-1}ATz + T^{-1}Bu$$
$$y = CTz + Du.$$

to obtain

$$\dot{z} = \tilde{A}z + \tilde{B}u$$
$$y = \tilde{C}z + \tilde{D}u$$

$$\boxed{\tilde{A} = T^{-1}AT, \tilde{B} = T^{-1}B, \tilde{C} = CT, \tilde{D} = D}$$

Model Representation

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{Y(s)}{X(s)}\frac{X(s)}{U(s)}$$

$$\frac{X(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \frac{Y(s)}{X(s)} = b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0$$

Choosing $x_1 = x^{(n-1)}, x_2 = x^{(n-2)}, \ldots, x_{n-1} = x^{(1)}, x_n = x$

$$A_c = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, C_c = \begin{bmatrix} b_{n-1} & b_{n-2} & \cdots & b_1 & b_0 \end{bmatrix}, D_c = 0$$

Controller Form

State Transformation

We are interested in finding the state transformation such that

$$\{A,B,C,D\} \xrightarrow[T]{} \{A_c,B_c,C_c,D_c\}$$

where

$$A_c = T^{-1}AT, B_c = T^{-1}B, C_c = CT, D_c = D$$

Such transformation is given by

$$T = \overline{C}\,\overline{C}_c^{-1}$$

Proof: In class

State Feedback

Bass-Gura Formula:

$$K = (\alpha - a)T_u^{-1}(a^*)\overline{C}^{-1}$$

where

$$a(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_2 s^2 + a_1 s + a_0$$

is the actual characteristic polynomial and

$$\alpha(s) = s^n + \alpha_{n-1}s^{n-1} + \alpha_{n-2}s^{n-2} + \cdots + \alpha_2 s^2 + \alpha_1 s + \alpha_0$$

is the desired characteristic equation.

State Feedback

And where

$$a \equiv \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 & a_0 \end{bmatrix}$$
$$\alpha \equiv \begin{bmatrix} \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix}$$
$$a^* \equiv \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{bmatrix}$$

and

$$\overline{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$ Controllability Matrix

$$T_u(a^*) = \begin{bmatrix} 1 & a_{n-1} & & a_1 & a_0 \\ 0 & 1 & a_{n-1} & & a_1 \\ \vdots & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$ Upper Toeplitz Matrix

State Feedback

Ackerman Formula:

$$K = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \overline{C}^{-1} \alpha(A)$$

where

$$\alpha(s) = s^n + \alpha_{n-1}s^{n-1} + \alpha_{n-2}s^{n-2} + \cdots + \alpha_2 s^2 + \alpha_1 s + \alpha_0$$

is the desired characteristic polynomial and

$$\overline{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$ Controllability Matrix

State Feedback

Mayne-Murdoch Formula:

$$k_i^d b_i^d = \frac{\prod_j \left(\lambda_i - \mu_j\right)}{\prod_{j \neq i} \left(\lambda_i - \lambda_j\right)}$$

where $\{\lambda_1, \ldots, \lambda_n\}$ are the (open-loop) eigenvalues of A, and $\{\mu_1, \ldots, \mu_n\}$ are the desired (closed-loop) eigenvalues of $A\text{-}BK$.

This formula assumes a diagonal or modal realization

$$\{A, B, C, D\} \xrightarrow[T]{} \{A_d, B_d, C_d, D_d\}$$

where

$$A_d = T^{-1}AT, B_d = T^{-1}B, C_d = CT, D_d = D$$

and

$$B_d = \begin{bmatrix} b_n^d & b_{n-1}^d & \cdots & b_1^d & b_0^d \end{bmatrix}^T$$

$$K_d = \begin{bmatrix} k_n^d & k_{n-1}^d & \cdots & k_1^d & k_0^d \end{bmatrix}^T$$

State Feedback

Since

$$x(t) = Tz(t) \Leftrightarrow T^{-1}x(t) = z(t)$$

Then,

$$u(t) = K_d z(t) = K_d T^{-1} x(t) \equiv Kx(t)$$

with

$$K = K_d T^{-1}$$

State Feedback

Examples :

State Feedback

We consider the linear, time-invariant system

$$\dot{x} = Ax + Bu + Ew,$$

where w is a disturbance (which may be measurable or not) modeled by

$$\dot{w} = A_w w$$

We consider a reference r modeled by

$$\dot{r} = A_r r$$

The dynamics of the tracking error e=x-r is given by

$$\dot{e} = Ae + (A - A_r)r + Ew + Bu = Ae + Fv + Bu$$

$$F = \begin{bmatrix} A - A_r & E \end{bmatrix}, v = \begin{bmatrix} r \\ w \end{bmatrix}$$

State Feedback

Let us consider a control law

$$u = -K_e e - K_r r - K_w w = -K_e e - K_v v$$

$$K_v = \begin{bmatrix} K_r & K_w \end{bmatrix}, v = \begin{bmatrix} r \\ w \end{bmatrix}$$

Then, the closed-loop error dynamics is given by

$$\dot{e} = (A - BK_e)e + (F - BK_v)v$$

In steady state, the constant error e_{ss} is given by

$$0 = (A - BK_e)e_{ss} + (F - BK_v)v \Rightarrow e_{ss} = (A - BK_e)^{-1}(F - BK_v)v$$

Performance requirements can be summarized as:

• The closed-loop system should be asymptotically stable

$$eig(A - BK_e) < 0$$

A linear combination of the error must be zero in steady state

$$y_{ss} = C_{ss}e_{ss} = C_{ss}(A - BK_e)^{-1}(F - BK_v)v = 0$$

State Feedback

We want the steady state zero condition to hold for any v. Then,

$$C_{ss}(A - BK_e)^{-1}(F - BK_v) = 0$$

or equivalently

$$C_{ss}(A - BK_e)^{-1}BK_v = C_{ss}(A - BK_e)^{-1}F$$

The unknown variable is K_v. If $C_{ss}(A - BK_e)^{-1}B$ is square,

$$K_v = \left[C_{ss}(A - BK_e)^{-1}B\right]^{-1} C_{ss}(A - BK_e)^{-1}F$$

If number of outputs > number of inputs $\Rightarrow$ overdetermined (there may not exist solution). If number of outputs < number of inputs $\Rightarrow$ underdetermined (there may exist more than one solution).