Nuclear Fusion and Radiation
Lecture 8 (Meetings 21, 22 & 23)

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Radioactivity

- The discovery of artificial, or induced, radioactivity started a new line of nuclear research and hundreds of artificial nuclei have been produced by many different nuclear reactions.

- A number of radioactive nuclides occur naturally on Earth:
  - One of the most important is $^{40}_{19}K$, which has an isotopic abundance of 0.0118% and a half-life of $1.28 \times 10^9 y$. Potassium is an essential element needed by plants and animals, and is an important source of human internal and external radiation exposure.

- Tritium ^3_1H and $^{14}_6C$ are produced by cosmic ray interactions in the upper atmosphere, and also can cause measurable human exposures. $^{14}_6C$ (half life $5730y$), which is the result of a neutron reaction with $^{14}_7N$ in the atmosphere, is incorporated into plants by photosynthesis. By measuring the decay of $^{14}_6C$ in ancient plant material, the age of the material can be determined.

- Other sources of terrestrial radiation are uranium, thorium, and their radioactive progeny. All elements with $Z > 83$ are radioactive. Uranium and thorium decay into daughter radionuclides, forming a series (or chain) of radionuclides that ends with a stable isotope of lead or bismuth.

Radioactivity

- In all nuclear interactions, including radioactive decay, there are several quantities that are always conserved or unchanged by the nuclear transmutation.

- The most important of these conservation laws include:

  - Conservation of charge, i.e., the number of elementary positive and negative charges in the reactants must be the same as in the products.

  - Conservation of the number of nucleons, i.e., A is always constant. With the exception of electron capture (EC) and $\beta^{\pm}$ radioactive decay, in which a neutron (proton) transmutes into a proton (neutron), the number of protons and neutrons is also generally conserved.

  - Conservation of mass / energy (total energy). Although, neither rest mass nor kinetic energy is generally conserved, the total (rest-mass energy equivalent plus kinetic energy) is conserved.

  - Conservation of linear momentum. This quantity must be conserved in all inertial frames of reference.

  - Conservation of angular momentum. The total angular momentum (or the spin) of the reacting particles must always be conserved.

Radioactivity

Decay Type	Reaction	Description
gamma (γ)	$^A_Z P^* \longrightarrow ^A_Z P + \gamma$	An excited nucleus decays to its ground state by the emission of a gamma photon.
alpha (α)	$^A_Z P \longrightarrow ^{A-4}_{Z-2} D + \alpha$	An α particle is emitted leaving the daughter with 2 fewer neutrons and 2 fewer protons than the parent.
negatron (β^-)	$^A_Z P \longrightarrow ^A_{z+1} D + \beta^- + \overline{\nu}$	A neutron in the nucleus changes to a proton. An electron (β^-) and an anti-neutrino ($\overline{\nu}$) are emitted.
positron (β^+)	$^A_Z P \longrightarrow ^A_{z-1} D + \beta^+ + \nu$	A proton in the nucleus changes into a neutron. A positron (β^+) and a neutrino (ν) are emitted.
electron capture (EC)	$^A_Z P + e^- \longrightarrow ^A_{z-1} D^* + \nu$	An orbital electron is absorbed by the nucleus, converts a nuclear proton into a neutron and a neutrino (ν), and, generally, leaves the nucleus in an excited state.
proton (p)	$^A_Z P \longrightarrow ^{A-1}_{Z-1} D + p$	A nuclear proton is ejected from the nucleus.
neutron (n)	$^A_Z P \longrightarrow ^{A-1}_Z D + n$	A nuclear neutron is ejected from the nucleus.
internal conversion (IC)	$^A_Z P^* \longrightarrow [^A_Z P]^+ + e^-$	The excitation energy of a nucleus is used to eject an orbital electron (usually a K-shell) electron.

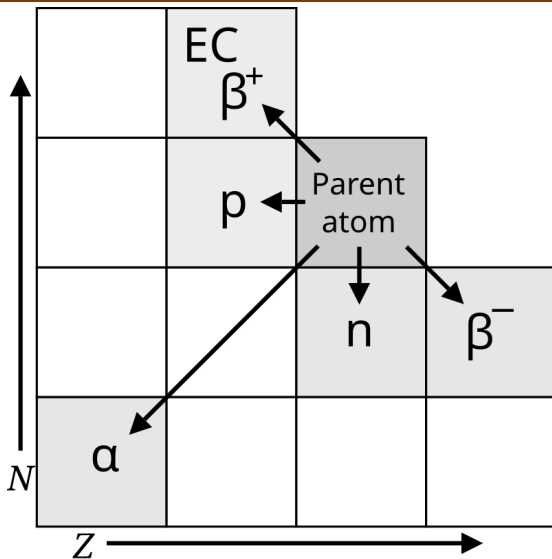Figure 1: Types of radioactive decay.

Radioactivity

Figure 2: Transition diagram for some key radioactive decays. N: neutrons, Z: protons.
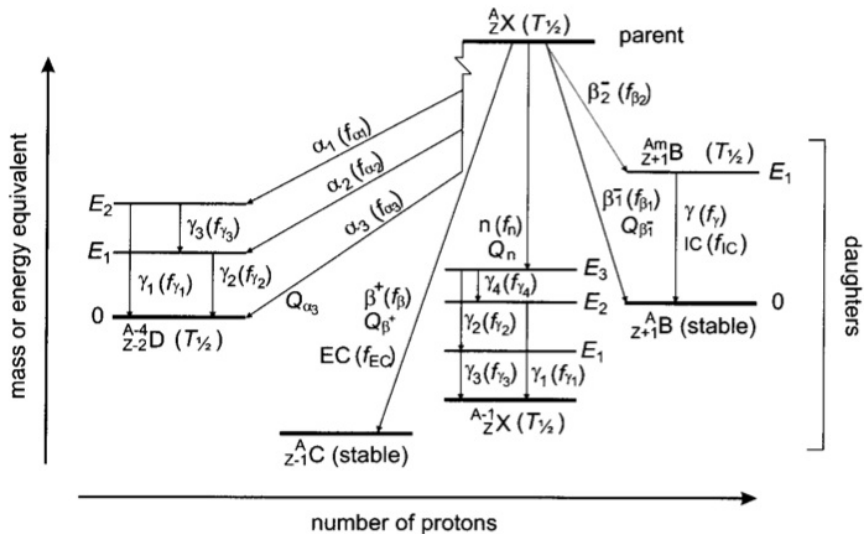
Figure 3: Radioactive decay diagram.

Radioactivity

- All radioactive decays, whatever the particles that are emitted or the rates at which they occur, are described by a single law: the *radioactive decay law*.

- The probability that an unstable parent nucleus will decay spontaneously into one or more particles of lower mass/ energy is independent of the past history of the nucleus and is the same for all radionuclides of the same type.

- There is no way of predicting whether or not a single nucleus will undergo decay in a given time period; however, we can predict the expected or average decay behavior of a very large number of identical radionuclides.

Radioactivity

- Consider a sample containing a very large number N of the same radionuclide.
- In a small time interval Δt, ΔN of the atoms undergo radioactive decay.
- The probability that any one of the radionuclides in the sample decays in Δt is thus $\Delta N/N$, where $\Delta N \triangleq N(t) - N(t + \Delta t) \geq 0$.
- If we performed such a measurement of the decay probability $\Delta N/N$ for different time intervals Δt and plot the ratio, we would obtain results such as those shown in the figure below.
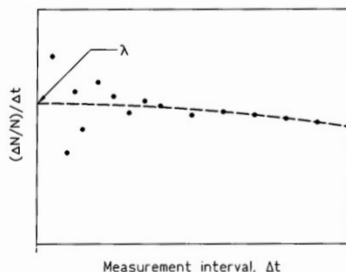

Figure 4: Measured decay rate $(\Delta N/N)/\Delta t$.

