Nuclear Fusion and Radiation

Lecture 2 (Meetings 3 & 4)

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Modern Physics Concepts

First three decades of the 20th century $\rightarrow$ Modern Physics

- Theory of special relativity, which extended Newtonian mechanics.
- Wave-particle duality, which says that both electromagnetic waves and atomic particles have dual wave and particle properties.
- Quantum mechanics, which revealed that the microscopic atomic world is far different from our everyday macroscopic world.

The results and insights provided by these three advances in physics are fundamental to an understanding of nuclear science and technology.

1: Theory of Relativity

Newton's second law, in the form originally stated by Newton, says that the rate of change of a body's momentum p equals the force F applied to it, i.e.,

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} \tag{1}$$

For a constant mass m, as assumed by Newton, this equation immediately reduces to the modern form of the second law,

$$F = ma, \tag{2}$$

where $a = dv/dt$ is the acceleration of the body.

In 1905 Einstein showed that (1) is still correct, but that the mass of a body is not constant and increases with the body's speed v. Therefore, (2) is a good approximation in some cases but not quite correct!

1: Theory of Relativity

Einstein showed that m varies with the body's speed as

$$m = \frac{m_o}{\sqrt{1 - v^2/c^2}} \tag{3}$$

where m_o is the body's "rest mass," i.e., the body's mass when it is at rest, and $c \approx 3 \times 10^8 m/s$ is the speed of light.

- Amazingly, Einstein's correction to the laws of motion was produced theoretically before being discovered experimentally, which is uncommon in physics.
- This may be explained by the fact that in our everyday world the difference between m and m_o is very small. Thus for engineering problems in our macroscopic world, relativistic effects can safely be ignored.
- However, at the atomic and nuclear level, these effects can be very important.

1: Theory of Relativity

Newton's words: "The motions of bodies included in a given space are the same amongst themselves, whether the space is at rest or moves uniformly forward in a straight line."

- This means that experiments made in a laboratory in uniform motion (i.e., non-accelerating motion) produce the same results as when the laboratory is at rest.
- But do all the laws of physics indeed remain the same in all non-accelerating (inertial) coordinate systems?

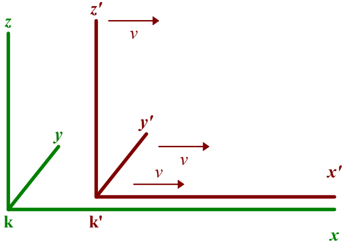


Figure: Coordinate Systems

Consider the two coordinate systems shown in Fig. 1. System k is at rest, while system k' is moving uniformly to the right with speed v. At $t = 0$, the origin of k' is at the origin of k. The coordinates of some point P are (x, y, z) in k and (x', y', z') in k'. Clearly, the primed and unprimed coordinates are related by

$$x' = x - vt; \quad y' = y; \quad z' = z; \quad t' = t. \tag{4}$$

Consider a force in the x-direction, F_x, acting on some "constant" mass m_o,

$$F_x = m_o \frac{d^2x'}{dt'^2} = m_o \frac{d^2(x - vt)}{dt^2} = m_o \frac{d^2x}{dt^2} - m_o v \frac{d^2t}{dt^2} = m_o \frac{d^2x}{dt^2}.$$

Thus the second law has the same form in both systems.

- Since the laws of motion are the same in all inertial coordinate systems, it follows that it is impossible to tell, from results of mechanical experiments, whether or not the system is moving.
- However, when Eqs. (4) are used to transform Maxwell's equations of electromagnetism (1870) to another inertial system, they have different form.
- Thus from optical experiments in a moving system, one should be able to determine the speed of the system.
- For many years Maxwell's equations were thought to be somehow incorrect, but eventually some scientists began to wonder if the problem was indeed the Galilean transformation in Eqs. (4).

Indeed, Lorentz observed in 1904 that if the transformation

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}; \quad y' = y; \quad z' = z; \quad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}; \tag{5}$$

is used, Maxwell's equations become the same in all inertial coordinate systems.

1: Theory of Relativity

- The Lorentz transformation is indeed strange, since it indicates that space and time are not independent quantities. Time in the k' system, as measured by an observer in the k system, is different from the time in the observer's system!

