# ME 433 – STATE SPACE CONTROL

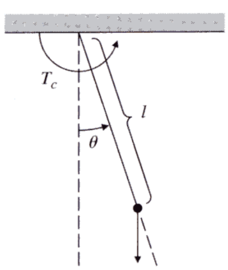
**Lecture 3**



# Dynamic Model

MECHANICAL SYSTEM:

$F = I\alpha$ Newton's law




damping coefficient

$$I\alpha = -lmg\sin\theta - b\omega + T_c$$

$\omega = \dot{\theta}$ angular velocity

$\alpha = \dot{\omega} = \ddot{\theta}$ angular acceleration

$I = ml^2$ moment of inertia

$$\ddot{\theta} = -\frac{b}{ml^2}\dot{\theta} - \frac{g}{l}\sin\theta + \frac{T_c}{ml^2}$$

Which are the equilibrium points when $T_c=0$?

At equilibrium: $\ddot{\theta} = \dot{\theta} = 0 \Rightarrow 0 = -\frac{g}{l}\sin\theta \Rightarrow \theta = 0, \pi$

Stable ($\theta = 0$)

Unstable ($\theta = \pi$)

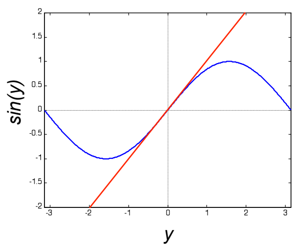
# Linearization

What happens around $\theta=0$?

$$\theta = y \Rightarrow \qquad \ddot{y} = -\frac{b}{ml^2}\dot{y} - \frac{g}{l}\sin(y) + \frac{T_c}{ml^2}$$

By Taylor Expansion:

$$\sin(y) = y + h.o.t. \Rightarrow \sin(y) \approx y$$




Linearized Equation:

$$\ddot{y} = -\frac{b}{ml^2}\dot{y} - \frac{g}{l}y + \frac{T_c}{ml^2}$$



# State-variable Representation

$$\ddot{y} = -\frac{b}{ml^2}\dot{y} - \frac{g}{l}y + \frac{T_c}{ml^2}$$ ⇨ Reduce to first order equations:

State Variable Representation

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \end{aligned} \Rightarrow \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{ml^2}x_2 - \frac{g}{l}x_1 + \frac{T_c}{ml^2} \end{aligned}$$

$$x \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, u \equiv T_c \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u = Ax + Bu$$

Is this state-space representation unique?

# Linearization

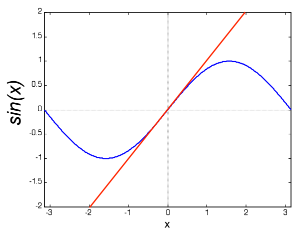
What happens around $\theta = \pi$?

$$\theta = \pi + x \Rightarrow \quad \ddot{x} = -\frac{b}{ml^2}\dot{x} - \frac{g}{l}\sin(\pi + x) + \frac{T_c}{ml^2}$$

$$\ddot{x} = -\frac{b}{ml^2}\dot{x} + \frac{g}{l}\sin(x) + \frac{T_c}{ml^2}$$

By Taylor Expansion:

$$\sin(x) = x + h.o.t. \Rightarrow \sin(x) \approx x$$




Linearized Equation:

$$\ddot{x} = -\frac{b}{ml^2}\dot{x} + \frac{g}{l}x + \frac{T_c}{ml^2}$$



# State-variable Representation

$$\ddot{x} = -\frac{b}{ml^2}\dot{x} + \frac{g}{l}x + \frac{T_c}{ml^2}$$

Reduce to first order equations:

State Variable Representation

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned} \Rightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{ml^2}x_2 + \frac{g}{l}x_1 + \frac{T_c}{ml^2} \end{aligned}$$

$$x \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, u \equiv T_c \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u = Ax + Bu$$

Is this state-space representation unique?

# State Transformation

We consider the linear, time-invariant system

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

We define the state transformation

$$x(t) = Tz(t) \Leftrightarrow T^{-1}x(t) = z(t)$$

Then we can write

$$T\dot{z} = ATz + Bu \Rightarrow \dot{z} = T^{-1}ATz + T^{-1}Bu$$
$$y = CTz + Du.$$

to obtain

$$\dot{z} = \tilde{A}z + \tilde{B}u$$
$$y = \tilde{C}z + Du$$

$$\boxed{\tilde{A} = T^{-1}AT, \tilde{B} = T^{-1}B, \tilde{C} = CT, \tilde{D} = D}$$

The state-space representation is NOT unique!