Radioactivity

- The statistical averaged decay probability per unit time, in the limit of infinitely small Δt, approaches a constant λ, i.e., we define

$$\lambda = \lim_{\Delta t \to 0} \frac{\Delta N/N}{\Delta t} \tag{1}$$

- Each radionuclide has its own characteristic decay constant λ which, from its definition, is the probability a radionuclide decays in a unit time for an infinitesimal time interval. The smaller λ, the more slowly the radionuclides decays. For stable nuclides, $\lambda = 0$.

- The decay constant for a radionuclide is independent of most experimental parameters such as temperature and pressure, since it depends only on the nuclear forces inside the nucleus.

Radioactivity

- Consider a sample composed of a large number of identical radionuclides with decay constant λ.

- With a large number of radionuclides ($N >>> 1$), one can use continuous mathematics to describe an inherently discrete process. Thus, $N(t)$ is interpreted as the average or expected number of radionuclides in the sample at time t, a continuous quantity.

- Then, the probability any one radionuclide decays in an interval dt is λdt. Thus, the expected number of decays in the sample that occur in dt at time t is $\lambda dt N(t)$. This must equal the change $-dN$ in the number of radionuclides in the sample (note that dN is negative since the population N decreases), i.e.,

$$-dN = \lambda N(t) dt \iff \frac{dN(t)}{dt} = -\lambda N(t) \tag{2}$$

- This is consistent with the definition of λ in (1) since

$$\lambda = \lim_{\Delta t \to 0} \frac{\Delta N/N}{\Delta t} = \frac{1}{N} \lim_{\Delta t \to 0} \frac{N(t) - N(t + \Delta t)}{\Delta t} = \frac{1}{N} \left(-\frac{dN(t)}{dt} \right) \tag{3}$$

Radioactivity

- The solution of this differential equation is

$$N(t) = N_o e^{-\lambda t} \qquad (4)$$

where N_o is number of radionuclides in sample at $t = 0$. This can be obtained by integrating (2) by separation of variables, i.e.

$$\int_{N_o}^{N(t)} \frac{dN}{N} = -\lambda \int_0^t dt \iff \ln N \big|_{N_o}^{N(t)} = -\lambda t \big|_0^t \iff \ln \frac{N(t)}{N_o} = -\lambda t \quad (5)$$

- This exponential decay of a radioactive sample is known as the radioactive decay law.

- Such an exponential variation with time not only applies to radionuclides, but to any process governed by a constant rate of change, such as decay of excited electron states of an atoms, the rate of growth of money earning compound interest, and the growth of human populations.

Radioactivity

- Any dynamic process governed by exponential decay (or growth) has a remarkable property. The time it takes for it to decay to one-half of (or to grow to twice) the initial value, $T_{1/2}$, is a constant called the *half-life*.

$$N(t) = N_o e^{-\lambda t} \rightarrow N(T_{1/2}) = N_o e^{-\lambda T_{1/2}} \equiv \frac{N_o}{2}$$

- Solving for $T_{1/2}$ yields

$$T_{1/2} = \frac{\ln 2}{\lambda} \approx \frac{0.693}{\lambda} \tag{6}$$

- Notice that the half-life is independent of time t.

Radioactivity

- After n half–lives, the initial number of radionuclides has then decreased by a multiplicative factor of $1/2^n$, i.e.,

$$N(nT_{1/2}) = \frac{1}{2^n} N_o$$

- The number of half–lives n needed for a radioactive sample to decay to a fraction ϵ of its initial value is

$$\epsilon \equiv \frac{N(nT_{1/2})}{N_o} = \frac{1}{2^n}$$

which, upon solving for n, yields $n = -\frac{\ln \epsilon}{\ln 2} \approx -1.44 \ln \epsilon$.

- Alternatively, by fitting (6) into (4), the radioactive decay law can be expressed in terms of the half-life as

$$N(t) = N_o \left(\frac{1}{2}\right)^{t/T_{1/2}} \tag{7}$$

Radioactivity

- From the exponential decay law, we can determine some useful <u>probabilities</u> and <u>averages</u>.

- If we have N_o identical radionuclides at $t = 0$, we expect to have $N(t) = N_o e^{-\lambda t}$ atoms at a later time t. Thus, the probability $\bar{P}$ that any one of the atoms does not decay in a time interval t is

$$\bar{P}(t) = \frac{N(t)}{N_o} = \frac{N_o e^{-\lambda t}}{N_o} = e^{-\lambda t} \tag{8}$$

- The probability P that a radionuclide does decay in a time interval t is

$$P(t) = 1 - \bar{P}(t) = 1 - \frac{N(t)}{N_o} = \frac{N_o - N(t)}{N_o} = \frac{\Delta N(t)}{N_o} = 1 - e^{-\lambda t} \tag{9}$$

Radioactivity

- As time interval becomes very small, i.e., $t \to \Delta t << 1$,

$$P(\Delta t) = 1 - e^{-\lambda \Delta t} = 1 - \left[1 - \lambda \Delta t + \frac{1}{2!}(\lambda \Delta t)^2 - \ldots \right] \approx \lambda \Delta t \qquad (10)$$

NOTE: We use the Taylor series expansion of $e^{-\lambda \Delta t}$.

- This approximation is in agreement with our earlier interpretation of the decay constant λ (see equation (1) where we defined $P(\Delta t) = \Delta N / N$) as being the decay probability per infinitesimal time interval.

- Let $p(t)dt$ be the probability a radionuclide, which exists at time $t = 0$, decays in the time interval between t and $t + dt$. Clearly,

$$p(t)dt = \bar{P}(t)P(dt) = \lambda e^{-\lambda t} dt \qquad (11)$$

where we have used (8) and (10) to write $\bar{P}(t) = e^{-\lambda t}$ and $P(dt) = \lambda dt$.

Radioactivity

- In radioactive samples, radionuclides decay at all times.

- We can note from exponential decay law that an infinite time is required for all radioactive atoms to decay.

- However, as time increases, fewer atoms decay.

- We can calculate the average lifetime of a radionuclide by using the decay probability distribution $p(t)dt$ in (11). Integrating by parts, the average or mean lifetime T_{avg} of a radionuclide is thus

$$
\begin{aligned}
T_{avg} = \int_0^\infty tp(t)dt &= \int_0^\infty t\lambda e^{-\lambda t}dt \\
&= -te^{-\lambda t}|_0^\infty + \int_0^\infty e^{-\lambda t}dt \\
&= \frac{1}{\lambda}
\end{aligned}
\tag{12}
$$

Radioactivity

- For detection and safety purposes, we are not really interested in the number of radioactive atoms in a sample; rather we are interested in the number of decays or transmutations per unit of time that occur within the sample.

- This decay rate, or activity $A(t)$, of a sample is given by

$$A(t) \triangleq -\frac{dN(t)}{dt} = \lambda N(t) = \lambda N_o e^{-\lambda t} = A_o e^{-\lambda t} \qquad (13)$$

where A_o is the activity at $t = 0$. Since the number of radionuclides in a sample decreases exponentially, the activity also decreases exponentially.

Radioactivity

- The SI unit used for activity is the becquerel (Bq) and is defined as one transformation per second.

- An older unit of activity, and one that is still sometimes encountered, is the curie (Ci) defined as $3.7 \times 10^{10} Bq$. One Ci is the approximate activity of one gram of $^{226}_{88}Ra$ (radium).

- In many instances, the activity of a radioactive sample is normalized to the mass or volume of the sample, e.g., curies / liter or Bq/g. This normalized activity is called the specific activity, which we denoted by $\hat{A}$. Many safety limits and regulations are based on the specific activity concept.