- Einstein showed that the Lorentz transformation (5) was the correct transformation relating all inertial coordinate systems in 1905 by using two postulates:

  – **The laws of physics are expressed by equations that have the same form in all coordinate systems moving at constant velocities relative to each other.**

  – **The speed of light in free space is the same for all observers and is independent of the relative velocity between the source and the observer.**

- Einstein demonstrated several amazing properties:

First Property: Time Dilation

1. The passage of time appears to slow in a system moving with respect to a stationary observer with speed v. The time t required for some physical phenomenon (e.g., the interval between two heart beats) in a moving inertial system appears to be longer (dilated) than the time t_o for the same phenomenon to occur in the stationary system. The relation between t and t_o is given by

$$t = \frac{t_o}{\sqrt{1 - v^2/c^2}} \quad (6)$$

where t_o is the "proper" time (stationary frame) and t is the "apparent" time (moving frame).

Proof: In class.

Consider a timing device that emits a pulse of light from a source (S) and then records the time the light takes to travel to a detector (D). Left: stationary detector. Right: moving detector with steady speed v in the x direction.

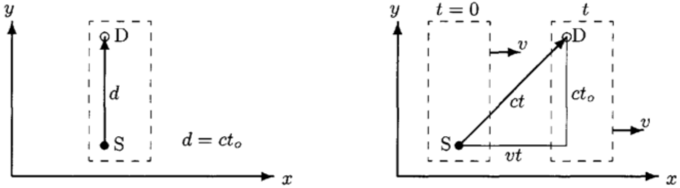


Figure: Dilation: t_o is *proper* time (stationary frame), t is *apparent* time (moving frame).

- *Proper* time t_o: Transit time measured by two events at the same location in observer's frame.
- *Apparent* time t: Transit time measured by two events at different locations in observer's frame.

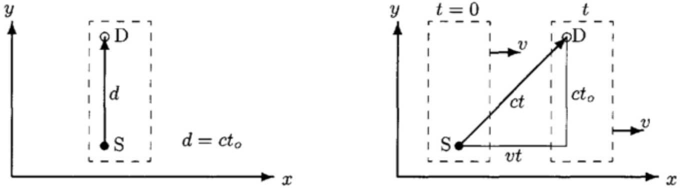


Figure: Dilation: t_o is *proper* time (stationary frame), t is *apparent* time (moving frame).

- Left: Travel time denoted by t_o (proper time)
  - Separation distance given by $d = ct_o$
- Right: Travel time denoted by t (apparent time)
  - Separation distance still given by $d = ct_o$
  - Distance traveled by the detector given by vt
  - Key postulate: "The speed of light in free space is the same for all observers and is independent of the relative velocity between the source and the observer." Therefore, distance traveled by light given by ct.

$$(ct)^2 = (ct_o)^2 + (vt)^2 \Rightarrow t = \frac{t_o}{\sqrt{1 - v^2/c^2}}$$

- Travel time for pulse of light appears to dilate as timing device moves with respect to stationary observer. Like all relativity effects, effect is reciprocal.

Second Property: Length Contraction

2. The length of a moving object with speed v in the direction of its motion appears smaller to an observer at rest, namely,

$$L = L_o\sqrt{1 - v^2/c^2} \tag{7}$$

where L_o is the "proper" length or length of the object when at rest and L is the "apparent" length.

Proof: In class.

1: Theory of Relativity

Consider a rod of length L_o for a stationary observer moving in the x direction with steady speed v and then measure its length by the time it takes to move past a reference pointer. Left: stationary pointer (observer and pointer are in the same frame of reference). Right: moving pointer (observer and pointer are in different frames of reference, the observer is moving with the bar).

- *Proper* time t_o: Transit time measured by two events at the same location in observer's frame.
- *Apparent* time t: Transit time measured by two events at different locations in observer's frame.

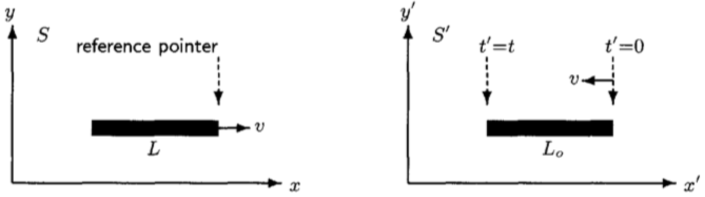


Figure: Contraction: L_o is *proper* length ($L_o = vt$), L is *apparent* length ($L = vt_o$).

