

# Nuclear Fusion and Radiation

## Lecture 5 (Meetings 9 & 10)

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# Power Balance in Fusion Plasmas

- Now that we have established methods for calculating reaction rates in a plasma, we are in a position to consider energy/power balances in a fusing plasma.
- Consider the following schematic representation of a fusing plasma:

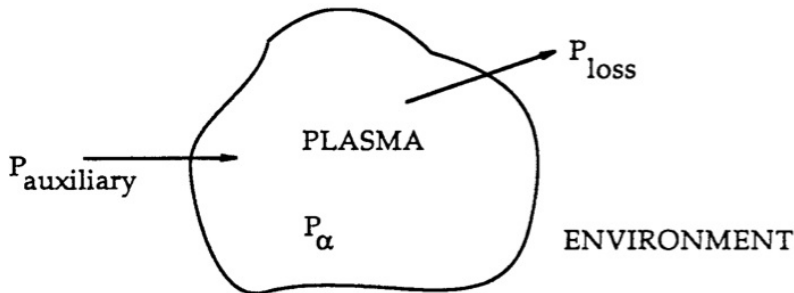


Figure 1: Power balance.

# Power Balance in Fusion Plasmas

- $P_{aux}$  denotes the external power input required to heat fuel to fusion conditions and to sustain temperature at fusion conditions.
- $P_{\alpha}$  denotes the power associated with  $DT$  produced  $\alpha$ 's assumed to be deposited within the plasma.
- $P_{loss}$  denotes the power loss from the plasma.

**What mechanisms might result in energy loss from the plasma?**

# Power Balance in Fusion Plasmas

- The temperature of the environment is much less than that of the plasma. Therefore, very steep gradients in temperature will exist between the plasma and its environment, driving heat out of the plasma (heat conduction).
- The density of ions and electrons in the environment is much less than in the plasma. Therefore, very steep gradients in density will exist between the plasma and its environment, driving particles out of the plasma (particle transport/diffusion). Note also that escaping particles carry energy out with them.
- Processes in the plasma can generate electromagnetic (EM) radiation and much of this radiation will not be absorbed in the plasma, but will escape from the plasma leading to radiative power loss from the plasma.

# Power Balance in Fusion Plasmas

- The plasma energy is given by

$$E_{content} = \int_{volume} \left( \frac{3}{2} n_i k T_i + \frac{3}{2} n_e k T_e \right) dV$$

$n_e/T_e, n_i/T_i$  : electron, ion density/temperature.

- The quasineutrality condition is written as

$$n_p = n_e$$

$n_p, n_e$  : number (i.e., density) of protons, electrons.

- Consider energy content of **purely hydrogenic** 50/50 DT plasma:
  - No ions other than fuel ions (no impurities or  $\alpha$  particles)), i.e.  $n_i = n_D + n_T$ .
  - All species have same temperature, i.e.  $T_i = T_e \triangleq T$ .
  - D and T have optimal 50/50 mixture, i.e.  $n_D = n_T$ .

# Power Balance in Fusion Plasmas

- For the assumed **purely hydrogenic** DT plasma, we have

$$n_p = n_D Z_D + n_T Z_T = n_D + n_T = n_i$$

- In this case, the quasineutrality conditions reduces to

$$n_i = n_e$$

- The overall plasma density is always defined as

$$n = n_i + n_e$$

- Therefore, for the assumed **purely hydrogenic** DT plasma,

$$n = n_i + n_e = 2n_e = 2n_i$$

- Recalling that  $T_i = T_e \triangleq T$ , we can then rewrite

$$E_{content} = \int_{volume} 3n_e kT dV = \int_{volume} \frac{3}{2} n kT dV$$

# Power Balance in Fusion Plasmas

- In general, densities and temperatures will be functions of position and time. After integrating over the volume, we can write

$$E_{content} = \frac{3}{2} \bar{n} k \bar{T} V$$

where  $\bar{n}$  and  $\bar{T}$  represent the appropriate volume average of  $n$  and  $T$ .