Radioactivity

- Of great practical importance is the determination of a radionuclide's half-life $T_{1/2}$, or, equivalently, its decay constant λ.
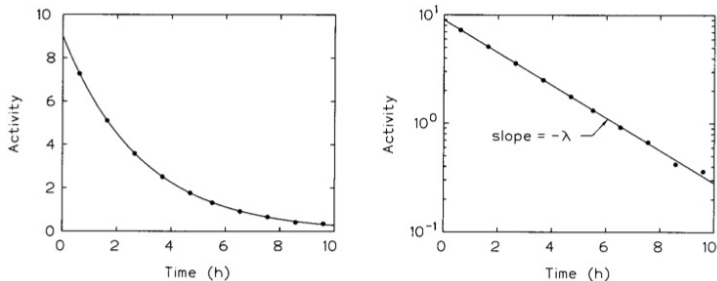

Figure 5: Linear vs. logarithmic activity plots.

- The decay constant λ is the slope of the $\ln A(t)$ line, i.e.

$$A(t) = \lambda N(t) = A_o e^{-\lambda t} \iff \ln A(t) = \ln A_o - \lambda t \qquad (14)$$

Radioactivity

- Some radionuclides decay by more than one process.

- To find the effective decay constant when the decay process has n competing decay modes, write the differential equation that models the rate of decay by denoting the decay constant for the ith mode by λ_i. Thus, the rate of decay of the parent radionuclide is

$$\frac{dN(t)}{dt} = -\lambda_1 N(t) - \cdots - \lambda_n N(t) = -\sum_{i=1}^{n} \lambda_i N(t) = -\lambda N(t) \qquad (15)$$

where $\lambda = \sum_{i=1}^{n} \lambda_i$ is the overall decay constant.

- The probability f_i that the nuclide will decay by the ith mode is

$$f_i = \frac{\text{decay rate by ith mode}}{\text{decay rate by all modes}} = \frac{\lambda_i}{\lambda} \qquad (16)$$

Radioactivity

Example: What is the value of the decay constant and the mean lifetime of ^{40}K (Potassium) (half-life 1.29 Gy)?

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Solution:

Using (6), the decay constant can be computed as

$$\lambda = \frac{\ln 2}{T_{1/2}} = \frac{\ln 2}{1.29 \times 10^9 y} = 5.37 \times 10^{-10} y^{-1}$$

Using (12), the average lifetime is $T_{avg} = 1/\lambda = 1.86 \times 10^9 y$.

Radioactivity

Example: The isotope ^{132}I (Iodine) decays by $\beta-$ emission to ^{132}Xe (Xenon) with a half-life of 2.3 h. (a) How long will it take for 7/8 of the original number of ^{132}I nuclides to decay? (b) How long will it take for a sample of ^{132}I to lose 95% of its activity?

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Solution:

(a) After 3 half-lives, the activity is $1/2^3 = 1/8$ of the original activity. Using (7), the time to obtain $1/8$ of the original activity is $t = 3 \times T_{1/2} = 6.9h$.

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(b) From the radioactive decay law (13), $A(t)/A(0) = e^{-\lambda t}$, upon solving for t, the time required to reach a specified value of $A(t)/A(0)$ is

$$t = -\frac{1}{\lambda} \ln \left[\frac{A(t)}{A(0)} \right]$$

In this problem $\lambda = \ln(2/T_{1/2}) = 0.3014h^{-1}$ (from (6)) and $A(t)/A(0) = 0.05$. Thus from the above equation we have

$$t = -\frac{1}{0.3014h^{-1}} \ln(0.05) = 9.34h.$$

Radioactivity

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Solution:

Because $A \triangleq \lambda N$ (from (13)), we have $N(atoms) = A(Bq)/\lambda(s^{-1})$.

For ^{24}Na we find from Table A.4 in the book that $T_{1/2} = 14.96h = 5.385 \times 10^4 s$. Then $\lambda = \ln 2/T_{1/2} = 1.287 \times 10^{-5} s^{-1}$ (from (6)).

Thus the number of atoms of ^{24}Na for a 1.20 MBq source is

$$N = \frac{A}{\lambda} = \frac{1.200 \times 10^6}{1.287 \times 10^{-5}} = 9.32 \times 10^{10} atoms.$$

Radioactivity

Example: What is the probability that ^{64}Cu (Copper) decays by positron emission? The decay constants for the three decay modes of this radioisotope are $\lambda_{\beta^+} = 0.009497h^{-1}$, $\lambda_{\beta^-} = 0.02129h^{-1}$, and $\lambda_{EC} = 0.02380h^{-1}$.

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Solution:

From (15), the overall decay constant is

$$\lambda = \lambda_{\beta^+} + \lambda_{\beta^-} + \lambda_{EC} = 0.009497h^{-1} + 0.02129h^{-1} + 0.02380h^{-1} = 0.05459h^{-1}$$

Using (16), the probability that an atom of ^{64}Cu eventually decays by positron emission is

$$\text{Probability of } \beta^+ \text{decay} = \lambda_{\beta^+}/\lambda = 0.009497/0.05459 = 0.174$$

Radioactivity

- A very important application of radioactive decay is the dating of geological and archaeological specimens.

- Measurements of daughter and parent concentrations in a specimen allow us to determine the sample's age because the decay rates of radionuclides in the sample serve as nuclear clocks running at a constant rate.

- If, at the time a sample was created, a known number of radioactive atoms of the same type, $N(0)$, was incorporated, the sample age t is readily found from the remaining number of these atoms, $N(t)$, and the radioactive decay law $N(t) = N(0)e^{-\lambda t}$:

$$t = -\frac{1}{\lambda} \ln \left(\frac{N(t)}{N(0)} \right) \tag{17}$$

Radioactivity

Method 1:

- Unfortunately, we never know $N(0)$. However, the initial ratio $N(0)/N_s$ of the radionuclide and some stable isotope of the same element can sometimes be estimated with reliability. This ratio also decays with the same radioactive decay law as the radionuclide. Thus,

$$t = -\frac{1}{\lambda} \ln \left(\frac{N(t)/N_s}{N(0)/N_s} \right) \qquad (18)$$

- The most common dating method using this approach is ^{14}C dating. The radionuclide ^{14}C has a half-life of $5730y$ and is introduced into the environment by cosmic-ray $^{14}N(n,p)^{14}C$ interactions in the atmosphere and removed by radioactive decay, leading if unperturbed to an equilibrium ratio of ^{14}C to all carbon atoms in the environment of about

$$N_{14}/N_C = 1.23 \times 10^{-12}. \qquad (19)$$

Radioactivity

- It is usually easier to measure the specific activity of ^{14}C in a sample, i.e., A_{14} per gram of carbon. This specific activity is proportional to the N_{14}/N_C ratio, since

$$\hat{A}_{14} = \frac{A_{14}}{g(C)} = \frac{\lambda_{14} N_{14}}{\frac{12 N_C}{N_a}} = \left(\frac{N_{14}}{N_C}\right) \frac{\lambda_{14} N_a}{12} = 0.237 \frac{Bq}{g(C)} = 6.4 \frac{pCi}{g(C)} \quad (20)$$

Note: $\left(\frac{N_{14}}{N_C}\right) \frac{\lambda_{14} N_a}{12} = (1.23 \times 10^{-12}) \frac{\ln 2}{5730 \times 365.25 \times 24 \times 3600 s} (6.022 \times 10^{23}) \frac{1}{12g} = 0.237 \frac{1}{gs} = 0.237 \frac{Bq}{g}$.

- Note that the activity of a radionuclide can be normalized per gram of radionuclide. In this case, a sample containing only $N(t)$ atoms of the radionuclide would have a mass $m(t) = N(t)M/N_a$, where M is the molar mass (g/mol) of the radionuclide and N_a is the Avogadro Number. The specific activity, on a per unit mass of the radionuclide, is then

$$\hat{A}(t) = \frac{A(t)}{m(t)} = \frac{\lambda N(t)}{N(t)M/N_a} = \frac{\lambda N_a}{M} = \text{constant.} \quad (21)$$

Radioactivity

- All isotopes of carbon are incorporated by a living (biological) organism, either through ingestion or photosynthesis, in the same proportion that exists in its environment. Once the entity dies, the $N_{14}(t)/N_C$ ratio decreases as the ^{14}C atoms decay.