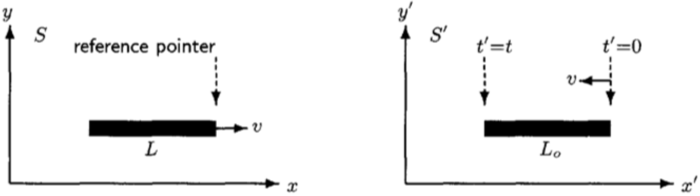


Figure: Contraction: L_o is *proper* length ($L_o = vt$), L is *apparent* length ($L = vt_o$).

- Left: Since the bar is moving with respect to the reference frame, we measure *apparent* length L. Since the transit time is measured at the same position of the reference frame, we measure the *proper* time t_o.

$$L = vt_o$$

- Right: Since the bar is stationary with respect to the reference frame, we measure *proper* length L_o. Since the transit time is measured at a different position of the reference frame, we measure the *apparent* time t.

$$L_o = vt \Rightarrow L_o = vt = \frac{vt_o}{\sqrt{1 - v^2/c^2}} = \frac{L}{\sqrt{1 - v^2/c^2}} \Leftrightarrow L = L_o\sqrt{1 - v^2/c^2}$$

Third Property: Mass Increase

3. The laws of motion are correct, as stated by Newton, if the mass of an object is made a function of the object's speed v, i.e.,

$$m = \frac{m_o}{\sqrt{1 - v^2/c^2}} \tag{8}$$

This result also shows that:

- No object can travel faster than the speed of light since the relativistic mass m must always be real.
- An object with a rest mass ($m_o > 0$) cannot travel at the speed of light; otherwise, its relativistic mass (and kinetic energy) would become infinite.

Proof: In class.

Consider an elastic collision between two identical balls and then measure the time for two balls thrown with speed u in the y direction to collide. Left: trajectories seen by an observer in stationary frame. Right: trajectories seen by an observer in moving frame with steady speed v in the x direction.

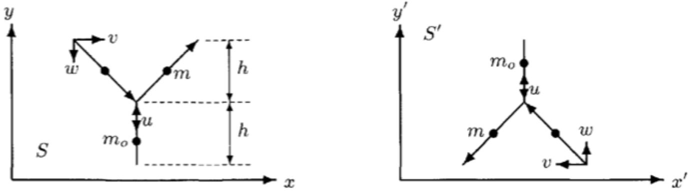


Figure: Mass Increase: m_o is the *rest* mass.

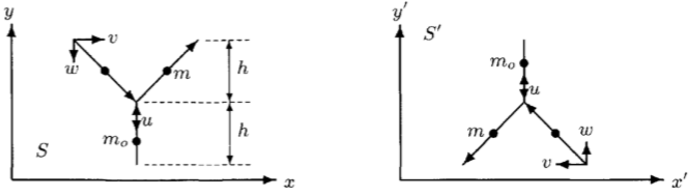


Figure: Mass Increase: m_o is the *rest* mass.

- By considering elastic collision between two bodies, we can infer that mass of a body varies with its speed. Momentum is conserved (elastic collision).
- Both experiments are at rest w.r.t. S and S'.
- S' is moving with speed v relative to S in the x-direction.
- Each observer throws identical spherical balls with speed u perpendicularly to the x axis. For launch speed $u << c$, each ball has mass m_o.
- Head-on collision occurs midway between them.

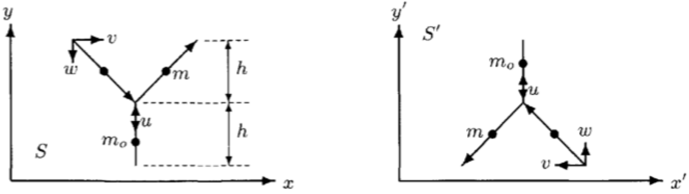


Figure: Mass Increase: m_o is the *rest* mass.

- For observer in S:
  - Travel time for own ball to collision point: *proper* time t_o
  - Travel time for colleague's ball to collision point: *apparent* time t
  - Ball from S takes $t_o = h/u$ to reach collision while ball from S' appears to take longer to reach collision, namely the *apparent* time t.
  - Therefore, $w = \frac{h}{t} = \frac{h\sqrt{1-v^2/c^2}}{t_o} = u\sqrt{1-v^2/c^2} \Leftrightarrow \frac{u}{w} = \frac{1}{\sqrt{1-v^2/c^2}}$.
- Key postulate: "Laws of physics are expressed by equations with same form in all coordinate systems moving at constant velocities relative to each other."
  - Momentum is conserved (elastic collision).

$$m_o u - mw = -m_o u + mw \Leftrightarrow m_o u = mw \Leftrightarrow m = m_o \frac{u}{w} = \frac{m_o}{\sqrt{1-v^2/c^2}}$$

Fourth Property: Energy-Mass Equivalence:

4. Energy and mass can be converted to each other, i.e.,

$$T = mc^2 - m_o c^2 \Rightarrow E \triangleq mc^2 = m_o c^2 + T \tag{9}$$

Indeed, all changes in energy of a system result in a corresponding change in the mass of the system. This equivalence of mass and energy plays a critical role in the understanding of nuclear technology.

