ME 433 – STATE SPACE CONTROL

Lecture 4

State Transformation

We consider the linear, time-invariant system

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

We define the state transformation

$$x(t) = Tz(t) \Leftrightarrow T^{-1}x(t) = z(t)$$

Then we can write

$$T\dot{z} = ATz + Bu \Rightarrow \dot{z} = T^{-1}ATz + T^{-1}Bu$$
$$y = CTz + Du.$$

to obtain

$$\dot{z} = \tilde{A}z + \tilde{B}u$$
$$y = \tilde{C}z + \tilde{D}u$$

$$\boxed{\tilde{A} = T^{-1}AT, \tilde{B} = T^{-1}B, \tilde{C} = CT, \tilde{D} = D}$$

The state-space representation is NOT unique!

Solution of State Equation

Time-invariant Dynamics:

We consider the linear, time-invariant, homogeneous system

$$\dot{x} = Ax + Bu$$

where A is a constant $n \times n$ matrix. The solution can be written as

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_{\tau}^{t} e^{A(t-\lambda)}Bu(\lambda)d\lambda$$

Time-variant Dynamics:

We consider the linear, time-variant, homogeneous system

$$\dot{x} = A(t)x + Bu$$

where A is a time-variant $n \times n$ matrix. The solution can be written as

$$x(t) = \Phi(t,\tau)x(\tau) + \int_{\tau}^{t} \Phi(t,\lambda)B(\lambda)u(\lambda)d\lambda$$

Transfer Function

We consider the linear, time-invariant system

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

We Laplace transform the state equation to obtain

$$sX(s) - x(0) = AX(s) + BU(s) \qquad \boxed{L[\dot{x}(t)] = sX(s) - x(0)}$$

$$X(s) = L[x(t)] \qquad U(s) = L[u(t)]$$

And we solve to obtain

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

We Laplace transform the output equation to obtain

$$Y(s) = CX(s) + DU(s)$$

Solution of State Equation

$$X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} BU(s)$$

We inverse Laplace transform to obtain

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

where $L\left[e^{At}\right] = (sI - A)^{-1}; \quad L\left[\int_0^t f(t-\tau) g(\tau) d\tau\right] = f(s) g(s)$

For LTI systems, the exponential matrix e^{At} is the transition matrix of the system. Its Laplace transform $\Phi(s)=(sI-A)^{-1}$ is called the resolvent of the system. Faddeeva's algorithm allows for a fast computation of $\Phi(s)$.

How can we compute the transition matrix e^{At} ?

1. Compute $sI-A$.
2. Obtain the resolvent $\Phi(s)$ by inverting $sI-A$.
3. Obtain the transition matrix by computing the inverse Laplace transform of the resolvent, element by element.

Transfer Function

Assuming $x(0)=0$ and combining state and output equations we obtain the transfer function

$$Y(s) = \left[C(sI - A)^{-1} B + D\right] U(s)$$

The transfer function does NOT depend on the state choice because it represents the input-output relationship.

Given two state space representations

$$\dot{x} = Ax + Bu, \qquad \dot{z} = \tilde{A}z + \tilde{B}u$$

$$y = Cx + Du. \qquad y = \tilde{C}z + \tilde{D}u$$

$$Y(s) = \left[\tilde{C}(sI - \tilde{A})^{-1} \tilde{B} + \tilde{D}\right] U(s) = \left[C(sI - A)^{-1} B + D\right] U(s)$$

Controllability and Observability

Let us consider the system

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

with

$$A = \begin{bmatrix} 2 & 3 & 2 & 1 \\ -2 & -3 & 0 & 0 \\ -2 & -2 & -4 & 0 \\ -2 & -2 & -2 & -5 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix}, C = \begin{bmatrix} 7 & 6 & 4 & 2 \end{bmatrix} D = 0$$

and transfer function

$$H(s) = C(sI - A)^{-1} B + D = \frac{s^3 + 9s^2 + 26s + 24}{s^4 + 10s^3 + 35s^2 + 50s + 4}$$

Controllability and Observability

The transfer function can be written as

$$H(s) = \frac{(s+2)(s+3)(s+4)}{(s+1)(s+2)(s+3)(s+4)} = \frac{1}{(s+1)}$$

What's going on? Let's see the system from a different angle. Consider:

$$x(t) = Tz(t) \Leftrightarrow T^{-1}x(t) = z(t)$$

with

$$T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

to obtain

Controllability and Observability

$$\dot{z} = \tilde{A}z + \tilde{B}u$$
$$y = \tilde{C}z + \tilde{D}u$$

with

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix},$$

$$\tilde{B} = T^{-1}B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \tilde{C} = CT = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}, \tilde{D} = 0$$

Controllability and Observability

Let us assume the following modal form:

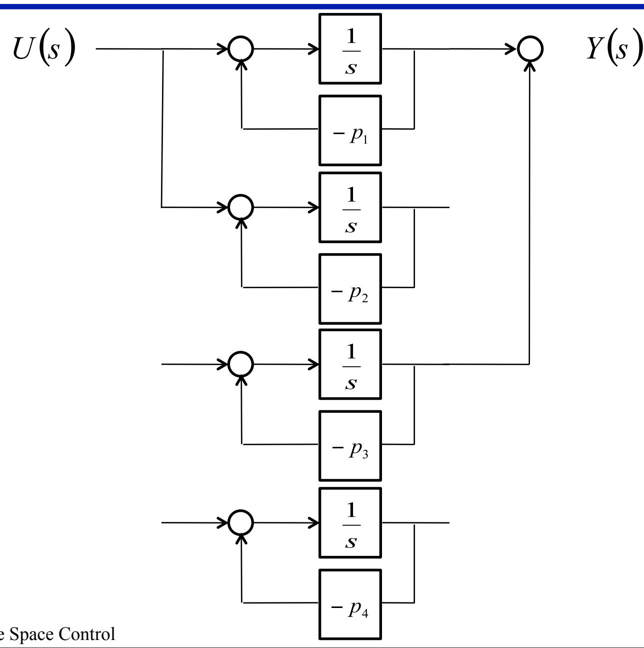
$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

$$A = \begin{bmatrix} -p_1 & 0 & 0 & 0 \\ 0 & -p_2 & 0 & 0 \\ 0 & 0 & -p_3 & 0 \\ 0 & 0 & 0 & -p_4 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} k_1 & 0 & k_3 & 0 \end{bmatrix} D = 0$$

$$Y(s) = \frac{k_1}{s+p_1}U(s) + \frac{0}{s+p_2}U(s) + \frac{k_3}{s+p_3}0 + \frac{0}{s+p_4}0$$

Only the controllable and observable mode appears in the transfer function

Controllability and Observability



Controllability

Problem Definition: "A system is said to be controllable if and only if it is possible, by means of the input, to transfer the system from any initial state $x(0)$ to any other state $x(t)$ in a finite time $t \geq 0$"

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Let us take

$$u(\tau) = B^T e^{A^T(t-\tau)}P^{-1}(t)\left[x(t) - e^{At}x(0)\right]$$

where

$$P(t) = \int_0^t e^{A(t-\tau)}BB^T e^{A^T(t-\tau)}d\tau$$

is invertible.

Controllability

Replacing u in the expression for x we will demonstrate that this is an input that drives the state from $x(0)$ to $x(t)$.

$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}B\left[B^T e^{A^T(t-\tau)}P^{-1}(t)\left[x(t) - e^{At}x(0)\right]\right]d\tau \\
&= e^{At}x(0) + \int_0^t e^{A(t-\tau)}BB^T e^{A^T(t-\tau)}d\tau P^{-1}(t)\left[x(t) - e^{At}x(0)\right] \\
&= e^{At}x(0) + P(t)P^{-1}(t)\left[x(t) - e^{At}x(0)\right] \\
&= x(t)
\end{aligned}
$$

It is indeed possible to show that this the only input that drives the state from $x(0)$ to $x(t)$.

Theorem: A system is controllable if and only if the matrix

$$P(t) = \int_0^t e^{A(t-\tau)}BB^T e^{A^T(t-\tau)}d\tau$$ Controllability Gramian

is nonsingular for some $t \geq 0$.

Proof: In class

Controllability

The time derivative of the Controllability Gramian

$$P(t) = \int_0^t e^{A(t-\tau)}BB^T e^{A^T(t-\tau)}d\tau$$

is given by (Leibniz's rule)

$$
\begin{aligned}
\frac{d}{dt}P(t) &= e^{A(t-\tau)}BB^T e^{A^T(t-\tau)}\Big|_{\tau=t} + \\
&+ \int_0^t \frac{d}{dt}\left[e^{A(t-\tau)}\right]BB^T e^{A^T(t-\tau)}d\tau + \int_0^t e^{A(t-\tau)}BB^T \frac{d}{dt}\left[e^{A^T(t-\tau)}\right]d\tau \\
&= BB^T + \int_0^t \frac{d}{dt}\left[e^{A(t-\tau)}\right]BB^T e^{A^T(t-\tau)}d\tau + \int_0^t e^{A(t-\tau)}BB^T \frac{d}{dt}\left[e^{A^T(t-\tau)}\right]d\tau
\end{aligned}
$$

Then, the Controllability Gramian satisfies

$$\dot{P}(t) = BB^T + AP(t) + P(t)A^T, \quad P(0) = 0$$ Lyapunov Equation

Controllability

Theorem: "A system is controllable if and only if the matrix

$$\bar{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$ Controllability Matrix

is full-rank."

Proof: In class

Observability

Problem Definition: "An unforced system is said to be observable if and only if it is possible to determine any (arbitrary initial) state $x(0)$ by using only a finite record, $y(\tau)$ for $0 \le \tau \le T$, of the output"

Theorem: "A system is observable if and only if the matrix

$$Q(t) = \int_0^t e^{A^T(t-\tau)} C^T C e^{A(t-\tau)} d\tau$$ Observability Gramian

is nonsingular for some $t \ge 0$."

The Observability Gramian satisfies

$$\dot{Q}(t) = C^T C + A^T Q(t) + Q(t) A, \quad Q(0) = 0$$ Lyapunov Equation

Theorem: "A system is observable if and only if the matrix

$$\bar{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$ Observability Matrix

is full-rank."

Controllability & Observability

Theorem (Popov-Belevitch-Hautus): *Eigenvector Tests*

A pair $\{A,B\}$ will be noncontrollable if and only if there exists a row vector $q \neq 0$ such that

$$qA = \lambda q, \qquad qB = 0$$

In other words, $\{A,B\}$ will be controllable if and only if there is no row (or left) eigenvector of A that is orthogonal to B.

A pair $\{C,A\}$ will be nonobservable if and only if there exists a column vector $p \neq 0$ such that

$$Ap = \lambda p, \qquad Cp = 0$$

In other words, $\{C,A\}$ will be observable if and only if there is no column (or right) eigenvector of A that is orthogonal to C.

Proof: In class

Controllability & Observability

Theorem (Popov-Belevitch-Hautus): *Rank Tests*

A pair $\{A,B\}$ will be controllable if and only if

$$rank\begin{bmatrix} sI - A & B \end{bmatrix} = n \qquad \text{for all } s$$

A pair $\{C,A\}$ will be observable if and only if

$$rank\begin{bmatrix} C \\ sI - A \end{bmatrix} = n \qquad \text{for all } s$$

Proof: In class

Controllability & Observability

Examples :