- Thus, a carbonaceous archaeological artifact, such as an ancient wooden axe handle or a mummy, had an initial $A_{14}(0)/g(C)$ ratio of about $6.4 pCi/g$. From a measurement of the present $N_{14}(t)/N_C$ ratio, the age of the artifact can be determined as

$$t = -\frac{1}{\lambda} \ln \left(\frac{N_{14}(t)/N_C}{N_{14}(0)/N_C} \right) = -\frac{1}{\lambda} \ln \left(\frac{A_{14}(t)/g(C)}{A_{14}(0)/g(C)} \right) \qquad (22)$$

Radioactivity

Example: What is the age of a sample of charcoal from an ancient fire that has a $A14(t)/g(C)$ ratio of $1.2 pCi/g$?

Example: What is the age of a sample of charcoal from an ancient fire that has a $A14(t)/g(C)$ ratio of $1.2pCi/g$?

Solution:

The age of the wood, from (22)

$$t = -\frac{1}{\lambda} \ln\left(\frac{A_{14}(t)/g(C)}{A_{14}(0)/g(C)}\right) = -\frac{1}{\lambda} \ln\left(\frac{1.2 \times 10^{-12}}{6.4 \times 10^{-12}}\right) = -\frac{5730yr}{\ln 2} \ln\left(\frac{1.2}{6.4}\right) \approx 13,800yr$$

Radioactivity

Method 2:

- Consider a sample containing an initial number of identical parent radioactive atoms $N_1(0)$, decaying to N_2 stable daughter atoms. For simplicity, let us assume there are initially no daughter nuclides in sample.

- If we further assume that there is no loss (or gain) of the parent N_1 and daughter N_2 atoms from the sample since its formation, the number of these atoms is then

$$N_1(t) = N_1(0)e^{-\lambda t} \text{ and } N_2(t) = N_1(0)[1 - e^{-\lambda t}]. \tag{23}$$

- Dividing the second equation by the first and solving the resulting equation for the sample age t yields

$$t = \frac{1}{\lambda} \ln \left(1 + \frac{N_2(t)}{N_1(t)} \right) \tag{24}$$

- The atom ratio $N_2(t)/N_1(t)$ in the sample is the same as the concentration ratio and can readily be found from mass spectroscopy or chemical analysis.

Radioactivity

Example: Long-lived ^{232}Th (Thorium) ($T_{1/2} = 14.05 \times 10^9 y$) decays through a series of much shorter lived daughters to the stable isotope ^{208}Pb (Lead). The number of atoms of ^{208}Pb in a geological rock sample, assuming no initial inventory of ^{208}Pb in the sample, equals the number of initial ^{232}Th atoms that have decayed since the rock was formed. The number of decayed ^{232}Th atoms in the form of intermediate daughters, which are in secular equilibrium and have not yet reached ^{208}Pb, is negligibly small. What is the age of a rock sample that is found to have $m_{Th} = 1.37g$ of ^{232}Th and $m_{Pb} = 0.31g$ of ^{208}Pb?

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Solution:

The corresponding atom ratio of these two isotopes is thus

$$\frac{N_{Pb}(t)}{N_{Th}(t)} = \frac{m_{Pb} N a / M_{Pb}}{m_{Th} N a / M_{Th}} \approx \frac{m_{Pb}}{m_{Th}} \frac{A_{Th}}{A_{Pb}}$$

If there is no ^{208}Pb initially in the rock, then from (24) the rock's age is

$$t = \frac{1}{\frac{\ln 2}{T_{1/2}}} \ln \left(1 + \frac{m_{Pb}}{m_{Th}} \frac{A_{Th}}{A_{Pb}} \right) = \frac{1}{\frac{\ln 2}{14.05 \times 10^9 yr}} \ln \left(1 + \frac{0.31}{1.37} \frac{232}{208} \right) = 4.56 \times 10^9 yr$$

Method 3:

- Often, however, the initial value $N_2(0)$ of stable daughter is not zero, and a slightly more refined analysis must be used. In this case, the number of atoms of a radioactive parent and a stable daughter in a sample at age t is

$$N_1(t) = N_1(0)e^{-\lambda t} \text{ and } N_2(t) = N_2(0) + N_1(0)[1 - e^{-\lambda t}]. \qquad (25)$$

- How do we handle the extra unknown variable?

- Suppose there exists in a sample N_2' atoms of another stable isotope of the same element as the daughter, one that is not formed as a product of another naturally occurring decay chain. The ratio $R = N_2/N_2'$ is known from the relative abundances of the stable isotopes of the element in question and is a result of the mix of isotopes in the primordial matter from which the solar system was formed.

Radioactivity

- In samples, without any of the radioactive parent, ratio remains constant, i.e., $N_2(t)/N_2'(t) = N_2(0)/N_2'(0) = R_o$.

- Samples in which R is observed to be larger than R_o must have been enriched as result of the parent decay.

- If we assume a sample has experienced no loss (or gain) of parent and daughter nuclides since its formation, then we may assume $N_2'(t) = N_2'(0)$.

- Dividing the previous equations by $N_2'(t) = N_2'(0)$ yields

$$\frac{N_1(t)}{N_2'(t)} = \frac{N_1(0)}{N_2'(0)}e^{-\lambda t} \text{ and } R(t) = R_o + \frac{N_1(0)}{N_2'(0)}[1 - e^{-\lambda t}]. \qquad (26)$$

Then,

$$R(t) = R_o + \frac{N_1(t)}{N_2'(t)}e^{\lambda t}[1 - e^{-\lambda t}] = R_o + \frac{N_1(t)}{N_2'(t)}[e^{\lambda t} - 1]. \qquad (27)$$

Radioactivity

- From this expression we can find the age, namely,

$$t = \frac{1}{\lambda} \ln \left\{ 1 + \frac{N_2'(t)}{N_1(t)} [R(t) - R_o] \right\} \tag{28}$$

- Thus, age of sample can be determined by three atom ratios (easier to obtain than absolute atom densities): $N_2'(t)/N_1(t)$ and $R(t) = N_2(t)/N_2'(t)$ (measurements) and the primordial relative isotopic ratio $R_o = N_2(0)/N_2'(0)$.

Radioactivity

Example: Some geological samples contain the long-lived radionuclide Rubidium ^{87}Rb (half-life $4.88 \times 10^{10}y$) which decays to stable ^{87}Sr. Strontium has another stable isotope ^{86}Sr. In samples without ^{87}Rb, the normal atomic ^{87}Sr to ^{86}Sr ratio is $R_o = 7.00/9.86 = 0.71$. What is the age of a rock that has an atomic ^{87}Sr to ^{86}Sr ratio $R(t) = 0.80$ and an atomic ^{87}Rb to ^{86}Sr ratio of 1.48?

Example: Some geological samples contain the long-lived radionuclide Rubidium ^{87}Rb (half-life $4.88 \times 10^{10} y$) which decays to stable ^{87}Sr. Strontium has another stable isotope ^{86}Sr. In samples without ^{87}Rb, the normal atomic ^{87}Sr to ^{86}Sr ratio is $R_o = 7.00/9.86 = 0.71$. What is the age of a rock that has an atomic ^{87}Sr to ^{86}Sr ratio $R(t) = 0.80$ and an atomic ^{87}Rb to ^{86}Sr ratio of 1.48?

Solution:

The age of this rock is estimated from (28) as

$$
t = \frac{1}{\frac{\ln 2}{T_{1/2}}} \ln\left\{1 + \frac{N_{86\,Sr}(t)}{N_{97\,Rb}(t)}[R(t) - R_o]\right\}
$$

$$
t = \frac{1}{\frac{\ln 2}{4.88 \times 10^{10} yr}} \ln\left\{1 + \frac{1}{1.48}[0.8 - 0.71]\right\}
$$

$$
= 4.16 \times 10^9 yr
$$

Radioactivity

- Transient behavior of number of atoms of a particular radionuclide in a sample depends on nuclide's rates of production and decay, initial values of it and its parents, and rate at which it escapes from the sample.

- The decay of radionuclides is often accompanied by the creation of new ones, either from the decay of a parent or from production by nuclear reactions such as cosmic ray interactions in the atmosphere or from neutron interactions in a nuclear reactor.

- If $Q(t)$ is rate at which the radionuclide is being created, the rate of change of number of radionuclides is

$$\frac{dN(t)}{dt} = -\text{rate of decay} + \text{rate of production} \qquad (29)$$

Radioactivity

- Therefore,

$$\frac{dN(t)}{dt} = -\lambda N(t) + Q(t) \tag{30}$$

- The most general solution of this differential equation is

$$N(t) = N_o e^{-\lambda t} + \int_0^t dt' Q(t') e^{-\lambda(t-t')}$$

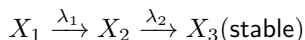where again N_o is the number of radionuclides at $t = 0$.

- For the special case that $Q(t) = Q_o$ (a constant production rate), this integral can be evaluated analytically to give

$$N(t) = N_o e^{-\lambda t} + \frac{Q_o}{\lambda}[1 - e^{-\lambda t}] \tag{31}$$

As $t \to \infty$ we see $N(t) \to N_e = Q_o/\lambda$.

Radioactivity

- Often a radionuclide decays to another radionuclide which in turn decays to yet another. The chain continues until a stable nuclide is reached. For simplicity, we consider a three component chain. Such three-member decay chains can be written schematically as

$$X_1 \xrightarrow{\lambda_1} X_2 \xrightarrow{\lambda_2} X_3(\text{stable})$$

- At $t = 0$ the number of atoms of each type in the sample under consideration is denoted by $N_i(0)$, $i = 1, 2, 3$. The differential decay equations for each species are (assuming no loss from or production in the sample)

Radioactivity

$$\frac{dN_1(t)}{dt} = -\lambda_1 N_1(t) \tag{32}$$

$$\frac{dN_2(t)}{dt} = -\lambda_2 N_2(t) + \lambda_1 N_1(t) \tag{33}$$

$$\frac{dN_3(t)}{dt} = \lambda_2 N_2(t) \tag{34}$$

With solutions

$$N_1(t) = N_1(0)e^{-\lambda_1 t} \tag{35}$$

$$N_2(t) = N_2(0)e^{-\lambda_2 t} + \frac{\lambda_1 N_1(0)}{\lambda_2 - \lambda_1}[e^{-\lambda_1 t} - e^{-\lambda_2 t}] \tag{36}$$

$$N_3(t) = N_3(0) + N_2(0)[1 - e^{-\lambda_2 t}] + \frac{N_1(0)}{\lambda_2 - \lambda_1}[\lambda_2(1 - e^{-\lambda_1 t}) - \lambda_1(1 - e^{-\lambda_2 t})] \tag{37}$$

Radioactivity

Example: A radioactive source is prepared by chemically separating Strontium ^{90}Sr from other elements. Initially, the source contains only ^{90}Sr ($T_{1/2} = 29.12y$), but this radionuclide decays to a radioactive daughter Yttrium ^{90}Y ($T_{1/2} = 64.0h$), which, after some time, reaches secular equilibrium with its parent. What is the time after the source is created that the activity of the daughter ^{90}Y is within 5% of that of the parent?

NOTE: Secular equilibrium is a situation in which the quantity of a radioactive isotope remains constant because its production rate (e.g., due to decay of a parent isotope) is equal to its decay rate.

$$\frac{dN_2(t)}{dt} = -\lambda_2 N_2(t) + \lambda_1 N_1(t) \text{ with } \frac{dN_2(t)}{dt} \equiv 0 \Rightarrow N_2 = \frac{\lambda_1}{\lambda_2} N_1 \qquad (38)$$

Radioactivity

Solution:
$$^{90}Sr(T_{1/2} = 29.12yr) \rightarrow^{90} Y(T_{1/2} = 64h) \rightarrow ???$$

Sr: Strontium, Y: Yttrium. From (35) and (36),

$$A_1(t) = A_1(0)e^{-\lambda_1 t}$$
$$A_2(t) = A_1(0)\frac{\lambda_2}{\lambda_2 - \lambda_1}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]$$

Then,

$$0.05 = \frac{A_1(t) - A_2(t)}{A_1(t)} = 1 - \frac{A_2(t)}{A_1(t)} = 1 - \frac{A_1(0)}{A_1(0)e^{-\lambda_1 t}}\frac{\lambda_2}{\lambda_2 - \lambda_1}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]$$
$$= 1 - \frac{\lambda_2}{\lambda_2 - \lambda_1}\left[1 - e^{(\lambda_2 - \lambda_1)t}\right],$$

With $\lambda_1 = \ln 2/T_{1/2}^{Sr} = 2.715 \times 10^{-6}h^{-1}$ and $\lambda_2 = \ln 2/T_{1/2}^{Y} = 1.083 \times 10^{-2}h^{-1}$,

$$t = \frac{1}{\lambda_2 - \lambda_1}\ln\left[1 - (1 - 0.05)\frac{\lambda_2 - \lambda_1}{\lambda_2}\right] = 11.52d$$

Radioactivity - Gamma Decay

- The gamma-decay reaction of an excited isotope of element P can be written as

$$^A_Z P^* \rightarrow {}^A_Z P + \gamma \qquad (39)$$

- Energy conservation for this nuclear reaction requires

$$M(^A_Z P^*)c^2 \equiv M(^A_Z P)c^2 + E^* = M(^A_Z P)c^2 + E_P + E_\gamma \qquad (40)$$

where E^* is the excitation energy (above the ground state) of the initial parent nucleus, E_γ is the energy of the gamma photon, and E_P is the recoil kinetic energy of the resulting ground-state nuclide.

- If the initial nuclide is at rest, the Q-value of this reaction is simply the sum of the kinetic energies of the products by definition, which yields

$$Q_\gamma = E_P + E_\gamma = E^* \qquad (41)$$

Radioactivity - Gamma Decay

- Linear momentum must also be conserved. Again with the parent at rest before the decay (i.e., with zero initial linear momentum), the gamma photon and recoil nucleus must move in opposite directions and have equal magnitudes of linear momentum.

- Since the photon has momentum $p_\gamma = E_\gamma/c$ and the recoil nuclide has momentum $M_p v_p = \sqrt{2M_p E_p}$, conservation of momentum requires

$$E_\gamma/c = \sqrt{2M_p E_p} \iff E_P = \frac{E_\gamma^2}{2M_p c^2} \tag{42}$$

where $M_P = M(_Z^A P)$. Since $Q_\gamma = E_P + E_\gamma$, we can write

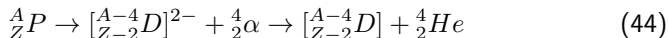$$E_\gamma = Q_\gamma \left[1 + \frac{E_\gamma}{2M_p c^2}\right]^{-1} \approx Q_\gamma = E^* \tag{43}$$

Radioactivity - Gamma Decay

- The approximation in this result follows from the fact that E_γ is at most $10 - 20 MeV$, while $2M_p c^2 \geq 4000 MeV$. Thus, in gamma decay, the kinetic energy of the recoil nucleus is negligible compared to the energy of the gamma photon and $E_\gamma \approx Q_\gamma = E^*$.

NOTE: Because of the comparatively large mass and low energy of the recoil atom, momentum of the recoil nucleus is computed using laws of classical mechanics.

Radioactivity - Alpha Decay

- In alpha decay, the nucleus of parent atom $_Z^A P$ emits an alpha particle. The resulting nucleus of daughter atom $_{Z-2}^{A-4} D$ has two fewer neutrons and two fewer protons.

- Initially, daughter still has Z electrons, two too many (i.e. it is a doubly negative ion $[_{Z-2}^{A-4} D]^{2-}$), but extra electrons quickly break away from atom, leaving it in neutral state.

- Fast moving doubly charged alpha particle quickly loses kinetic energy by ionizing/exciting atoms along its path and acquires 2 electrons to become neutral $_2^4 He$ atom.

- Since atomic number of daughter is different from that of parent, the daughter is a different chemical element.

- The alpha decay reaction is thus represented by

$$_Z^A P \rightarrow [_{Z-2}^{A-4} D]^{2-} + _2^4 \alpha \rightarrow [_{Z-2}^{A-4} D] + _2^4 He \tag{44}$$

Radioactivity - Alpha Decay

- The Q-value of a nuclear reaction is defined as the decrease in the rest mass energy (or increase in kinetic energy) of the product nuclei. For radioactive decay, the Q-value is sometimes called the *disintegration energy*.

- For alpha decay we have

$$
\begin{aligned}
Q_\alpha/c^2 &= M(^A_Z P) - \left[M([^{A-4}_{Z-2}D]^{2-}) + M(^4_2\alpha) \right] \\
&\approx M(^A_Z P) - \left[M([^{A-4}_{Z-2}D]) + 2m_e + M(^4_2\alpha) \right] \\
&\approx M(^A_Z P) - \left[M([^{A-4}_{Z-2}D]) + M(^4_2 He) \right]
\end{aligned}
\tag{45}
$$

- In this reduction to atomic masses, the binding energies of the two electrons in the daughter ion and in the He atom have been neglected since these are small (several eV) compared to the Q-value (several MeV).

Radioactivity - Alpha Decay

- For alpha decay to occur, Q_α, must be positive, i.e.

$$M(^A_Z P) > M([^{A-4}_{Z-2} D]) + M(^4_2 He) \qquad (46)$$

- The disintegration energy Q_α equals the kinetic energy of the decay products. How this energy is divided between the daughter atom and the α particle is determined from the conservation of momentum.

- Momentum of parent nucleus was zero before the decay, and thus, from conservation of linear momentum, the total momentum of the products must also be zero.

- The alpha particle and the daughter nucleus must, therefore, leave the reaction site in opposite directions with equal magnitudes of their linear momentum to ensure the vector sum of their momenta is zero.

Radioactivity - Alpha Decay

- Conservation of energy requires

$$Q_\alpha = E_D + E_\alpha = \frac{1}{2}M_D v_D^2 + \frac{1}{2}M_\alpha v_\alpha^2 \qquad (47)$$

- Conservation of linear momentum requires

$$M_D v_D = M_\alpha v_\alpha \qquad (48)$$

where $M_D \equiv M([{}_{Z-2}^{A-4}D])$ and $M_\alpha = M({}_2^4 He)$.

- These equations in the two unknowns v_D and v_α, can be solved to obtain the kinetic energies of the products, i.e.

$$Q_\alpha = \frac{1}{2}\frac{M_\alpha^2}{M_D}v_\alpha^2 + \frac{1}{2}M_\alpha v_\alpha^2 = \frac{1}{2}M_\alpha v_\alpha^2 \left[\frac{M_\alpha}{M_D} + 1\right] = E_\alpha \left[\frac{M_\alpha + M_D}{M_D}\right] \qquad (49)$$

Radioactivity - Alpha Decay

- Hence, the kinetic energy of the alpha particle is

$$E_\alpha = Q_\alpha \left[\frac{M_D}{M_D + M_\alpha} \right] \approx Q_\alpha \left[\frac{A_D}{A_D + A_\alpha} \right] \qquad (50)$$

- Notice that, in alpha decay, the alpha particle is emitted with a well defined energy. The recoiling nucleus carries off the remainder of the available kinetic energy. We can note that $E_D = Q_\alpha - E_\alpha$, so that from the above result

$$E_D = Q_\alpha \left[\frac{M_\alpha}{M_D + M_\alpha} \right] \approx Q_\alpha \left[\frac{A_\alpha}{A_D + A_\alpha} \right] \qquad (51)$$

Radioactivity - Alpha Decay

- The above calculation assumes that the alpha decay proceeds from the ground state of the parent nucleus to the ground state of the daughter nucleus.

- As discussed earlier, sometimes the daughter nucleus is left in an excited nuclear state (which ultimately relaxes to the ground state by gamma emmision).

- Nuclear mass of daughter is greater than the ground state mass by the mass equivalent of excitation energy.

- In these cases, the Q_α-value is reduced by the excitation energy E^* of the excited state, i.e.,

$$Q_\alpha/c^2 \approx M(_Z^A P) - \left[M([_{Z-2}^{A-4}D]) + M(_2^4 He) \right] - E^*/c^2 \tag{52}$$

- For the alpha decay to take place, $M(_Z^A P)$ must be greater than $M([_{Z-2}^{A-4}D]) + M(_2^4 He) + E^*/c^2$.
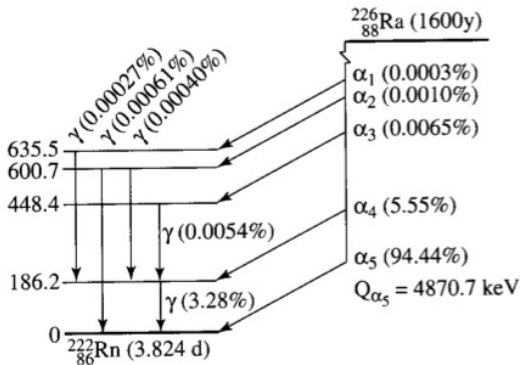
Figure 6: Energy levels for α decay of ^{226}Ra.

Example: What is the initial kinetic energy of the alpha particle produced in the radioactive decay $^{226}_{88}Ra \rightarrow \, ^{222}_{86}Rn + \, ^4_2He$? ($Ra$: Radium, Rn: Radon).

Example: What is the initial kinetic energy of the alpha particle produced in the radioactive decay $^{226}_{88}Ra \rightarrow ^{222}_{86}Rn + ^4_2He$? ($Ra$: Radium, Rn: Radon).

Solution:

The Q_α value in mass units (i.e., the mass defect) is, from (45) as

$$
\begin{aligned}
Q_\alpha &\approx \left\{ M(^{226}_{88}Ra) - \left[M([^{222}_{86}Rn]) + M(^4_2He) \right] \right\} c^2 \\
&= [226.025402 - 222.107571 - 4.00260325]u \times 931.5 Mev/u \\
&= 0.005228u \times 931.5 Mev/u = 4.870 MeV
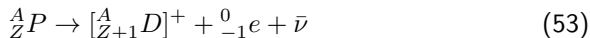\end{aligned}
$$

The kinetic energy of the alpha particle from (50) is

$$
E_\alpha = Q_\alpha \left[\frac{M_D}{M_D + M_\alpha} \right] \approx Q_\alpha \left[\frac{A_D}{A_D + A_\alpha} \right] = 4.870 MeV \left[\frac{222}{222 + 4} \right] = 4.784 MeV
$$

The remainder of the Q_α energy is the kinetic energy of the product nucleus (daughter), $^{222}_{86}Rn$, namely, $4.870 MeV - 4.783 MeV = 0.087 MeV$.

Radioactivity - Beta Decay

- Many neutron-rich radioactive nuclides decay by changing a neutron in the parent (P) nucleus into a proton and emitting an energetic electron.

- Different names are used for this decay: electron decay, beta minus decay, negatron decay, negative electron decay, negative beta decay, or simply beta decay.

- The ejected electron is called a *beta particle* denoted by β^-. The daughter atom, with one more proton in the nucleus, initially lacks one orbital electron, and thus is a single charged positive ion, denoted by $[{}_{Z+1}^{A}D]^+$. However, the daughter quickly acquires an extra orbital electron from the surrounding medium.