Proof: In class.

1: Theory of Relativity

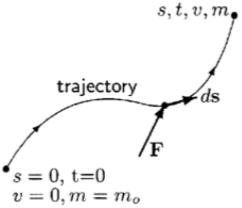


Figure: Force F accelerates particle along s.

- By conservation of energy, the work done on this particle as it moves along the path of length s must equal the kinetic energy T at the end of the path.
- The work done by $\mathbf{F}$ (a vector) on the particle as it moves through a displacement $d\mathbf{s}$ (a vector) is $\mathbf{F} \cdot \mathbf{ds}$.
- The kinetic energy is then given by

$$T = \int_0^s \mathbf{F} \cdot \mathbf{ds} \Rightarrow T = mc^2 - m_o c^2 \Rightarrow E \triangleq mc^2 = T + m_o c^2$$

Reduction to Classical Mechanics:

The Taylor Series expansion w.r.t. $x \triangleq \frac{v}{c}$ around $x_0 \equiv 0$ $(v << c)$ shows that

$$\frac{1}{\sqrt{1 - v^2/c^2}} = (1 - v^2/c^2)^{-1/2} = 1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} + \cdots \xrightarrow[\frac{v}{c} \to 0]{} 1 + \frac{v^2}{2c^2}$$

Then,

$$\begin{aligned} T = mc^2 - m_o c^2 &= m_o c^2 \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right) \\ &\approx m_o c^2 \left(\left[1 + \frac{v^2}{2c^2} \right] - 1 \right) = \frac{1}{2} m_0 v^2 \end{aligned}$$

Thus the relativistic kinetic energy reduces to the classical expression for kinetic energy if $v << c$, a reassuring result since the validity of classical mechanics is well established in the macroscopic world.

NOTE: The Taylor series of a function $f(x)$ at a point x_o is given by

$$f(x) = f(x_0) + \frac{f'(x_o)}{1!}(x - x_o) + \frac{f''(x_o)}{2!}(x - x_o)^2 + \frac{f'''(x_o)}{3!}(x - x_o)^3 + \cdots$$

Relationship Between Kinetic Energy and Momentum:

Both classically and relativistically, the momentum p of a particle is given by $p = mv$.

- Classical physics:

$$T = \frac{mv^2}{2} = \frac{p^2}{2m} \Rightarrow p = \sqrt{2mT} \qquad (10)$$

- Relativistic physics: Squaring (8) we obtain $m^2 = \frac{m_o^2}{1-v^2/c^2} = \frac{m_o^2 c^2}{c^2-v^2}$. Then, $m^2c^2 - m^2v^2 = m_o^2c^2$ or alternatively

$$p^2 \equiv (mv)^2 = (mc)^2 - (m_oc)^2 = \frac{1}{c^2}[(mc^2)^2 - (m_oc^2)^2] \qquad (11)$$

  Finally, using (9) for mc^2, i.e. $mc^2 = T + m_oc^2$, we obtain

$$p = \frac{1}{c}\sqrt{T^2 + 2Tm_oc^2} \qquad (12)$$

2: Wave-particle Duality

- Matter (e.g., electrons) and radiation (e.g., X rays) both have wave–like and particle–like properties.
- This dichotomy, known as the wave–particle duality principle, is a cornerstone of modern physics.
- For some phenomena, a wave description works best; for others, a particle model is appropriate.
- Three pioneering experiments helped to establish the wave-particle nature of matter:
  - Photoelectric Effect
  - Compton Scattering
  - Electron Scattering

Photoelectric Effect:

- In 1887, Heinrich Hertz discovered that, when metal surfaces were irradiated with light, “electricity” was emitted.
- In 1898, Joseph John Thomson showed that these emissions were electrons (thus the term photoelectrons).