- For convenience, we drop the upper-bar notation from now on.
- All variables are time-dependent and volume-averaged. With the appropriate volume averages, we can write the following power balance for our plasma:

$$\frac{d(\frac{3}{2}nkTV)}{dt} = P_{aux}V + P_{\alpha}V - P_LV$$

- At steady state,  $d()/dt = 0$ , and

$$P_{aux}V + P_{\alpha}V = P_LV \iff P_{aux} + P_{\alpha} = P_L$$

# Power Balance in Fusion Plasmas

- For macroscopic calculations, the term  $P_L$  is generally defined in terms of the energy content of the plasma and a global/ overall energy confinement time,  $\tau_E$ . Thus,

$$P_L \triangleq \frac{\frac{3}{2}nkT}{\tau_E}$$

- The above equation says nothing about the physics or mechanisms for energy loss, it simply defines a global energy confinement time

$$\tau_E \triangleq \frac{\frac{3}{2}nkT}{P_L}$$

- For the **purely hydrogenic** DT plasma ( $n = 2n_e$ ),

$$P_L = \frac{3n_e kT}{\tau_E}, \quad \tau_E = \frac{3n_e kT}{P_L}$$



# Power Balance in Fusion Plasmas

- The  $\alpha$ -power density,  $P_\alpha$ , can be written as

$$P_\alpha = n_D n_T \langle \sigma V \rangle E_\alpha$$

where it is understood that the densities and the reaction rate parameter are volume averages.  $E_\alpha = 3.5$  MeV denotes the fraction of energy released by the  $DT$  fusion reactor that is carried by the  $\alpha$  particle.

- For the **purely hydrogenic** 50/50 DT plasma ( $n_D = n_T = n_e/2$ ),

$$P_\alpha = \frac{n_e^2}{4} \langle \sigma V \rangle E_\alpha$$

# Power Balance in Fusion Plasmas

- We can also define a plasma condition in which the plasma is energy/ power self sufficient, a situation in which the losses,  $P_L$ , are exactly balanced by the  $\alpha$ -power,  $P_\alpha$ . Thus,

$$P_L = P_\alpha \iff P_{aux} = 0$$

This condition is called “ignition”.

- In terms of plasma parameters the ignition condition can be written as

$$n_D n_T \langle \sigma V \rangle E_\alpha = \frac{\frac{3}{2} n k T}{\tau_E}$$

# Power Balance in Fusion Plasmas

- For the **purely hydrogenic** 50/50 DT plasma ( $n_e = n_i = n_D + n_T$ ,  $n = n_i + n_e = 2n_e = 2n_i$ ,  $n_D = n_T = \frac{n_i}{2} = \frac{n_e}{2}$ ),

$$\frac{n_e^2}{4} \langle \sigma V \rangle E_\alpha = \frac{3n_e kT}{\tau_E}$$

where the volume average symbol has been dropped for convenience.

- Rearranging the above yields

$$n_e \tau_E \Big|_{IGN}^{DT} = \frac{12kT}{\langle \sigma V \rangle_{DT} E_\alpha}$$
$$n \tau_E \Big|_{IGN}^{DT} = \frac{24kT}{\langle \sigma V \rangle_{DT} E_\alpha}$$

- Since

$$\langle \sigma V \rangle_{DT} = f(kT)$$

the ignition parameter is a function of  $kT$  ( $n_e \tau_E \Big|_{IGN}^{DT} = g(kT)$ ).

# Power Balance in Fusion Plasmas

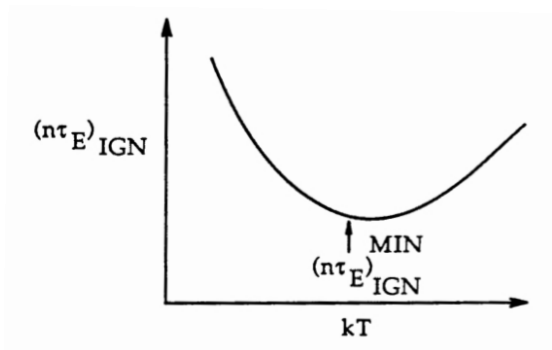


Figure 2: Ignition parameter.

- Note that  $n_e\tau_E|_{IGN}$  as defined here is a requirement on the product  $n_e\tau_E$ . As we shall see later the value of  $n_e\tau_E$  actually achieved for a given plasma must be compared with the required  $n_e\tau_E$  to judge if the plasma can ignite.

# Energy Gain in a Fusion Reaction

- We now turn to fusion power/ energy gain in a fusing plasma. In discussing energy gain in our beam-target reactor system, we defined the gain in terms of the ratio of fusion energy out to injected energy in the beam. For fusing plasmas we have a similar definition:

$$\text{Power Gain} \equiv \frac{P_f}{P_{aux}} \equiv Q$$

- Historically the symbol  $Q$  has been used for the power gain (Note that this symbol is also used to denote energy release in fusion reactions!!!). For our steady-state plasma,

$$P_{aux} = P_L - P_\alpha \iff Q = \frac{P_f}{P_L - P_\alpha}$$

- Note that at ignition,  $Q \rightarrow \infty$  ( $P_L = P_\alpha$ ).

# Energy Gain in a Fusion Reaction

- We can divide numerator and denominator of the above equation by  $P_\alpha$ , yielding

$$Q = \frac{P_f/P_\alpha}{P_L/P_\alpha - 1}$$

- In a  $DT$  reaction,

$$P_f = n_D n_T \langle \sigma V \rangle E_f$$

where  $E_f = 17.6$  MeV.

- What is the value of  $P_f/P_\alpha$  for the  $DT$  reaction?

$$\frac{P_f}{P_\alpha} = \frac{n_D n_T \langle \sigma V \rangle E_f}{n_D n_T \langle \sigma V \rangle E_\alpha} = \frac{E_f}{E_\alpha} = \frac{17.6}{3.5} \sim 5$$

- What is the value of  $P_L/P_\alpha$  for the 50/50  $DT$  reaction in the case of a **purely hydrogenic** plasma?

$$\frac{P_L}{P_\alpha} = \frac{\frac{3n_e kT}{\tau_E}}{\frac{n_e^2}{4} \langle \sigma V \rangle_{DT} E_\alpha} = \frac{12kT}{\langle \sigma V \rangle_{DT} E_\alpha} \frac{1}{n_e \tau_E}$$

# Energy Gain in a Fusion Reaction

- But

$$\frac{12kT}{\langle \sigma V \rangle_{DT} E_\alpha} \equiv n_e \tau_E \Big|_{IGN}^{DT} = g(kT)$$

Thus, at a given  $kT$  value

$$\frac{P_L}{P_\alpha} = \frac{n_e \tau_E \Big|_{IGN}^{DT}}{n_e \tau_E \Big|_{ACTUAL}^{DT}}$$

Then,

$$Q_{DT} = \frac{5}{\frac{n_e \tau_E \Big|_{IGN}^{DT}}{n_e \tau_E \Big|_{ACT}^{DT}} - 1}$$

- Various goals of the fusion program are discussed in terms of values of  $Q$ .

# Energy Gain in a Fusion Reaction

- The goal of “breakeven” is defined as  $Q = 1 \Leftrightarrow P_{aux} = P_f$ . In this case,

$$Q_{DT} = \frac{5}{\frac{n_e \tau_E|_{IGN}^{DT}}{n_e \tau_E|_{ACT}^{DT}} - 1} \equiv 1,$$

yields

$$\frac{n_e \tau_E|_{IGN}^{DT}}{n_e \tau_E|_{ACT(Q=1)}^{DT}} = 6 \Rightarrow n_e \tau_E|_{ACT(Q=1)}^{DT} = \frac{1}{6} n_e \tau_E|_{IGN}^{DT}$$

- For  $Q_{DT} = 1$ , the achieved  $n_e \tau$  must be  $\sim 1/6$  of that required for ignition. At  $kT = 10keV$ ,

$$n_e \tau_E|_{IGN}^{DT} \sim 3 \times 10^{20} \frac{s}{m^3} \Rightarrow n_e \tau_E|_{ACT(Q=1)}^{DT} \sim 5 \times 10^{19} \frac{s}{m^3}$$

- The goal of breakeven, achieving as much fusion power release as auxiliary power into the plasma, has been the major goal of the fusion program for over forty years. However, it should be emphasized that  $Q = 1$  will not yield net electricity,  $Q > 1$  is required for a useful reactor.



