Nuclear Fusion and Radiation

Lecture 5 (Meetings 9, 10, 11 & 12) - Problem 2

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Power Balance in Fusion Plasmas

Problem: Consider a fully ionized DT plasma with $n_e = 10^{20}/m^3$, $kT_e = kT_i = kT = 10keV$, and $n_D = n_T \triangleq n_{DT}$. Impurity concentration is negligible.

(a) (50 points) What is the α's build-up concentration $r_\alpha = n_\alpha / n_e$ that makes the Bremsstrahlung power loss equal to a quarter of the α-power production (i.e., $P_b = f_\alpha P_\alpha$ with $f_\alpha = 0.25$)?

(b) (25 points) What is the necessary auxiliary power (kW) to achieve breakeven condition $(Q = 1)$?

(c) (25 points) What is the necessary energy confinement time τ_E to achieve steady-state operation?

Hint: Use $C = 5 \times 10^{-37} \frac{Wm^3}{keV^{1/2}}$ as Bremsstrahlung power constant, $< \sigma v >_{DT}^{10keV} = 1.09 \times 10^{-22} \frac{m^3}{s}$ and $E_\alpha = 3.5MeV$. Remember that $1eV = 1.60217733 \times 10^{-19} J$.

(a) (50 points) What is the α's build-up concentration $r_\alpha = n_\alpha / n_e$ that makes the Bremsstrahlung power loss equal to a quarter of the α-power production (i.e., $P_b = f_\alpha P_\alpha$ with $f_\alpha = 0.25$)?

The Bremsstrahlung power loss is given by

$$P_b = C n_e^2 T^{1/2} Z_{eff}. \quad (1)$$

The effective average charge of the ions in the plasma is defined as:

$$Z_{eff} = \sum_i \frac{n_i \langle \bar{Z}_i \rangle^2}{n_e} = \frac{n_D \langle \bar{Z}_D \rangle^2 + n_T \langle \bar{Z}_T \rangle^2 + n_\alpha \langle \bar{Z}_\alpha \rangle^2}{n_e} = \frac{n_D + n_T}{n_e} + \frac{n_\alpha}{n_e} Z_\alpha^2.$$

Assuming only α-particle impurities and applying the charge neutrality condition we obtain:

$$\begin{aligned} n_e &= n_D Z_D + n_T Z_T + n_\alpha Z_\alpha \\ 1 &= \frac{n_D + n_T}{n_e} + \frac{n_\alpha}{n_e} Z_\alpha \\ \frac{n_D + n_T}{n_e} &= 1 - \frac{n_\alpha}{n_e} Z_\alpha. \end{aligned}$$

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This allows us to calculate

$$Z_{eff} = 1 - \frac{n_\alpha}{n_e} Z_\alpha + \frac{n_\alpha}{n_e} Z_\alpha^2 = 1 + \frac{n_\alpha}{n_e} Z_\alpha \left(Z_\alpha - 1 \right) = 1 + 2\frac{n_\alpha}{n_e}, \qquad (2)$$

where we have used the fact that $\langle \bar{Z}_D \rangle = Z_D = 1$, $\langle \bar{Z}_T \rangle = Z_T = 1$, $\langle \bar{Z}_\alpha \rangle = Z_\alpha = 2$. Substituting into (1) we obtain

$$P_b = C n_e^2 T^{1/2} \left[1 + 2\frac{n_\alpha}{n_e} \right] = 15.8 \times 10^{-3} MW/m^3 \left[1 + 2\frac{n_\alpha}{n_e} \right].$$

Assuming $n_D = n_T \triangleq n_{DT}$, the charge neutrality condition gives

$$n_{DT} = \frac{1}{2} \left[1 - 2\frac{n_\alpha}{n_e} \right] n_e \iff 2 n_{DT} = (1 - 2 r_\alpha) n_e, \qquad (3)$$

and the α-particle power produced would be

$$\begin{aligned} P_\alpha &= n_D n_T \langle \sigma v \rangle E_\alpha \\ &= \left[\frac{1}{2} \left(1 - 2\frac{n_\alpha}{n_e} \right) n_e \right]^2 \langle \sigma v \rangle_{DT}^{10keV} E_\alpha \\ &= 0.1526 MW/m^3 \left[1 - 2\frac{n_\alpha}{n_e} \right]^2. \end{aligned}$$