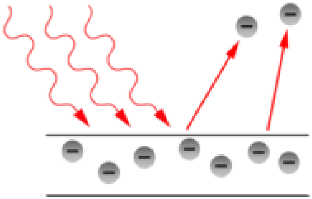


Figure: Photoelectric Effect

Photoelectric Effect:

Classical (wave theory) description of light: light energy is absorbed by the metal surface, a bound electron is freed, a photoelectron "boils" off the surface.

If light were a wave:

1. Photoelectrons should be produced by light of all frequencies (wave amplitude determines energy).
2. At low intensities, a time lag would be expected between the start of irradiation and the emission of a photoelectron (time to absorb sufficient energy).
3. As the light intensity (i.e., wave amplitude) increases, more energy is absorbed per unit time and, hence, the photoelectron emission rate should increase.
4. The kinetic energy of the photoelectron should increase with the light intensity since more energy is absorbed.

Photoelectric Effect:

However, experimental results differed dramatically with these expectations.

It was observed:

1. For each metal there is a minimum light frequency below which no photoelectrons are emitted no matter how high the intensity.
2. There is no time lag between the start of irradiation and the emission of photoelectrons, no matter how low the intensity.
3. The intensity of the light affects only the emission rate of photoelectrons.
4. The kinetic energy of the photoelectron depends only on light frequency and not on its intensity. The higher the frequency, the more energetic is the photoelectron.

Photoelectric Effect:

- In 1905, Albert Einstein assumed that light energy consists of photons or "quanta of energy," each with an energy

$$E_p = h\nu \tag{13}$$

where h is Planck's constant $(6.62 \times 10^{-34} Js)$ and ν is the light frequency.

- He further assumed that the energy associated with each photon interacts as a whole, i.e., either all the energy is absorbed by an atom or none is.

- With this "particle" model for the light, the maximum kinetic energy of a photoelectron would be

$$E = h\nu - A, \tag{14}$$

where A is the amount of energy (the so-called work function) required to free an electron from the metal.

Photoelectric Effect:

- If $h\nu < A$, no photoelectrons are produced.
- Increasing light intensity only increases number of photons hitting current collector metal surface per unit time and, thus, rate of photoelectron emission.
- Electromagnetic waves traveling at speed of light c with wavelength λ and frequency ν satisfy $c = \lambda\nu \iff \nu = c/\lambda$.

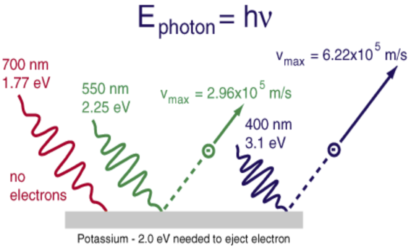


Figure: Photoelectric Effect

Photoelectric Effect:

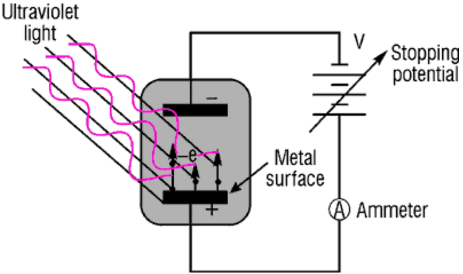


Figure: Photoelectric Effect

At a critical voltage V_o (no current $\Rightarrow$ the ammeter shows 0 Amps), the kinetic energy E of the photoelectron equals the potential energy $V_o e$ (e is the electron charge) the photoelectron must overcome, i.e.,

$$V_o e = E \Rightarrow V_o e = h\nu - A, \quad V_o = \frac{h\nu}{e} - \frac{A}{e}. \tag{15}$$

Photoelectric Effect:

- In 1912, Arthur Llewelyn Hughes showed that, for a given metallic surface (a given A), V_o was a linear function of ν.
- In 1916, Robert Andrews Millikan, verified that plots of V_o versus ν had a slope of h/e, from which h was calculated.

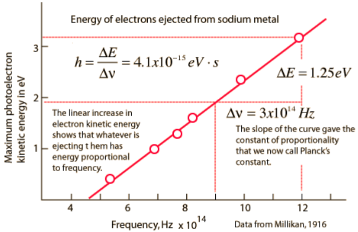


Figure: Photoelectric Effect

- **Thus, Einstein was able to explain the Photoelectric Effect by exploiting the quantum nature of radiant energy.**

Compton Scattering:

- Other experimental observations showed that light, besides having quantized energy characteristics, must have another particle–like property, namely, momentum.

- According to the wave model of electromagnetic radiation, radiation should be scattered from an electron with no change in wavelength.

- However, in 1922, Arthur Holly Compton observed that X rays scattered from electrons had a decrease in the wavelength $\Delta\lambda = \lambda_f - \lambda_i$ proportional to $(1 - \cos(\theta))$ where θ was the scattering angle.

Compton Scattering:

To explain this observation, it was necessary to treat X rays as particles with a linear momentum $p = h/\lambda$ and energy $E = h\nu = hc/\lambda = pc$ (remember that electromagnetic waves traveling at the speed of light c with wavelength λ and frequency ν satisfy $c = \lambda\nu \Rightarrow \nu = c/\lambda$).

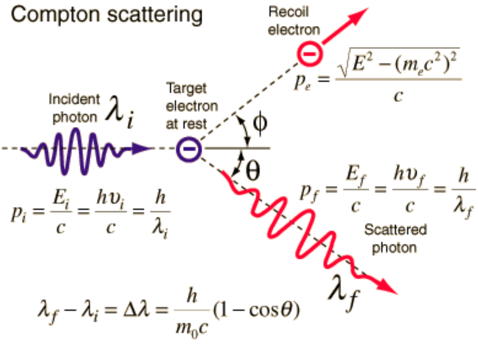


Figure: Compton Scattering

Compton Scattering:

In an X-ray scattering interaction, the energy and momentum before scattering must equal the energy and momentum after scattering, i.e., both energy (scalar) and momentum (vector) are conserved:

$$p_i c + m_o c^2 = p_f c + mc^2 \tag{16}$$

$$\mathbf{p}_i = \mathbf{p}_f + \mathbf{p}_e \iff p_e^2 = p_i^2 + p_f^2 - 2p_i p_f \cos(\theta) \tag{17}$$

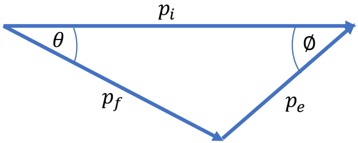


Figure: Compton Scattering

where m_o and m are the at-rest and relativistic masses of the electron respectively. Note that equation (17) employs the *law of cosines*.

Compton Scattering:

Note that the recoil electron may be relativistic. From (11) we can write, $p_e^2 = (mc)^2 - (m_o c)^2 \Rightarrow E = mc^2 = c\sqrt{p_e^2 + (m_o c)^2}$. We replace this last expression for mc^2 in (16) to write

$$p_i c + m_o c^2 = p_f c + c\sqrt{p_e^2 + (m_o c)^2} \tag{18}$$

and solve to obtain

$$p_e^2 = (p_i - p_f + m_o c)^2 - (m_o c)^2 = (p_i - p_f)^2 + 2(p_i - p_f)m_o c \tag{19}$$

Finally, we replace this expression for p_e^2 in (17) to write

$$(p_i - p_f)^2 + 2(p_i - p_f)m_o c = p_i^2 + p_f^2 - 2p_i p_f \cos(\theta) \tag{20}$$

and obtain

$$\frac{1}{p_f} - \frac{1}{p_i} = \frac{1}{m_o c}(1 - \cos(\theta)) \tag{21}$$

Compton Scattering:

Then since $\lambda = h/p$, this result gives the decrease in the scattered wavelength as

$$\lambda_f - \lambda_i = \frac{h}{m_o c}(1 - \cos(\theta)) \tag{22}$$

With the photon relations $\lambda = c/\nu$ and $E = h\nu$ ($\Rightarrow \lambda = ch/E$), we can write

$$\frac{1}{E_f} - \frac{1}{E_i} = \frac{1}{m_o c^2}(1 - \cos(\theta)) \tag{23}$$

Thus, Compton was able to predict the wavelength change of scattered X rays by using a particle model for the X rays, a prediction which could not be obtained with a wave model.

Electron Scattering:

- In 1924, Louis de Broglie postulated that, since light had particle properties, then particles should have wave properties. Because photons had a discrete energy $E = h\nu$ and momentum $p = h/\lambda$, de Broglie suggested that a particle, because of its momentum, should have an associated wavelength $\lambda = h/p$.

- In 1927, Clinton Davisson and Lester Germer confirmed that electrons did indeed behave like waves with de Broglie's predicted wavelength. In their experiment, they illuminated the surface of a Ni crystal by a perpendicular beam of $54eV$ electrons and measured the number of electrons $N(\phi)$ reflected at different angles ϕ from the incident beam.

