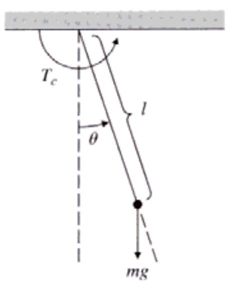
ME 343 – Control Systems

Lecture 34

November 11, 2009

Modeling

MECHANICAL SYSTEM:



$$F = I\alpha$$ Newton's law

damping coefficient

$$I\alpha = -lmg\sin\theta - b\omega + T_c$$

$\omega = \dot{\theta}$ angular velocity

$\alpha = \dot{\omega} = \ddot{\theta}$ angular acceleration

$I = ml^2$ moment of inertia

$$\ddot{\theta} = -\frac{b}{ml^2}\dot{\theta} - \frac{g}{l}\sin\theta + \frac{T_c}{ml^2}$$

Which are the equilibrium points when T_c=0?

Stable

At equilibrium: $\ddot{\theta} = \dot{\theta} = 0 \Rightarrow 0 = -\frac{g}{l}\sin\theta \Rightarrow \theta = 0, \pi$

Unstable

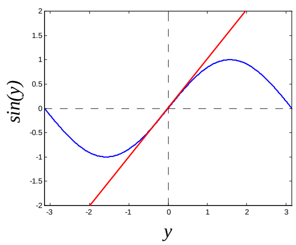
Linearization

What happens around $\theta=0$?

$$\theta = y \Rightarrow \quad \ddot{y} = -\frac{b}{ml^2}\dot{y} - \frac{g}{l}\sin(y) + \frac{T_c}{ml^2}$$

By Taylor Expansion:

$$\sin(y) = y + h.o.t. \Rightarrow \sin(y) \approx y$$



Linearized Equation:

$$\ddot{y} = -\frac{b}{ml^2}\dot{y} - \frac{g}{l}y + \frac{T_c}{ml^2}$$

Transfer Function Modeling

$$u \equiv T_c \Rightarrow \quad \ddot{y} = -\frac{b}{ml^2}\dot{y} - \frac{g}{l}y + \frac{T_c}{ml^2}$$

$$\ddot{y} + \frac{b}{ml^2}\dot{y} + \frac{g}{l}y = \frac{u}{ml^2}$$

Laplace Transform

$$\mathcal{L}\left\{\frac{d^m f(t)}{dt^m}\right\} = s^m F(s),\ U(s) = \mathcal{L}\{u\},\ Y(s) = \mathcal{L}\{y\}$$

Transfer Function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\frac{1}{ml^2}}{s^2 + \frac{b}{ml^2}s + \frac{g}{l}}$$

Characteristic Equation

Transfer Function Modeling

$T_c = 0 \Rightarrow \ddot{y} + \frac{b}{ml^2}\dot{y} + \frac{g}{l}y = 0$ What is the solutions $y(t)$?

Characteristic Equation

$$\lambda^2 + \frac{b}{ml^2}\lambda + \frac{g}{l} = 0 \Rightarrow \lambda_{1,2} = \frac{-\frac{b}{ml^2} \pm \sqrt{\left(\frac{b}{ml^2}\right)^2 - 4\frac{g}{l}}}{2}$$

$$G(s) = \frac{1/ml^2}{s^2 + \frac{b}{ml^2}s + \frac{g}{l}} \qquad x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

The dynamics of the system is given by the roots of the denominator (poles) of the trasfer function

real$(\lambda_1, \lambda_2) < 0 \Rightarrow$ STABLE SYSTEM

Linearization

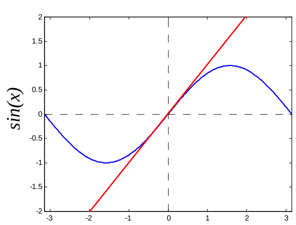
What happens around $\theta = \pi$?

$$\theta = \pi + x \Rightarrow \quad \ddot{x} = -\frac{b}{ml^2}\dot{x} - \frac{g}{l}\sin(\pi + x) + \frac{T_c}{ml^2}$$

$$\ddot{x} = -\frac{b}{ml^2}\dot{x} + \frac{g}{l}\sin(x) + \frac{T_c}{ml^2}$$

By Taylor Expansion:

$$\sin(x) = x + h.o.t. \Rightarrow \sin(x) \approx x$$



Linearized Equation:

$$\ddot{x} = -\frac{b}{ml^2}\dot{x} + \frac{g}{l}x + \frac{T_c}{ml^2}$$

State-Space Modeling

$$\ddot{x} = -\frac{b}{ml^2}\dot{x} + \frac{g}{l}x + \frac{T_c}{ml^2}$$ Reduce to first order equations:

State Variable Representation

$$\begin{matrix} x_1 = x \\ x_2 = \dot{x} \end{matrix} \Rightarrow \begin{matrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{b}{ml^2}x_2 + \frac{g}{l}x_1 + \frac{T_c}{ml^2} \end{matrix}$$

$$x \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, u \equiv T_c \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{b}{ml^2} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u = Ax + Bu$$

$$eig(A) = \{\lambda : |\lambda I - A| = 0\} = \left\{\lambda : \lambda^2 + \frac{b}{ml^2}\lambda - \frac{g}{l} = 0\right\}$$

Characteristic Equation

State-Space Modeling

$$T_c = 0 \Rightarrow \ddot{x} + \frac{b}{ml^2}\dot{x} - \frac{g}{l}x = 0$$ What is the solution $x(t)$?

Characteristic Equation

$$\lambda^2 + \frac{b}{ml^2}\lambda - \frac{g}{l} = 0 \Rightarrow \lambda_{1,2} = \frac{-\frac{b}{ml^2} \pm \sqrt{\left(\frac{b}{ml^2}\right)^2 + 4\frac{g}{l}}}{2}$$

$$eig(A) = \{\lambda : |\lambda I - A| = 0\} = \left\{\lambda : \lambda^2 + \frac{b}{ml^2}\lambda - \frac{g}{l} = 0\right\} \qquad x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

The dynamics of the system is given by the eigenvalues of the system matrix

real(eig(A)) > 0 (real(λ_1,λ_2) > 0) $\Rightarrow$ INSTABILITY