Setting $P_b = f_\alpha P_\alpha$ gives

$$\begin{aligned} C n_e^2 T^{1/2} \left(1 + 2\frac{n_\alpha}{n_e}\right) &= f_\alpha \frac{n_e^2}{4} \langle \sigma v \rangle_{DT}^{10keV} E_\alpha \left(1 - 2\frac{n_\alpha}{n_e}\right)^2 \\ \left(1 + 2\frac{n_\alpha}{n_e}\right) &= f_\alpha \frac{\frac{n_e^2}{4} \langle \sigma v \rangle_{DT}^{10keV} E_\alpha}{C n_e^2 T^{1/2}} \left(1 - 2\frac{n_\alpha}{n_e}\right)^2 \end{aligned}$$

Defining

$$c_\alpha = f_\alpha \frac{\frac{n_e^2}{4} \langle \sigma v \rangle_{DT}^{10keV} E_\alpha}{C n_e^2 T^{1/2}}, \quad r_\alpha = \frac{n_\alpha}{n_e},$$

we can write

$$\begin{aligned} (1 + 2r_\alpha) &= c_\alpha (1 - 2r_\alpha)^2 \\ (1 + 2r_\alpha) &= c_\alpha \left(1 - 4r_\alpha + 4r_\alpha^2\right) \\ 0 &= 4c_\alpha r_\alpha^2 - (4c_\alpha + 2) r_\alpha + (c_\alpha - 1) \end{aligned}$$

which can be expressed as

$$9.6633 r_\alpha^2 - 11.6633 r_\alpha + 1.4158. \tag{4}$$

The solution of (4) is given by

$$r_{\alpha_{1,2}} = \frac{11.6633 \pm \sqrt{11.6633^2 - 4 \cdot 9.6633 \cdot 1.4158}}{2 \cdot 9.6633} = \begin{cases} 1.0700 \\ 0.1369 \end{cases}$$

The solution $r_\alpha = n_\alpha / n_e = 1.0700$ is not physically possible since n_α must be smaller than n_e from the quasi-neutrality condition. Note also that if we take $r_\alpha = n_\alpha / n_e = 1.0700$, we have that $n_D = \frac{1}{2}\left[1 - 2\frac{n_\alpha}{n_e}\right] n_e < 0$ which has no physical meaning. Therefore, we conclude that the valid solution is

$$r_\alpha = \frac{n_\alpha}{n_e} = 0.1369. \quad (5)$$

(b) (25 points) What is the necessary auxiliary power (kW) to achieve breakeven condition ($Q = 1$)?

$$Q = \frac{P_f}{P_{aux}} = \frac{5P_\alpha}{P_{aux}}$$

A breakeven condition ($Q = 1$) implies that $P_{aux} = 5P_\alpha = 402.83\frac{\text{kW}}{m^3}$.

(c) (25 points) What is the necessary energy confinement time τ_E to achieve steady-state operation?

The power balance equation in steady state is written as

$$P_{aux} + P_\alpha = P_L + P_b$$

Therefore,

$$P_L = P_{aux} + P_\alpha - P_b = 463.25\text{kW}$$

Since

$$P_L = \frac{\frac{3}{2}nkT}{\tau_E} \iff \tau_E = \frac{\frac{3}{2}nkT}{P_L}, \qquad (6)$$

and using (3) and (5) we can write

$$\begin{aligned} n = n_e + n_i &= n_e + n_D + n_T + n_\alpha = n_e + 2n_{DT} + n_\alpha \\ &= n_e + (1 - 2r_\alpha)n_e + r_\alpha n_e = (2 - r_\alpha)n_e, \end{aligned}$$

we can finally compute

$$
\begin{aligned}
\tau_E &= \frac{\frac{3}{2}nkT}{P_L} \\
&= \frac{\frac{3}{2}(2-r_\alpha)n_e kT}{P_L} \\
&= \frac{\frac{3}{2}\times(2-0.1369)\times 10^{20}/m^3\times 10\times 10^3 eV\times 1.602\times 10^{-19}J/eV}{4.6325\times 10^5 W/m^3} \\
&= 0.9664s
\end{aligned}
$$