Electron Scattering:

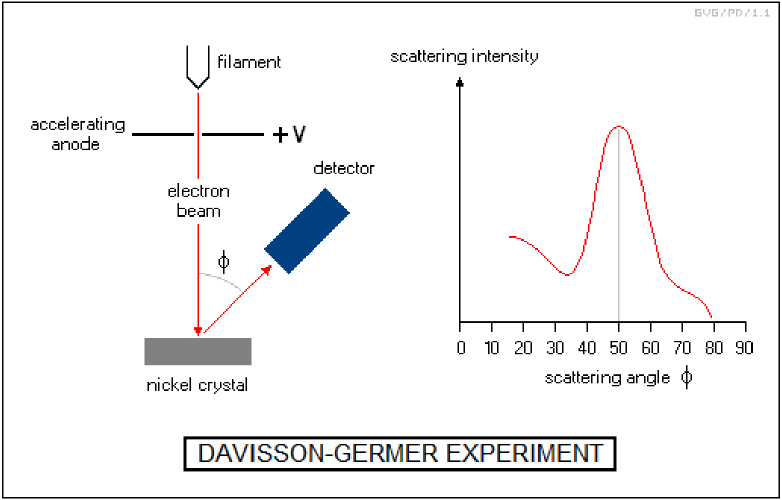


Figure: Electron Scattering

Electron Scattering:

- According to the particle model, electrons should be scattered by individual atoms isotropically and $N(\phi)$ should exhibit no structure.
- However, $N(\phi)$ was observed to have a peak near 50°.
- This observation could only be explained by recognizing the peak as a constructive interference wave phenomenon. Specifically, two reflected electron waves are in phase (constructively interfere) if the difference in their path lengths is an integral number of wavelengths.
- This experiment and many similar ones clearly demonstrated that electrons (and other particles such as atoms) have wave–like properties.

3: Quantum Mechanics

- The demonstration that particles (point objects) also had wave properties led to another major advance of modern physics. Because a material object such as an electron has wave properties, it should obey some sort of wave equation.
- Indeed, Erwin Rudolf Josef Alexander Shrodinger in 1925 showed that atomic electrons could be well described as standing waves around the nucleus.
- Further, the electron associated with each wave could have only a discrete energy.
- The branch of physics devoted to this wave description of particles is called quantum mechanics.

Schrödinger Wave Equation:

To illustrate Schrödinger's wave equation, we begin with an analogy to the standing waves produced by a plucked string anchored at both ends.

The wave equation that describes the displacement $\Psi(x,t)$ as a function of position x from one end of the string, which has length L, and at time t is

$$\frac{\partial^2 \Psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \Psi(x,t)}{\partial t^2}. \tag{24}$$

Here v is the wave speed. There are infinitely many discrete solutions to this homogeneous partial differential equation, subject to the boundary condition $\Psi(0,t) = \Psi(L,t) = 0$. We will further assume in the analysis below that the initial condition is given by $\Psi(x,0) = 0$.

- The solution of the wave equation (24) is separable, i.e., it has the form $\Psi(x,t) = \psi(x)T(t)$, which yields

$$\psi''T - \frac{1}{v^2}\psi T'' = 0 \iff \frac{v^2\psi''}{\psi} = \frac{T''}{T} \equiv -\gamma^2 \tag{25}$$

- The solution for $T'' + \gamma^2 T = 0$ is given by $T(t) = A\sin(2\pi\nu t), \gamma \triangleq 2\pi\nu$.
  - Characteristic equation: $r^2 + \gamma^2 = 0 \Rightarrow r = \pm j\gamma$
  - General solution: $T(t) = A\sin(\gamma t) + B\cos(\gamma t) = A\sin(2\pi\nu t) + B\cos(2\pi\nu t)$
  - If $T(0) = 0 \Rightarrow B = 0 \Rightarrow T(t) = A\sin(2\pi\nu t)$

- The equation of ψ then takes the form (recall $v = \lambda\nu$)

$$\frac{\partial^2\psi(x)}{\partial x^2} + \frac{4\pi^2\nu^2}{v^2}\psi(x) = 0 \iff \frac{\partial^2\psi(x)}{\partial x^2} + \frac{4\pi^2}{\lambda^2}\psi(x) = 0 \tag{26}$$

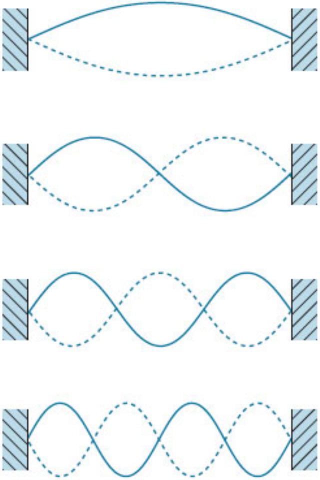
3: Quantum Mechanics

- The solution for $\psi'' + \frac{4\pi^2}{\lambda^2}\psi = 0$ is given by $\psi(x) = C\sin\left(2\pi\frac{n}{2L}x\right)$.
  - Characteristic equation: $r^2 + \frac{4\pi^2}{\lambda^2} = 0 \Rightarrow r = \pm j\frac{2\pi}{\lambda}$
  - General solution: $\psi(x) = C\sin(\frac{2\pi}{\lambda}x) + D\cos(\frac{2\pi}{\lambda}x)$
  - If $\psi(0) = 0 \Rightarrow D = 0 \Rightarrow \psi(x) = C\sin(\frac{2\pi}{\lambda}x)$
  - If $\psi(L) = 0 \Rightarrow \frac{2\pi}{\lambda}L = n\pi \Rightarrow \lambda = \frac{2L}{n} \Rightarrow \psi(x) = C\sin(2\pi\frac{n}{2L}x)$
- The final solution takes the form

$$\Psi(x,t) = E\sin\left(2\pi\frac{n}{2L}x\right)\sin\left(2\pi\frac{nv}{2L}t\right) \tag{27}$$

- The frequencies $\nu = \frac{v}{\lambda} = \frac{nv}{2L}, n = 1, 2, 3, \ldots$ are discrete.
- The first mode is called the fundamental solution.

3: Quantum Mechanics



mode	wavelength	frequency
first	$2L$	$\frac{v}{2L}$
second	L	$\frac{v}{L}$
third	$\frac{2L}{3}$	$\frac{3v}{2L}$
fourth	$\frac{L}{2}$	$\frac{2v}{L}$

Figure: Wave solution of a vibrating string

3: Quantum Mechanics

- Generalizing the wave equation to describe the dynamics of an electron bound to an atomic nucleus in a three-dimensional space, we obtain the steady-state Schrödinger wave equation, which is the fundamental equation of quantum mechanics:

$$-\frac{h^2}{8\pi^2 m}\nabla^2\psi(x,y,z) + V(x,y,z)\psi(x,y,z) = E\psi(x,y,z) \qquad (28)$$

- The nucleus produces an electric field or electric force on the electron.
- The electron with mass m has a total energy E, kinetic energy T, and a potential energy V such that $T = E - V$.
- The wavelength of the electron is $\lambda = h/p = h/\sqrt{2mT} = h/\sqrt{2m(E-V)}$ (assuming the electron is nonrelativistic).
- The electron around a nucleus has only discrete values of $E = E_n$ (verified experimentally).

3: Quantum Mechanics

- The non-trivial solution $\psi_n(x, y, z)$ of (28) when $E = E_n$ (an eigenvalue of the equation) is called a wave function.
- In general, this is a complex quantity which extends over all space, and may be thought of as the relative amplitude of a wave associated with the particle described by (28).
- The wave function is normalized as

$$\int\int\int \psi(x, y, z)\psi^*(x, y, z)dV = 1.$$

- The square of the amplitude of the wave function $|\psi|^2$ gives the probability of finding the particle at any position in space.
- Thus, the probability that the particle is in some small volume dV around the point (x, y, z) is given by $|\psi(x, y, z)|^2 dV = \psi(x, y, z)\psi^*(x, y, z)dV$.
- The normalization condition requires that the particle be somewhere in space.

The Uncertainty Principle:

- Within quantum mechanics it is no longer possible to say that a particle is at a particular location. We can say only that the particle has a probability $\psi\psi^* dV$ of being in dV.
- In 1927, Werner Heisenberg postulated that if one attempts to measure both a particle's position along the x-axis and its momentum, there will be an uncertainty Δx in the measured position and an uncertainty Δp in the momentum given by

$$\Delta x \Delta p \geq \frac{h}{2\pi} \tag{29}$$

- This limitation is a direct consequence of the wave properties of a particle and can be derived rigorously from Schrödinger's wave equation. However, a more phenomenological approach is to consider an attempt to measure the location of an electron with very high accuracy (by observing a system, the system is necessarily altered).