# Energy Gain in a Fusion Reaction

- Another goal of the fusion program is to achieve plasma conditions such that the  $\alpha$ -power deposited in the plasma is of the same order of magnitude as the auxiliary power to the plasma—this is an interesting physics regime since now the  $\alpha$ -power plays a major role in heating the plasma. For this condition,

$$P_{aux} = P_{\alpha} \Leftrightarrow Q = \frac{P_f}{P_{aux}} = \frac{P_f}{P_{\alpha}} = 5$$

- Therefore,

$$Q_{DT} = \frac{5}{\frac{n_e \tau_E|_{IGN}^{DT}}{n_e \tau_E|_{ACT}^{DT}} - 1} \equiv 5,$$

which implies

$$\frac{n_e \tau_E|_{IGN}^{DT}}{n_e \tau_E|_{ACT(Q=5)}^{DT}} = 2.$$

- For  $Q_{DT} = 5$ , the achieved  $n_e \tau$  must be  $\sim 1/2$  of that required for ignition. At  $kT = 10keV$ , this results in the requirement

$$n_e \tau_E|_{ACT(Q=5)}^{DT} = \frac{1}{2} n_e \tau_E|_{IGN}^{DT} \sim 1.5 \times 10^{20} \frac{s}{m^3}$$

# Particle Balances

- Up to now we have been considering power balances. It is also important to consider particle balances in a fusing plasma. Schematically,

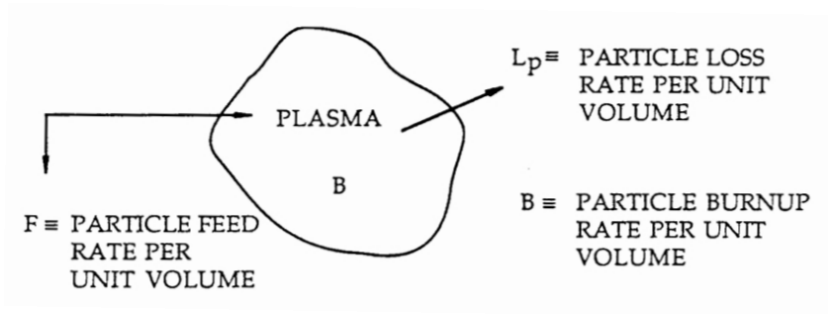


Figure 3: Particle balance.

# Particle Balances

- Thus, for the assumed **purely hydrogenic** DT plasma,

$$\frac{d(n_D + n_T)}{dt} = F - L_p - B$$

where  $n_D + n_T = n_i$  represents the appropriate volume average of the fuel density.

- As with the  $P_L$  term in the power balance we define a global/macroscopic particle confinement time,  $\tau_p$ , such that

$$L_p \equiv \frac{(n_D + n_T)}{\tau_p}$$

- The burnup rate is given by

$$B \equiv \frac{n_e^2}{4} \langle \sigma V \rangle \times 2 = \frac{n_e^2}{2} \langle \sigma V \rangle = \frac{n_i^2}{2} \langle \sigma V \rangle = \frac{(n_D + n_T)^2}{2} \langle \sigma V \rangle$$

Again, average symbol has been dropped for convenience.

