ME 343 – Control Systems

Lecture 39

November 23, 2009

State Feedback

We consider the linear, time-invariant system

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

and we look for a state gain K such that

$$u = -Kx$$

In this case we have the closed-loop system

$$\dot{x} = Ax - BKx = (A - BK)x$$

We should note that we can modify the dynamics (eigenvalues) of the system by state feedback. If the system is controllable, it is always possible to find a state gain K to set the eigenvalues of the closed-loop system at arbitrary values.

State Feedback

We consider the linear, time-invariant system

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

We define the state transformation

$$x(t) = Tz(t) \Leftrightarrow T^{-1}x(t) = z(t)$$

Then we can write

$$T\dot{z} = ATz + Bu \Rightarrow \dot{z} = T^{-1}ATz + T^{-1}Bu$$
$$y = CTz + Du.$$

to obtain

$$\dot{z} = \tilde{A}z + \tilde{B}u$$
$$y = \tilde{C}z + \tilde{D}u$$

$$\boxed{\tilde{A} = T^{-1}AT, \tilde{B} = T^{-1}B, \tilde{C} = CT, \tilde{D} = D}$$

State Feedback

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{Y(s)}{X(s)}\frac{X(s)}{U(s)}$$

$$\frac{X(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \frac{Y(s)}{X(s)} = b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0$$

Choosing $x_1 = x^{(n-1)}, x_2 = x^{(n-2)}, \ldots, x_{n-1} = x^{(1)}, x_n = x$

$$A = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} b_{n-1} & b_{n-2} & \cdots & b_1 & b_0 \end{bmatrix}, D = 0$$

Controller Form

State Feedback

$$\begin{aligned}\dot{x} &= Ax + Bu,\\ y &= Cx + Du.\end{aligned} \quad \xrightarrow[\substack{x = Tx_c \\ T = \overline{C}T_u(a)}]{} \quad \begin{aligned}\dot{x}_c &= A_c x_c + B_c u,\\ y &= C_c x_c + D_c u.\end{aligned}$$

where $a(s)$ is the characteristic polynomial

$$\overline{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$ Controllability Matrix

$$T_u = \begin{bmatrix} 1 & a_{n-1} & & a_1 & a_0 \\ 0 & 1 & a_{n-1} & & a_1 \\ \vdots & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$ Upper Toeplitz Matrix

State Feedback

Bass-Gura Formula:

$$K = (\alpha - a)T_u^{-1}(a)\overline{C}^{-1}$$

where $a(s)$ is the actual characteristic polynomial and $\alpha(s)$ is the desired characteristic equation.

$$\overline{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$ Controllability Matrix

$$T_u = \begin{bmatrix} 1 & a_{n-1} & & a_1 & a_0 \\ 0 & 1 & a_{n-1} & & a_1 \\ \vdots & \ddots & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$ Upper Toeplitz Matrix

State Feedback

Ackerman Formula:

$$K = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \overline{C}^{-1} \alpha(A)$$

where $\alpha(s)$ is the desired characteristic equation.

$$\overline{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$ Controllability Matrix

State Feedback

Mayne-Murdoch Formula:

$$k_i b_i = \frac{\prod_j \left(\lambda_i - \mu_j\right)}{\prod_{j \neq i} \left(\lambda_i - \lambda_j\right)}$$

where $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of A, and $\{\mu_1, \ldots, \mu_n\}$ are the desired eigenvalues of A-BK.