- The general β^- decay reaction is thus written as

$$
{}_{Z}^{A}P \rightarrow [{}_{Z+1}^{A}D]^+ + {}_{-1}^{0}e + \bar{\nu} \tag{53}
$$

Radioactivity - Beta Decay

- Here $\bar{\nu}$ is an antineutrino, a chargeless particle with very little, if any, rest mass.

- That a third product particle is involved with β^- decay is implied from the observed energy and momentum of the emitted β^- particle.

- If the decay products were only the daughter nucleus and the β^- particle, then, as in α decay, conservation of energy and linear momentum would require that the decay energy be shared in very definite proportions between them.

- However, β^- particles are observed to be emitted with a continuous distribution of energies that has a well defined maximum energy (see figure below).
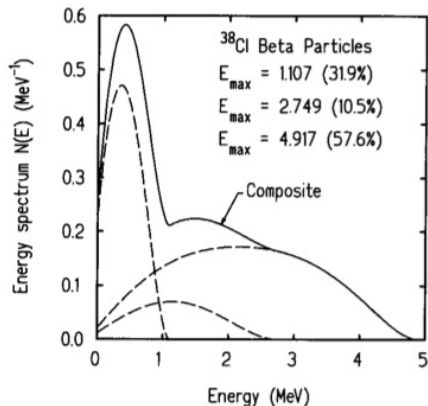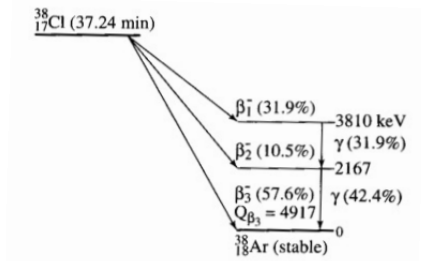
Radioactivity - Beta Decay

Figure 7: Energy spectra of ^{38}Cl β^- particles (Cl: Chlorine, Ar: Argon). Three groups of β^- particles are emitted.

Radioactivity - Beta Decay

- Pauli suggested in 1933 that at least three particles must be produced in a β^- decay. The vector sum of the linear momenta of three products can be zero without any unique division of the decay energy among them.

- In 1934 Fermi used Pauli's suggestion of a third neutral particle to produce a beta-decay theory which explained well the observed beta-particle energy distributions.

- This mysterious third particle, which Fermi named the neutrino (lit. "little neutral one"), has since been verified experimentally. Nowadays it is extensively studied by physicists developing fundamental theories of universe.

- The maximum energy of the β^- spectrum corresponds to a case in which the neutrino obtains zero kinetic energy, and the decay energy is divided between the daughter nucleus and the β^- particle.

Radioactivity - Beta Decay

- The beta decay energy is readily obtained from the Q-value of the decay reaction. Specifically,

$$
\begin{aligned}
Q_{\beta^-}/c^2 &= M(^A_Z P) - \left[M([^A_{Z+1}D]^+) + m_{\beta^-} + m_{\bar{\nu}} \right] \\
&\approx M(^A_Z P) - \left[\left\{ M([^A_{Z+1}D]) - m_e \right\} + m_{\beta^-} + m_{\bar{\nu}} \right] \\
&\approx M(^A_Z P) - M(^A_{Z+1}D)
\end{aligned}
\tag{54}
$$

Where we have used $m_e = m_{\beta^-}$ and $m_{\bar{\nu}} \equiv 0$.

- For β^- decay to occur spontaneously, Q_{β^-} must be positive or, equivalently, the mass of the parent atom must exceed that of the daughter atom, i.e.,

$$
M(^A_Z P) > M(^A_{Z+1}D)
\tag{55}
$$

Radioactivity - Beta Decay

- Often in β^- decay, the nucleus of the daughter is left in an excited state. For example, ^{38}Cl decays both to the ground state of the daughter ^{38}Ar as well as to two excited states.
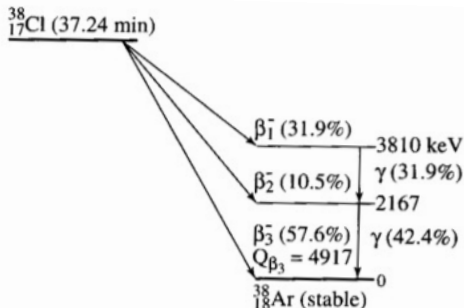

Figure 8: Decay diagram of ^{38}Cl (Cl: Chlorine, Ar: Argon). Three groups of β^- particles are emitted.

Radioactivity - Beta Decay

- The resulting β^- energy spectrum is a composite of the β^- particles emitted in the transition to each energy level of the daughter.

- For a decay to an energy level E^* above the ground, the mass of the daughter atom $M(^A_{Z+1}D)$ must be replaced by the mass of the excited daughter $M(^A_{Z+1}D^*) \approx M(^A_{Z+1}D) + E^*/c^2$. Thus,

$$Q_{\beta^-}/c^2 = M(^A_Z P) - M(^A_{Z+1}D) - E^*/c^2 \qquad (56)$$

- Because the kinetic energy of the parent nucleus is zero, the Q_{β^-} decay energy must be divided among the kinetic energies of the products. The maximum kinetic energy of the β^- particle occurs when the antineutrino obtains negligible energy. In this case, since the mass of the β^- particle is much less than that of the daughter nucleus, $Q_{\beta^-} = E_D + E_{\beta^-} \approx E_{\beta^-}$ or $(E_{\beta^-})_{max} \approx Q_{\beta^-}$.

Example: The radionuclide ^{41}Ar decays by β^- emission to an excited level of ^{41}K that is 1.293 MeV above the ground state. What is the maximum kinetic energy of the emitted β^- particle?
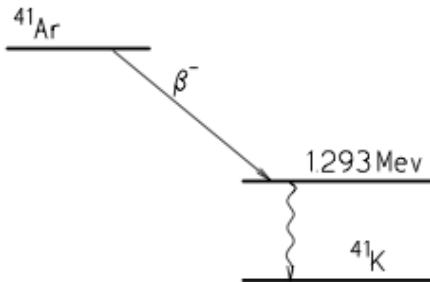

Figure 9: Decay diagram of ^{41}Ar (Ar: Argon, K: Potassium.

Solution:

We first find the Q_{β^-} for this decay from (56) with the atomic masses in App. B in the book. The result is

$$
\begin{aligned}
Q_{\beta^-} &= \left\{ M(^{41}_{18}Ar) - M(^{41}_{19}K) + E^*/c^2 \right\} c^2 \\
&= [40.9645008 - 40.96182597]u \times 931.5 MeV/u - 1.293 MeV \\
&= 2.492 MeV - 1.293 MeV \\
&= 1.199 MeV
\end{aligned}
$$

The maximum kinetic energy the beta particle can have equals this Q-value, i.e., $(E_\beta)_{max} = 1.199 MeV$.

Radioactivity - Positron Decay

- Nuclei that have too many protons for stability often decay by changing a proton into a neutron.

- In this decay mechanism, an anti-electron or *positron* β^+ or $^0_{+1}e$, and a neutrino ν are emitted.

- The daughter atom, with one less proton in the nucleus, initially has one too many orbital electrons, and thus is a negative ion, denoted by $[^A_{Z-1}D]^-$.

- However, the daughter quickly releases the extra orbital electron to the surrounding medium and becomes a neutral atom. The β^+ decay reaction is written as

$$^A_Z P \rightarrow [^A_{Z-1}D]^- + {}^0_{+1}e + \nu \tag{57}$$

Radioactivity - Positron Decay

- The positron has the same physical properties as an electron, except that it has one unit of positive charge. The positron β^+ is the antiparticle of the electron.

- The neutrino ν is required (as in β^- decay) to conserve energy and linear momentum since the β^+ particle is observed to be emitted with a continuous spectrum of energies up to some maximum value $(E_{\beta^+})_{max}$. The neutrino ν is the antiparticle of the antineutrino $\bar{\nu}$ produced in beta-minus decay.

