ME 343 – Control Systems

Lecture 37

November 18, 2009

Solution of State Space Equation

Solution by the Laplace Transform:

We consider the linear, time-invariant system

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

We Laplace transform the state equation to obtain

$$sX(s) - x(0) = AX(s) + BU(s) \qquad L[\dot{x}(t)] = sX(s) - x(0)$$

$$X(s) = L[x(t)], \quad U(s) = L[u(t)]$$

And we solve to obtain

$$X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} BU(s)$$

Solution of State Space Equation

$$X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} BU(s)$$

We inverse Laplace transform to obtain

$$\boxed{x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau}$$

where we have used that

$$L\left[e^{At}\right] = (sI - A)^{-1}$$

$$L\left[\int_0^t f(t-\tau) g(\tau) d\tau\right] = f(s) g(s)$$

State Transformation

We consider the linear, time-invariant system

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

We define the state transformation

$$x(t) = Tz(t) \Leftrightarrow T^{-1} x(t) = z(t)$$

Then we can write

$$T\dot{z} = ATz + Bu \Rightarrow \dot{z} = T^{-1} ATz + T^{-1} Bu$$
$$y = CTz + Du.$$

to obtain

$$\dot{z} = \tilde{A} z + \tilde{B} u$$
$$y = \tilde{C} z + \tilde{D} u$$

$$\boxed{\tilde{A} = T^{-1} AT, \tilde{B} = T^{-1} B, \tilde{C} = CT, \tilde{D} = D}$$

State Transformation

The transfer function (input-output relationship) does NOT depend on state choice

$$\frac{Y(s)}{U(s)} = \tilde{C}\left(sI - \tilde{A}\right)^{-1}\tilde{B} + \tilde{D} = C\left(sI - A\right)^{-1}B + D$$

Proof:

Controllability and Observability

Let us assume the following modal form:

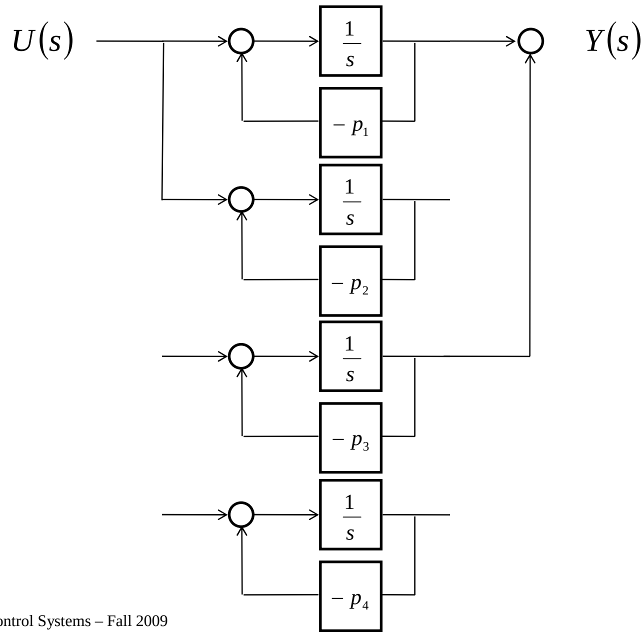
$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

$$A = \begin{bmatrix} -p_1 & 0 & 0 & 0 \\ 0 & -p_2 & 0 & 0 \\ 0 & 0 & -p_3 & 0 \\ 0 & 0 & 0 & -p_4 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} k_1 & 0 & k_3 & 0 \end{bmatrix}, D = 0$$

$$Y(s) = \frac{k_1}{s+p_1}U(s) + \frac{0}{s+p_2}U(s) + \frac{k_3}{s+p_3}0 + \frac{0}{s+p_4}0$$

Only the controllable and observable mode appears in the transfer function

Controllability and Observability



Controllability

Problem Definition: "A system is said to be controllable if and only if it is possible, by means of the input, to transfer the system from any initial state *x*(0) to any other state *x*(*t*) in a finite time $t \geq 0$."

Theorem: "A system is controllable if and only if the matrix

$$\overline{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$ Controllability Matrix

is full-rank."

Observability

Problem Definition: "An unforced system is said to be observable if and only if it is possible to determine any (arbitrary initial) state $x(0)$ by using only a finite record, $y(\tau)$ for $0 \le \tau \le T$, of the output"

Theorem: "A system is controllable if and only if the matrix

$$\overline{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Observability Matrix

is full-rank."

Controllability and Observability

Examples: