Nuclear Fusion and Radiation

Lecture 5 (Meetings 9, 10, 11 & 12) - Problem 1

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Power Balance in Fusion Plasmas

Problem: (100 points) Consider a DT plasma with $n_e = 10^{20}/m^3$, $n_D = n_T \triangleq n_{DT}$, $kT_e = kT_i = kT = 10keV$, energy confinement time $\tau_E = 2.4282s$, α particle build-up concentration $r_\alpha = n_\alpha / n_e = 0.15$, and impurity concentration $r_I = n_I / n_e$ with atomic number $Z_I = 8$.

(a) (20 points) Give an expression for the DT density (n_{DT}) in terms of n_e.

(b) (10 points) Give an expression for the plasma density n in terms of n_e.

(c) (10 points) Give an expression for the plasma Z_{eff}.

(d) (10 points) Give an expression for the Bremsstrahlung power loss.

(e) (20 points) Give a value for the r_I needed to achieve $P_\alpha = 5.8733 \times 10^4 W/m^3$.

(f) (20 points) Give a value for the auxiliary power needed to achieve steady-state operation.

(g) (10 points) What is the Q for this plasma?

(a) Quasi-neutrality condition:

$$\begin{aligned} n_e &= n_p \\ n_e &= n_D Z_D + n_T Z_T + n_\alpha Z_\alpha + n_I Z_I \\ n_e &= n_D + n_T + 2n_\alpha + n_I Z_I \\ n_e &= n_D + n_T + 2r_\alpha n_e + r_I n_e Z_I \end{aligned}$$

Assuming $n_D = n_T \triangleq n_{DT} \iff n_D + n_T = 2n_{DT}$,

$$n_e = 2n_{DT} + 2r_\alpha n_e + Z_I r_I n_e, \tag{1}$$

which yields

$$n_{DT} = \frac{1}{2}\left(1 - 2r_\alpha - Z_I r_I\right) n_e \tag{2}$$

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(b) The total plasma density is given by $n = n_e + n_i$, where

$$n_i = \sum_j n_j^{ion} = n_D + n_T + n_\alpha + n_I = 2n_{DT} + r_\alpha n_e + r_I n_e. \tag{3}$$

Thus, using (1) and (3),

$$\begin{aligned} n = n_e + n_i &= (2n_{DT} + 2r_\alpha n_e + Z_I r_I n_e) + (2n_{DT} + r_\alpha n_e + r_I n_e) \\ &= 4n_{DT} + 3r_\alpha n_e + (Z_I + 1) r_I n_e \end{aligned}$$

Using the expression for n_{DT} from (2), we can finally write

$$\begin{aligned} n &= 4\left(\frac{1}{2}\left(1 - 2r_\alpha - Z_I r_I\right) n_e\right) + 3r_\alpha n_e + (Z_I + 1) r_I n_e \\ &= \left(2 - r_\alpha - (Z_I - 1) r_I\right) n_e \end{aligned} \tag{4}$$

(c) Effective Z:

$$
\begin{aligned}
Z_{eff} = \sum_i \frac{n_i \langle Z_i \rangle^2}{n_e} &= \frac{n_D Z_D^2 + n_T Z_T^2 + n_\alpha Z_\alpha^2 + n_I Z_I^2}{n_e} \\
&= \frac{n_D + n_T}{n_e} + \frac{4n_\alpha + Z_I^2 n_I}{n_e} \\
&= \frac{2n_{DT}}{n_e} + \frac{4n_\alpha + Z_I^2 n_I}{n_e} \\
&= \frac{2n_{DT}}{n_e} + 4r_\alpha + Z_I^2 r_I.
\end{aligned}
$$

Using (2), we can write

$$
\begin{aligned}
Z_{eff} &= (1 - 2r_\alpha - Z_I r_I) + 4r_\alpha + Z_I^2 r_I \\
&= 1 + 2r_\alpha + Z_I (Z_I - 1) r_I \qquad (5)
\end{aligned}
$$

(d) Bremsstrahlung Power Density:

$$P_b = C n_e^2 \sqrt{T} Z_{eff}$$

Using (5), we can write

$$P_b = C n_e^2 \sqrt{T} \left[1 + 2r_\alpha + Z_I \left(Z_I - 1\right) r_I\right] \quad (6)$$

(e) α Power Density:

$$\begin{aligned} P_\alpha &= n_D n_T \langle \sigma v \rangle_{DT} E_\alpha \\ &= n_{DT}^2 \langle \sigma v \rangle_{DT} E_\alpha \end{aligned}$$

Using (2), we write

$$P_\alpha = \frac{n_e^2}{4} \left(1 - 2r_\alpha - Z_I r_I\right)^2 \langle \sigma v \rangle_{DT} E_\alpha$$

Then,

$$\frac{4P_\alpha}{\langle \sigma v \rangle_{DT} E_\alpha n_e^2} = \left(1 - 2r_\alpha - Z_I r_I\right)^2$$

Assuming $1 - 2r_\alpha - Z_I r_I > 0$,

$$r_I = \frac{1}{Z_I} \left[1 - 2r_\alpha - \sqrt{\frac{4P_\alpha}{\langle \sigma v \rangle_{DT} E_\alpha n_e^2}} \right]$$

$$r_I = \frac{1}{8} \left[1 - 2 \cdot 0.15 - \sqrt{\frac{4 \cdot 5.8733 \times 10^4 W/m^3}{1.09 \times 10^{-22} m^3/s \times 3.5 \times 10^6 eV \times 1.602 \times 10^{-19} J/eV \times \left(10^{20}/m^3\right)^2}} \right] = 0.01$$

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(f) Steady-state Power Density Balance:

$$\begin{aligned} P_{aux} + P_\alpha &= P_L + P_b \\ P_{aux} &= P_L + P_b - P_\alpha \end{aligned}$$

From the problem's conditions, $P_\alpha = 5.8733 \times 10^4 W/m^3$.

Using (6), now we can compute

$$\begin{aligned} P_b &= C n_e^2 \sqrt{T} \left[1 + 2r_\alpha + Z_I \left(Z_I - 1\right) r_I\right] \\ &= 5\times 10^{-37} W m^3/\sqrt{keV} \left(10^{20}/m^3\right)^2 \sqrt{10keV} \left[1 + 2\times 0.15 + 8\left(8-1\right) 0.01\right] \\ &= 2.9409 \times 10^4 W/m^3 \end{aligned}$$

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Using (4), the total density can now be computed as

$$n = (2 - r_\alpha - (Z_I - 1)r_I)\, n_e = (2 - 0.15 - (8-1)0.01)\, n_e = 1.78 n_e \quad (7)$$

Therefore, using (4) and (7), the power losses can be written as

$$\begin{aligned}
P_L = \frac{\frac{3}{2}nkT}{\tau_E} &= \frac{\frac{3}{2}\,(2 - r_\alpha - (Z_I - 1)r_I)\, n_e kT}{\tau_E} \\
&= \frac{\frac{3}{2}1.78 n_e kT}{\tau_E} \\
&= \frac{\frac{3}{2} \times 1.78 \times 10^{20}/m^3 \times 10 \times 10^3 eV \times 1.602 \times 10^{-19} J/eV}{2.4282 s} \\
&= 1.7616 \times 10^5 W/m^3
\end{aligned}$$

Finally, we can write

$$P_{aux} = P_L + P_b - P_\alpha = 1.4683 \times 10^5 W/m^3$$

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(g) Using the fact that $P_f = 5P_\alpha$ in a DT plasma, we can obtain

$$Q = \frac{P_f}{P_{aux}} = \frac{5P_\alpha}{P_{aux}} = \frac{5 \times 5.8733 \times 10^4 W/m^3}{1.4683 \times 10^5 W/m^3} = 2$$