- The decay energy is readily obtained from the Q-value of the decay reaction. Specifically,

$$
\begin{aligned}
Q_{\beta^+}/c^2 &= M(^A_Z P) - \left[M([^A_{Z-1}D]^-) + m_{\beta^+} + m_\nu \right] \\
&\approx M(^A_Z P) - \left[\left\{ M([^A_{Z-1}D]) + m_e \right\} + m_{\beta^+} + m_\nu \right] \\
&\approx M(^A_Z P) - M(^A_{Z-1}D) - 2m_e
\end{aligned}
$$

Radioactivity - Positron Decay

- The binding energy of the electron to the daughter ion and the neutrino mass have been neglected. If the daughter nucleus is left in an excited state, the excitation energy E^* must also be included in the Q_{β^+} calculation, namely,

$$Q_{\beta^+}/c^2 = M(_Z^A P) - M(_{Z-1}^A D) - 2m_e - E^*/c^2 \qquad (58)$$

- Thus, for β^+ decay to occur spontaneously, Q_{β^+}, must be positive, i.e.,

$$M(_Z^A P) > M(_{Z-1}^A D) + 2m_e + E^*/c^2 \qquad (59)$$

- The maximum energy of the emitted positron occurs when the neutrino acquires negligible kinetic energy, so that the Q_{β^+} energy is shared by the daughter atom and the positron.

Radioactivity - Positron Decay

- Because the daughter atom is so much more massive than the positron (by factors of thousands), almost all the $Q_{\beta+}$ energy is transferred as kinetic energy to the positron. Thus $(E_{\beta+})_{max} = Q_{\beta+}$.

- The emitted positron loses its kinetic energy by ionizing and exciting atomic electrons as it moves through the surrounding medium.

- Eventually, it captures an ambient electron, forming for a brief instant a pseudo-atom called *positronium* before they annihilate each other. Their entire rest mass energy $2m_e c^2$ is converted into photon energy (the kinetic energy at the time of annihilation usually being negligible).

- Before the annihilation, there is zero linear momentum, and there must be no net momentum remaining; thus, two photons traveling in opposite directions must be created, each with energy $E = m_e c^2 = 0.511 MeV$.

Radioactivity - Electron Capture

- In the quantum mechanical model of the atom, the orbital electrons have a finite (but small) probability of spending some time inside the nucleus, the innermost K-shell electrons having the greatest probability.

- It is possible for an orbital electron, while inside the nucleus, to be captured by a proton, which is thus transformed into a neutron.

- Conceptually we can visualize this transformation of the proton as $p + {}_{-1}^{0}e \to n + \nu$, where the neutrino is again needed to conserve energy and momentum. The general electron capture (EC) decay reaction is written as

$$
{}_{Z}^{A}P \to {}_{Z-1}^{A}D^* + \nu \tag{60}
$$

where the daughter is generally left in an excited nuclear state with energy E^* above ground level.

Radioactivity - Electron Capture

- Unlike in most other types of radioactive decay, no charged particles are emitted. The only nuclear radiations emitted are gamma photons produced when the excited nucleus of the daughter relaxes to its ground state. As the outer electrons cascade down in energy to fill the inner shell vacancy, X rays are also emitted.

- The decay energy is readily obtained from the Q-value of the decay reaction.

$$
\begin{aligned}
Q_{EC}/c^2 &= M(^A_Z P) - \left[M(^A_{Z-1} D^*) + m_\nu \right] \\
&= M(^A_Z P) - \left[M(^A_{Z-1} D) + m_\nu + E^*/c^2 \right] \\
&\approx M(^A_Z P) - M(^A_{Z-1} D) - E^*/c^2
\end{aligned}
$$

- Thus, for EC decay to occur spontaneously, Q_{EC} must be positive, i.e.,

$$
M(^A_Z P) > M(^A_{Z-1} D) + E^*/c^2 \tag{61}
$$

Radioactivity - Neutron Decay

- A few neutron-rich nuclides decay by emitting a neutron producing a different isotope of the same parent element. Generally, the daughter nucleus is left in an excited state which subsequently emits gamma photons as it returns to its ground state. This decay reaction is

$$
{}_{Z}^{A}P \rightarrow {}_{Z}^{A-1}D^* + {}_{0}^{1}n \tag{62}
$$

- The decay energy is readily obtained from the Q-value of the decay reaction.

$$
\begin{aligned}
Q_n/c^2 &= M({}_{Z}^{A}P) - \left[M({}_{Z}^{A-1}D^*) + m_n \right] \\
&= M({}_{Z}^{A}P) - M({}_{Z}^{A-1}D) - m_n - E^*/c^2
\end{aligned}
$$

where E^* is the initial excitation energy of the daughter nucleus.

- Thus, for neutron decay to occur to even the ground state ($E^* = 0$) of the daughter,

$$
M({}_{Z}^{A}P) > M({}_{Z}^{A-1}D) + m_n \tag{63}
$$

Radioactivity - Proton Decay

- A few proton-rich radionuclides decay by emission of a proton. In such decays, the daughter atom has an extra electron (i.e., it is a singly charged negative ion). This extra electron is subsequently ejected from the atom's electron cloud to the surroundings and the daughter returns to an electrically neutral atom. The proton decay reaction is thus

$$
{}_Z^A P \rightarrow [{}_{Z-1}^{A-1}D^*]^- + {}_1^1 p \tag{64}
$$

- Thus, the Q-value for this reaction is

$$
\begin{aligned}
Q_p/c^2 &= M({}_Z^A P) - \left[M([{}_{Z-1}^{A-1}D^*]^-) + m_p \right] \\
&\approx M({}_Z^A P) - \left[M({}_{Z-1}^{A-1}D^*) + m_e + m_p \right] \\
&\approx M({}_Z^A P) - \left[M({}_{Z-1}^{A-1}D) + E^*/c^2 + m_e + m_p \right] \\
&\approx M({}_Z^A P) - M({}_{Z-1}^{A-1}D) - M({}_1^1 H) - E^*/c^2
\end{aligned}
$$

- Thus, for proton decay to occur and leave the daughter in the ground state ($E^* = 0$), it is necessary that

$$
M({}_Z^A P) > M({}_{Z-1}^{A-1}D) + M({}_1^1 H) \tag{65}
$$

Radioactivity - Internal Conversion

- Often the daughter nucleus is left in an excited state, which decays (usually within about $10^{-9}s$) to the ground state by the emission of one or more gamma photons.

- However, the excitation may also be transferred to an atomic electron (usually a K-shell electron) causing it to be ejected from the atom leaving the nucleus in the ground state but the atom singly ionized with an inner electron-shell vacancy. Symbolically,

$$_Z^A P^* \to [_Z^A P]^+ + _1^0 e \tag{66}$$

- The inner electrons are very tightly bounded to the nucleus with large binding energies BE_e^K for K-shell electrons in heavy atoms. The amount of kinetic energy shared by the recoil ion and the ejected electron should take this into account.

Radioactivity - Internal Conversion

- The Q value for the IC decay is then calculated as follows:

$$
\begin{aligned}
Q_{IC}/c^2 &= M(_Z^A P^*) - \left[M([_Z^A P]^+) + m_e \right] \\
&\approx \left\{ M(_Z^A P) + E^*/c^2 \right\} - \left[\left\{ M(_Z^A P) - m_e + BE_e^K/c^2 \right\} + m_e \right] \\
&\approx \left[E^* - BE_e^K \right]/c^2
\end{aligned}
$$

- This decay energy is divided between the ejected electron and the daughter ion.

- To conserve the zero initial linear momentum, the daughter and IC electron must divide the decay energy as

$$
\begin{aligned}
E_e &= \left(\frac{M(_Z^A P)}{M(_Z^A P) + m_e} \right) \left[E^* - BE_e^K \right] \approx E^* - BE_e^K \\
E_D &= \left(\frac{m_e}{M(_Z^A P) + m_e} \right) \left[E^* - BE_e^K \right] \approx 0
\end{aligned}
$$