# Solution of State Equation

Time-invariant Dynamics:

We consider the linear, time-invariant, homogeneous system

$$\dot{x} = Ax$$

where $A$ is a constant $n \times n$ matrix. The solution can be written as

$$x(t) = e^{At}c$$

where

$$e^{At} = I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{3!} + \cdots$$

We can note that

$$\frac{de^{At}}{dt} = Ae^{At}$$

Then,

$$\dot{x} = Ae^{At}c = Ax$$

# Solution of State Equation

Let us assume that $x(\tau)$ is known. Then,

$$x(\tau) = e^{A\tau}c \Rightarrow c = e^{-A\tau}x(\tau)$$

The homogeneous solution can be finally written as

$$x(t) = e^{A(t-\tau)}x(\tau)$$

We consider now the linear, time-invariant, non-homogeneous system

$$\dot{x} = Ax + Bu$$

We assume a "particular" solution of the form

$$x(t) = e^{At}c(t)$$

Then,

$$\dot{c}(t) = e^{-At}Bu(t) \Rightarrow c(t) = \int_T^t e^{\ A\lambda}Bu(\lambda)d\lambda$$

and

$$x(t) = \int_T^t e^{A(t-\lambda)}Bu(\lambda)d\lambda$$



# Solution of State Equation

The overall solution can be written as

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_T^t e^{A(t-\lambda)}Bu(\lambda)d\lambda$$

At $t=\tau$

$$x(\tau) = x(\tau) + \int_T^\tau e^{A(\tau-\lambda)}Bu(\lambda)d\lambda \Rightarrow T = \tau$$

We finally can write the solution to the state equation as

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_\tau^t e^{A(t-\lambda)}Bu(\lambda)d\lambda$$

and the system output as

$$y(t) = Cx(t) + Du(t) = Ce^{A(t-\tau)}x(\tau) + \int_\tau^t Ce^{A(t-\lambda)}Bu(\lambda)d\lambda + Du(t)$$

Note that $B$, $C$ and $D$ can be functions of time.

# Solution of State Equation

Time-variant Dynamics:

We consider the linear, time-variant, homogeneous system

$$\dot{x} = A(t)x$$

where $A$ is a time-variant $n \times n$ matrix. The solution can be written as

$$x(t) = \Phi(t,\tau)x(\tau)$$

where $\Phi(t,\tau)$ is known as the "state-transition" matrix. For time-invariant systems, the state transition matrix is only function of $t$-$\tau$,

$$\Phi(t,\tau) = e^{A(t-\tau)}$$

We can write the state and output as

$$x(t) = \Phi(t,\tau)x(\tau) + \int_{\tau}^{t} \Phi(t,\lambda)B(\lambda)u(\lambda)d\lambda$$

$$y(t) = C(t)x(t) + D(t)u(t)$$

$$= C(t)\Phi(t,\tau)x(\tau) + \int_{\tau}^{t} \Phi(t,\lambda)C(\lambda)B(\lambda)u(\lambda)d\lambda + D(t)u(t)$$



# Solution of State Equation

Properties of the state transition matrix:

$$\dot{\Phi}(t,\tau) = A(t)\Phi(t,\tau)$$

$$\Phi(t,t) = I$$

$$\Phi(t_3,t_1) = \Phi(t_3,t_2)\Phi(t_2,t_1)$$

$$\Phi(\tau,t) = [\Phi(t,\tau)]^{-1}$$

For time-invariant systems: $\Phi(t_2,t_1) = \Phi(t_2 - t_1)$

$$\Phi(0) = I \qquad\qquad e^{A0} = I$$

$$\Phi(t)\Phi(\tau) = \Phi(t+\tau) \quad \Rightarrow \quad e^{At}e^{A\tau} = e^{A(t+\tau)}$$

$$\Phi^{-1}(t) = \Phi(-t) \qquad\qquad (e^{At})^{-1} = e^{-At}$$

NOTE: $e^{At}e^{Bt} = e^{(A+B)t}$ only if $AB = BA$.

# Solution of State Equation

Solution by the Laplace Transform:

We consider the linear, time-invariant system

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

We Laplace transform the state equation to obtain

$$sX(s) - x(0) = AX(s) + BU(s) \qquad L[\dot{x}(t)] = sX(s) - x(0)$$

$$X(s) = L[x(t)] \qquad U(s) = L[u(t)]$$

And we solve to obtain

$$X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} BU(s)$$



# Solution of State Equation

$$X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} BU(s)$$

We inverse Laplace transform to obtain

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

where we have used that

$$L\left[e^{At}\right] = (sI - A)^{-1}$$

$$L\left[\int_0^t f(t-\tau) g(\tau) d\tau\right] = f(s) g(s)$$

# State Transformation

$$X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} BU(s)$$

We Laplace transform the output equation to obtain

$$Y(s) = CX(s) + DU(s)$$

Assuming $x(0)=0$ and combining state and output equations we obtain the transfer function

$$Y(s)/U(s) = C(sI - A)^{-1} B + D$$

The transfer function does NOT depend on the state choice because it represents the input-output relationship.