# Particle Balances

- At steady state,  $d(n_D + n_T)/dt = 0$  and

$$F = L_p + B = \frac{(n_D + n_T)}{\tau_p} + \frac{(n_D + n_T)^2}{2} \langle \sigma V \rangle$$

- Defining  $n_{DT} \triangleq n_D = n_T$ , we can rewrite

$$F = \frac{(2n_{DT})}{\tau_p} + \frac{(2n_{DT})^2}{2} \langle \sigma V \rangle$$

- It is also useful to define a quantity called the fractional burnup,  $f_b$ , where

$$f_b \equiv \frac{B}{F} = \frac{\frac{(2n_{DT})^2}{2} \langle \sigma V \rangle}{\frac{2n_{DT}}{\tau_p} + \frac{(2n_{DT})^2}{2} \langle \sigma V \rangle} = \frac{n_{DT} \langle \sigma V \rangle}{\frac{1}{\tau_p} + n_{DT} \langle \sigma V \rangle}$$

## What is this quantity?

- Remember that the mean life of a particle between fusion events is given by

$$\tau_{DT} = \frac{1}{n_{DT} \langle \sigma V \rangle}$$

Thus,

$$f_b = \frac{\frac{1}{\tau_{DT}} \tau_p}{\frac{1}{\tau_p} + \frac{1}{\tau_{DT}}} = \frac{\frac{\tau_p}{\tau_{DT}}}{1 + \frac{\tau_p}{\tau_{DT}}}$$

- For ignition at  $10keV$  with standard plasma,  $n_e = 10^{20}/m^3$ :

$$n_e \tau_E |_{IGN} \sim 3 \times 10^{20} \frac{s}{m^3}, \quad \tau_E \sim 3s$$

- Assume  $\tau_p = 2\tau_E \sim 6s$ . From previous calculations  $\tau_{DT} \sim 180s$ . Thus,

$$f_b = \frac{\frac{6}{180}}{1 + \frac{6}{180}} \sim 0.032$$

About 3.2% of the feed is burned up and the rest appears as particle loss!!

# Radiative Processes in Plasmas

- We previously examined power/energy balances in plasmas on a macroscopic scale and we lumped all energy loss phenomena into a single term:

$$P_L \equiv \frac{\frac{3}{2}nkT}{\tau_E}$$

- For the **purely hydrogenic** DT plasma ( $n = 2n_e$ )

$$P_L = \frac{3n_e kT}{\tau_E}$$

- We noted  $P_L$  might consist of at least three contributions:
  - diffusion losses  $\propto dn/dx$  (and whatever energy the “leaking” particle carries)
  - conduction losses  $\propto dT/dx$
  - radiative losses
- We will now turn to a discussion of the radiative losses.

# Radiative Processes in Plasmas

- There are a number of processes by which EM radiation can be produced in, and eventually lost from, a plasma.
- For fusing plasmas the most important process is in general Bremsstrahlung (braking radiation). Whenever a charge experiences an acceleration/ deceleration (i.e., change in velocity), it will radiate electromagnetic energy.
- In a plasma, coulomb scattering collisions will cause acceleration/ deceleration of charged particles and will therefore result in the emission of Bremsstrahlung radiation.

# Radiative Processes in Plasmas

- Consider a scattering collision between target ion  $Z_1$  and projectile (ion or electron)  $Z_2$ . The acceleration,  $a_2$ , of  $Z_2$  produced by the collision is:

$$a_2 = \frac{F_{12}}{M_2}$$

where

$$F_{12} = \text{Force between } Z_1 \text{ and } Z_2 \propto Z_1 Z_2.$$

Therefore

$$a_2 \propto \frac{Z_1 Z_2}{M_2}$$

- It can be shown that the intensity of the emitted radiation is proportional to  $\|a_2\|^2$  and thus the Bremsstrahlung radiation intensity is proportional to  $(Z_1 Z_2 / M_2)^2$ .



# Radiative Processes in Plasmas

- In a plasma, the primary source of Bremsstrahlung radiation is that resulting from electron acceleration / deflections (note the mass dependence of the intensity) and essentially all from electron-ion collisions.
- It can be shown that the Bremsstrahlung radiation power density,  $P_b$ , is given by:

$$P_b^Z = CT_e^{1/2} n_e \sum_{\text{ions } i} n_i \langle \bar{Z}_i \rangle^2$$

where  $C$  is a constant and  $\langle \bar{Z}_i \rangle$  represents the average charge state of species  $i$ , which need not be fully ionized.

