ME 343 – Control Systems

Lecture 36

November 16, 2009

Model Representation

Scalar Differential Equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y = b_m u^{(m)} + b_{m-1} u^{(m-1)} + \cdots + b_1 \dot{u} + b_0 u$$

Transfer Function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

State Variable Representation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Transfer Function

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1} B + D$$

Model Representation: Diff. Eq. → State Space

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}=\frac{Y(s)}{X(s)}\frac{X(s)}{U(s)}$$

$$\frac{X(s)}{U(s)}=\frac{1}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0},\ \frac{Y(s)}{X(s)}=b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_1 s+b_0$$

Choosing $x_1=x^{(n-1)}, x_2=x^{(n-2)},\ldots,x_{n-1}=x^{(1)}, x_n=x$

$$A=\begin{bmatrix}-a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \cdot & \cdot & & \cdot & \cdot\\ 0 & 0 & \cdots & 1 & 0\end{bmatrix}, B=\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, C=\begin{bmatrix}b_{n-1} & b_{n-2} & \cdots & b_1 & b_0\end{bmatrix}, D=0$$

Controller Form

Model Representation: Diff. Eq. → State Space

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}$$

$$\left(s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0\right)Y(s)=\left(b_{n-1}s^{n-1}+b_{n-2}s^{n-2}+\cdots+b_1 s+b_0\right)U(s)$$

$$Y(s)=s^{-1}\left(b_{n-1}U(s)-a_{n-1}Y(s)\right)+s^{-2}\left(b_{n-2}U(s)-a_{n-2}Y(s)\right)+\cdots$$
$$+s^{1-n}\left(b_1U(s)-a_1Y(s)\right)+s^{-n}\left(b_0U(s)-a_0Y(s)\right)$$

Choosing $x_1=y^{(n-1)}, x_2=y^{(n-2)},\ldots,x_{n-1}=y^{(1)}, x_n=y$

$$A=\begin{bmatrix}0 & 0 & \cdots & 0 & -a_0\\ 1 & 0 & \cdots & 0 & -a_1\\ 0 & 1 & \cdots & 0 & \vdots\\ \cdot & \cdot & \ddots & \cdot & -a_{n-2}\\ 0 & 0 & \cdots & 1 & -a_{n-1}\end{bmatrix}, B=\begin{bmatrix}b_0\\ b_1\\ \vdots\\ b_{n-2}\\ b_{n-1}\end{bmatrix}, C=\begin{bmatrix}0 & 0 & \cdots & 0 & 1\end{bmatrix}, D=0$$

Observer Form

Model Representation: Diff. Eq. $\rightarrow$ State Space

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{k_1}{s+p_1} + \cdots + \frac{k_n}{s+p_n}$$

Choosing

$$\dot{x}_i = -p_i x_i + u$$
$$y_i = k_i x_i$$

$$\boxed{y = \sum_{i=1}^{n} y_i}$$

$$A = \begin{bmatrix} -p_1 & 0 & \cdots & 0 & 0 \\ 0 & -p_2 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \cdot & \cdot & \cdot & -p_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & -p_n \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} k_1 & k_2 & \cdots & k_{n-1} & k_n \end{bmatrix}, D = 0$$

Modal Form