Given two state space representations

$$\dot{x} = Ax + Bu, \qquad \dot{z} = \tilde{A}z + \tilde{B}u$$

$$y = Cx + Du. \qquad y = \tilde{C}z + Du$$

$$Y(s)/U(s) = \tilde{C}\left(sI - \tilde{A}\right)^{-1} \tilde{B} + \tilde{D} = C(sI - A)^{-1} B + D$$

Proof: In class



# Model Representation

Scalar Differential Equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y = b_m u^{(m)} + b_{m-1} u^{(m-1)} + \cdots + b_1 \dot{u} + b_0 u$$

Transfer Function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

State Variable Representation

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Transfer Function

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1} B + D$$

# Model Representation

Example:

$$\dot{p} = -2p + 2q + 24u$$
$$\dot{q} = 4p - 9q + 4\dot{u} + 2u$$

Find: - Transfer function between $q(t)$ and $u(t)$
- Scalar ODE for $q(t)$

Scalar Differential Equation → Transfer Function

↑ ↑

State Variable Representation → Transfer Function



# Model Representation

Example:

$$\dddot{y} + 3\ddot{y} + 4\dot{y} + 2y = u$$

Find: - State variable representation

Scalar Differential Equation

↓

State Variable Representation

# Model Representation

Example: $$\frac{Y(s)}{U(s)} = \frac{6}{s^2 + 5s + 6}$$

Find: - State variable representation

Transfer function

↓

State Variable Representation



# Model Representation

Example: Find: - State variable representation

$$\dddot{y} + 3\ddot{y} + 4\dot{y} + 2y = 2\dot{u} + u$$

Scalar Differential Equation

↓

State Variable Representation

$$\frac{Y(s)}{U(s)} = \frac{s + 6}{s^2 + 5s + 6}$$

Transfer function

↓

State Variable Representation

# Model Representation

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}=\frac{Y(s)}{X(s)}\frac{X(s)}{U(s)}$$

$$\frac{X(s)}{U(s)}=\frac{1}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0},\ \frac{Y(s)}{X(s)}=b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_1s+b_0$$

Choosing $x_1=x^{(n-1)}, x_2=x^{(n-2)},\ldots,x_{n-1}=x^{(1)}, x_n=x$

$$A=\begin{bmatrix}-a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \cdot & \cdot & & \cdot & \cdot\\ 0 & 0 & \cdots & 1 & 0\end{bmatrix}, B=\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, C=\begin{bmatrix}b_{n-1} & b_{n-2} & \cdots & b_1 & b_0\end{bmatrix} D=0$$

Controller Form



# Model Representation

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}$$

$$\left(s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0\right)Y(s)=\left(b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_1s+b_0\right)U(s)$$

$$Y(s)=s^{-1}\left(b_{n-1}U(s)-a_{n-1}Y(s)\right)+s^{-2}\left(b_{n-2}U(s)-a_{n-2}Y(s)\right)+\cdots$$
$$+s^{1-n}\left(b_1U(s)-a_1Y(s)\right)+s^{-n}\left(b_0U(s)-a_0Y(s)\right)$$

Choosing $x_1=y^{(n-1)}, x_2=y^{(n-2)},\ldots,x_{n-1}=y^{(1)}, x_n=y$

$$A=\begin{bmatrix}0 & 0 & \cdots & 0 & -a_0\\ 1 & 0 & \cdots & 0 & -a_1\\ 0 & 1 & \cdots & 0 & \vdots\\ \cdot & \cdot & \ddots & \cdot & -a_{n-2}\\ 0 & 0 & \cdots & 1 & -a_{n-1}\end{bmatrix}, B=\begin{bmatrix}b_0\\b_1\\\vdots\\b_{n-2}\\b_{n-1}\end{bmatrix}, C=\begin{bmatrix}0 & 0 & \cdots & 0 & 1\end{bmatrix} D=0$$

Observer Form

# Model Representation

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_m s^m + b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=\frac{k_1}{s+p_1}+\cdots+\frac{k_n}{s+p_n}$$

Choosing

$$\dot{x}_i = -p_i x_i + u$$
$$y_i = k_i x_i$$

$$y=\sum_{i=1}^{n} y_i$$

$$A=\begin{bmatrix}-p_1 & 0 & \cdots & 0 & 0\\ 0 & -p_2 & \cdots & 0 & 0\\ 0 & 0 & \ddots & 0 & 0\\ \cdot & \cdot & \cdot & -p_{n-1} & 0\\ 0 & 0 & \cdots & 0 & -p_n\end{bmatrix}, B=\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}, C=\begin{bmatrix}k_1 & k_2 & \cdots & k_{n-1} & k_n\end{bmatrix} D=0$$

Modal Form