- In discussing plasma radiation it is useful to define the effective atomic number of the plasma as

$$Z_{eff} = \frac{\sum_i n_i \langle \bar{Z}_i \rangle^2}{n_e}$$

# Radiative Processes in Plasmas

- The Bremsstrahlung radiation power density is rewritten as

$$P_b^Z = C n_e^2 T_e^{1/2} Z_{eff} f$$

- Note that for a **purely hydrogenic**  $DT$  plasma

$$\langle \bar{Z}_i \rangle = 1 \text{ and } \sum_i n_i \langle \bar{Z}_i \rangle^2 = n_D + n_T$$

and by charge neutrality

$$n_e = n_D + n_T = n_i.$$

Then,  $Z_{eff} \equiv 1$ , and

$$P_b = C T_e^{1/2} n_e^2$$

# Radiative Processes in Plasmas

- Once again, the Bremsstrahlung radiation power density is rewritten as

$$P_b^Z = \underbrace{C n_e^2 T_e^{1/2}}_{\text{Hydrogenic Term}} Z_{eff}^2$$

where  $[T_e] = keV$ . The Bremsstrahlung constant is given by

$$C = 5 \times 10^{-37} \frac{W m^3}{\sqrt{keV}} = 5 \times 10^{-43} \frac{MW m^3}{\sqrt{keV}}$$

- Consider the standard purely hydrogenic plasma,  $n_e = n_i = 10^{20}/m^3$ ,  $T_e = T_i = 10keV$ , where  $Z_{eff} = 1$ . Then,

$$P_b^{Z=1} = 5 \times 10^{-43} (10^{20})^2 (10)^{1/2} \sim 0.016 MW/m^3$$

- In addition, recall that for the standard plasma  $P_\alpha \sim 0.15 MW/m^3$ . Thus, for this case  $P_b \sim P_\alpha/10$ .

# Power Balance with $P_b$

- The power balance equation in Slide 7 needs to be modified to include the effect of the radiation loss.
- In this case, we can write the following power balance for our plasma:

$$\frac{d(\frac{3}{2}nkTV)}{dt} = P_{aux}V + P_{\alpha}V - P_LV - P_bV$$

- At steady state,  $d()/dt = 0$ , and

$$P_{aux}V + P_{\alpha}V = P_LV + P_bV \iff P_{aux} + P_{\alpha} = P_L + P_b$$

# Power Balance with $P_b$

- We will now examine the power balance equation for a purely hydrogenic 50/50 DT plasma ( $Z_{eff} = 1$ ) with  $P_b$  included explicitly for the ignition case ( $P_{aux} = 0$ ).
- For this case,

$$\begin{aligned} P_\alpha &= P_L + P_b \\ \frac{n_e^2}{4} \langle \sigma V \rangle_{DT} E_\alpha &= \frac{3n_e kT}{\tau_E} + CT_e^{1/2} n_e^2 Z_{eff} \end{aligned}$$

- This yields,

$$n_e \tau_E \Big|_{IGN, P_b}^{DT} = \frac{12kT}{\langle \sigma V \rangle_{DT} E_\alpha - 4CT_e^{1/2}}$$

- Comparing the above with the result in the absence of radiation loss,

$$n_e \tau_E \Big|_{IGN}^{DT} = \frac{12kT}{\langle \sigma V \rangle_{DT} E_\alpha}$$

we note that  $n_e \tau_E \Big|_{IGN, P_b}^{DT} > n_e \tau_E \Big|_{IGN}^{DT}$  (tougher requirement!).

# Power Balance with $P_b$

- Consider the ratio

$$\frac{n_e \tau_E |_{IGN, Pb}^{DT}}{n_e \tau_E |_{IGN}^{DT}} = \frac{\langle \sigma V \rangle_{DT} E_\alpha}{\langle \sigma V \rangle_{DT} E_\alpha - 4CT^{1/2}}$$

- Multiply numerator and denominator by  $n_e^2/4$

$$\frac{n_e \tau_E |_{IGN, Pb}^{DT}}{n_e \tau_E |_{IGN}^{DT}} = \frac{\frac{n_e^2}{4} \langle \sigma V \rangle_{DT} E_\alpha}{\frac{n_e^2}{4} \langle \sigma V \rangle_{DT} E_\alpha - Cn_e^2 T^{1/2}}$$

- For the standard case this yields

$$\frac{n_e \tau_E |_{IGN, Pb}^{DT}}{n_e \tau_E |_{IGN}^{DT}} = \frac{0.15}{0.15 - 0.016} = 1.12$$

- The required  $n_e \tau_E |_{IGN}$  increases by 12% compared to the case where Bremsstrahlung radiation loss is neglected!

# Power Balance with $P_b$

- We will now examine the power balance equation for a non-purely hydrogenic 50/50 DT plasma (plasma with impurities ( $Z_{eff} \neq 1$ )) with  $P_b$  included explicitly for the ignition case ( $P_{aux} = 0$ ).
- We consider the case with iron impurity (which could be introduced into the plasma from surrounding walls). Assume  $n_{Fe}/n_e = 0.01(1\%)$  and  $\langle \bar{Z}_{Fe} \rangle = 14$ . Thus, iron ( $Z = 26$ ) is not fully ionized. We assume that this value of the charge state is obtained from spectroscopic measurements.
- Thus,

$$\begin{aligned} Z_{eff} &= \frac{\sum_i n_i \langle \bar{Z}_i \rangle^2}{n_e} = \frac{n_D + n_T + n_{Fe} \langle \bar{Z}_{Fe} \rangle^2}{n_e} \\ &= \frac{n_D + n_T}{n_e} + \frac{n_{Fe} \langle \bar{Z}_{Fe} \rangle^2}{n_e} \end{aligned}$$

# Power Balance with $P_b$

- We are given the second term in  $Z_{eff}$ , how do we obtain  $(n_D + n_T)/n_e$ ?

We apply charge neutrality!!

- Charge neutrality requires that the number of electrons per unit volume equals the number of positive charges per unit volume. Thus,

$$\begin{aligned}n_e &= n_p \\n_e &= n_D Z_D + n_T Z_T + n_{Fe} \langle \bar{Z}_{Fe} \rangle \quad (Z_D = Z_T = 1)\end{aligned}$$

- Dividing both sides of this equation by  $n_e$  yields

$$1 = \frac{n_D + n_T}{n_e} + \frac{n_{Fe} \langle \bar{Z}_{Fe} \rangle}{n_e}$$

and

$$\frac{n_D + n_T}{n_e} = 1 - \frac{n_{Fe} \langle \bar{Z}_{Fe} \rangle}{n_e}$$



## Power Balance with $P_b$

- Thus,

$$Z_{eff} = 1 - \frac{n_{Fe}}{n_e} \langle \bar{Z}_{Fe} \rangle + \frac{n_{Fe}}{n_e} \langle \bar{Z}_{Fe} \rangle^2$$

For our case

$$Z_{eff} = 1 - (0.01)(14) + (0.01)(14)^2 = 2.82$$

- With the standard plasma

$$P_b^{Fe} = P_b^{Z=1} 2.82 = (0.016)(2.82) = 0.045 MW$$

- Now that we have already analyzed the impact of the iron impurities on the radiation losses, it is also of interest to consider the impact of the iron impurities on both the fuel density and the plasma power density.

# Power Balance with $P_b$

- We still assume  $n_D = n_T \equiv n_{DT}$ . The neutrality condition

$$\frac{n_D + n_T}{n_e} = 1 - \frac{n_{Fe}}{n_e} < \bar{Z}_{Fe} >$$

yields (note that  $n_D + n_T = 2n_{DT}$ )

$$n_{DT}^{Fe} = \frac{1}{2} \left[ 1 - \frac{n_{Fe}}{n_e} < \bar{Z}_{Fe} > \right] n_e \iff n_{DT}^{Fe} = 0.43n_e$$

- In the case with no impurities (**purely hydrogenic** plasma), the neutrality condition  $n_e = n_D + n_T = 2n_{DT}$  yields  $n_{DT}^{Z=1} = 0.5n_e$ .
- Then,

$$\frac{P_{\alpha}^{Fe}}{P_{\alpha}^{Z=1}} = \frac{(0.43n_e)^2}{(0.5n_e)^2} \sim 0.74$$

- For the standard plasma,

$$P_{\alpha}^{Fe} = 0.74 \times P_{\alpha}^{Z=1} = 0.74 \times 0.15 \sim 0.11 \text{ MW/m}^3$$

# Power Balance with $P_b$

- In this case, the power balance equation is written as

$$\begin{aligned}P_\alpha &= P_L + P_b \\n_D n_T \langle \sigma V \rangle_{DT} E_\alpha &= \frac{\frac{3}{2} n k T}{\tau_E} + C T_e^{1/2} n_e^2 Z_{eff} \\(0.43 n_e)^2 \langle \sigma V \rangle_{DT} E_\alpha &= \frac{\frac{3}{2} (1.87 n_e) k T}{\tau_E} + 2.82 C T_e^{1/2} n_e^2\end{aligned}$$

where we have used

$$\begin{aligned}n = n_e + n_i &= n_e + n_D + n_T + n_{Fe} \\&= n_e + 2n_{DT}^{Fe} + 0.01n_e \\&= n_e + 2(0.43)n_e + 0.01n_e \\&= 1.87n_e\end{aligned}$$

# Power Balance with $P_b$

- This yields,

$$n_e \tau_E |_{IGN, P_b}^{Fe} = \frac{\frac{3}{2} \frac{1.87}{0.43^2} kT}{\langle \sigma V \rangle_{DT} E_\alpha - \frac{2.82}{0.43^2} CT^{1/2}}$$

- Comparing the above with the result in the absence of impurities,

$$n_e \tau_E |_{IGN, P_b}^{DT} = \frac{12kT}{\langle \sigma V \rangle_{DT} E_\alpha - 4CT^{1/2}}$$

we compute the ratio

$$\frac{n_e \tau_E |_{IGN, P_b}^{Fe}}{n_e \tau_E |_{IGN, P_b}^{DT}} = \frac{\frac{3}{2} \frac{1.87}{0.43^2}}{12} \frac{\langle \sigma V \rangle_{DT} E_\alpha - 4CT^{1/2}}{\langle \sigma V \rangle_{DT} E_\alpha - \frac{2.82}{0.43^2} CT^{1/2}}$$

# Power Balance with $P_b$

- Multiply numerator and denominator by  $n_e^2/4$ , we obtain

$$\frac{n_e \tau_E |_{IGN, P_b}^{Fe}}{n_e \tau_E |_{IGN, P_b}^{DT}} = \frac{\frac{3}{2} \frac{1.87}{0.43^2}}{12} \frac{\frac{n_e^2}{4} \langle \sigma V \rangle_{DT} E_\alpha - n_e^2 C T^{1/2}}{\frac{n_e^2}{4} \langle \sigma V \rangle_{DT} E_\alpha - \frac{2.82}{4 \times 0.43^2} n_e^2 C T^{1/2}}$$

- For the standard case this yields

$$\frac{n_e \tau_E |_{IGN, P_b}^{Fe}}{n_e \tau_E |_{IGN, P_b}^{DT}} = \frac{\frac{3}{2} \frac{1.87}{0.43^2}}{12} \frac{0.15 - 0.016}{0.15 - \frac{2.82}{4 \times 0.43^2} 0.016} \approx 1.9$$

The required  $n_e \tau_E |_{IGN}$  increases by  $\sim 90\%$  compared to the case where there are no impurities!

- Impurities (high  $Z$  materials) are deleterious to the plasma power balance in two ways:
  - ① Enhancement of  $P_b$  by  $Z_{eff}$  factor
  - ② Reduction of  $P_\alpha$  by fuel depletion effect
- This turns into an increase in the  $n_e \tau_E |_{IGN}$  requirement!