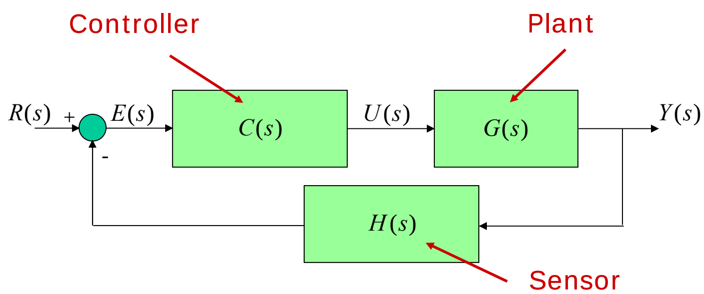
ME 343 – Control Systems

Lecture 28

October 28, 2009

Root Locus



$$C(s) = KD(s) \Rightarrow \quad \frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)H(s)} = \frac{C(s)G(s)}{1 + KL(s)}$$

Writing the loop gain as $KL(s)$ we are interested in tracking the closed-loop poles as "gain" K varies

Root Locus

Characteristic Equation:

$$1 + KL(s) = 0$$

The roots (zeros) of the characteristic equation are the closed-loop poles of the feedback system!!!

The closed-loop poles are a function of the "gain" K

Writing the loop gain as

$$L(s) = \frac{b(s)}{a(s)} = \frac{s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

The closed loop poles are given indistinctly by the solution of:

$$1 + KL(s) = 0, \quad 1 + K\frac{b(s)}{a(s)} = 0, \quad a(s) + Kb(s) = 0, \quad L(s) = -\frac{1}{K}$$

Root Locus

$$\text{RL} = \text{zeros}\{1 + KL(s)\} = \text{roots}\{\text{den}(L) + K\text{num}(L)\}$$

when K varies from 0 to ∞ (positive Root Locus) or from 0 to $-\infty$ (negative Root Locus)

$$K > 0 : L(s) = -\frac{1}{K} \Leftrightarrow \begin{cases} |L(s)| = \frac{1}{K} & \text{Magnitude condition} \\ \angle L(s) = 180^\circ & \text{Phase condition} \end{cases}$$

$$K < 0 : L(s) = -\frac{1}{K} \Leftrightarrow \begin{cases} |L(s)| = -\frac{1}{K} & \text{Magnitude condition} \\ \angle L(s) = 0^\circ & \text{Phase condition} \end{cases}$$

Phase and Magnitude of a Transfer Function

$$G(s)=\frac{b_m s^m + b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}{s^n + a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}$$

$$G(s)=K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

The factors K, $(s\text{-}z_j)$ and $(s\text{-}p_k)$ are complex numbers:

$$(s-z_j)=r_j^z e^{i\phi_j^z}, \qquad j=1\ldots m$$

$$(s-p_k)=r_k^p e^{i\phi_k^p}, \qquad k=1\cdots p$$

$$K=|K|e^{i\phi^K}$$

$$G(s)=|K|e^{i\phi^K}\ \frac{r_1^z e^{i\phi_1^z}\, r_2^z e^{i\phi_2^z}\cdots r_m^z e^{i\phi_m^z}}{r_1^p e^{i\phi_1^p}\, r_2^p e^{i\phi_2^p}\cdots r_n^p e^{i\phi_n^p}}$$

Phase and Magnitude of a Transfer Function

$$\begin{aligned}G(s)&=|K|e^{i\phi^K}\ \frac{r_1^z e^{i\phi_1^z}\, r_2^z e^{i\phi_2^z}\cdots r_m^z e^{i\phi_m^z}}{r_1^p e^{i\phi_1^p}\, r_2^p e^{i\phi_2^p}\cdots r_n^p e^{i\phi_n^p}}\\ &=|K|e^{i\phi^K}\ \frac{r_1^z r_2^z\cdots r_m^z e^{i\left(\phi_1^z+\phi_2^z+\cdots+\phi_m^z\right)}}{r_1^p r_2^p\cdots r_n^p e^{i\left(\phi_1^p+\phi_2^p+\cdots+\phi_n^p\right)}}\\ &=|K|\frac{r_1^z r_2^z\cdots r_m^z}{r_1^p r_2^p\cdots r_n^p}\, e^{i\left[\phi^K+\left(\phi_1^z+\phi_2^z+\cdots+\phi_m^z\right)-\left(\phi_1^p+\phi_2^p+\cdots+\phi_n^p\right)\right]}\end{aligned}$$

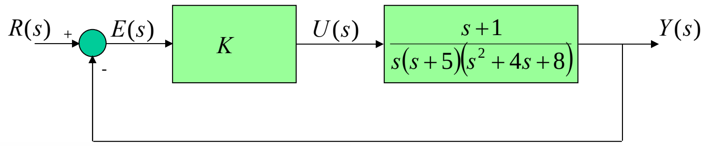
Now it is easy to give the phase and magnitude of the transfer function:

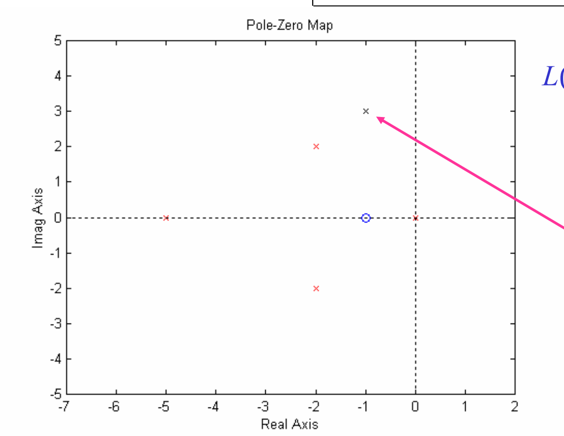
$$|G(s)|=|K|\frac{r_1^z r_2^z\cdots r_m^z}{r_1^p r_2^p\cdots r_n^p},$$

$$\angle G(s)=\phi^K+\left(\phi_1^z+\phi_2^z+\cdots+\phi_m^z\right)-\left(\phi_1^p+\phi_2^p+\cdots+\phi_n^p\right)$$

Root Locus by Phase Condition

Example:





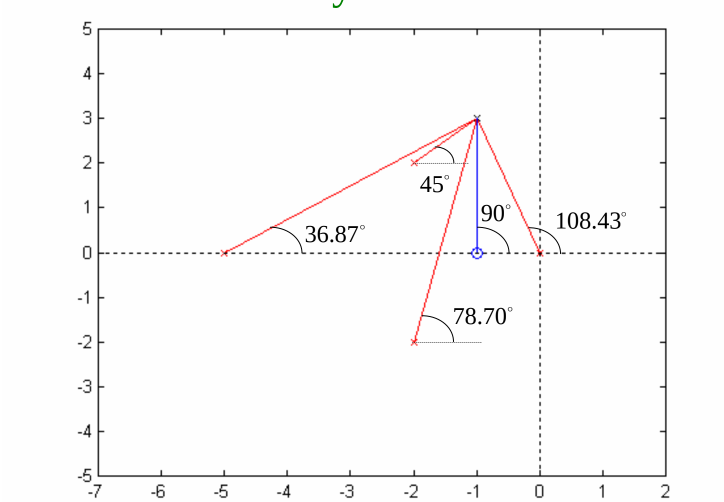
$$L(s) = \frac{s+1}{s(s+5)(s^2+4s+8)}$$

$$= \frac{s+1}{s(s+5)(s+2+2i)(s+2-2i)}$$

$s_o = -1+3i$

belongs to the locus?

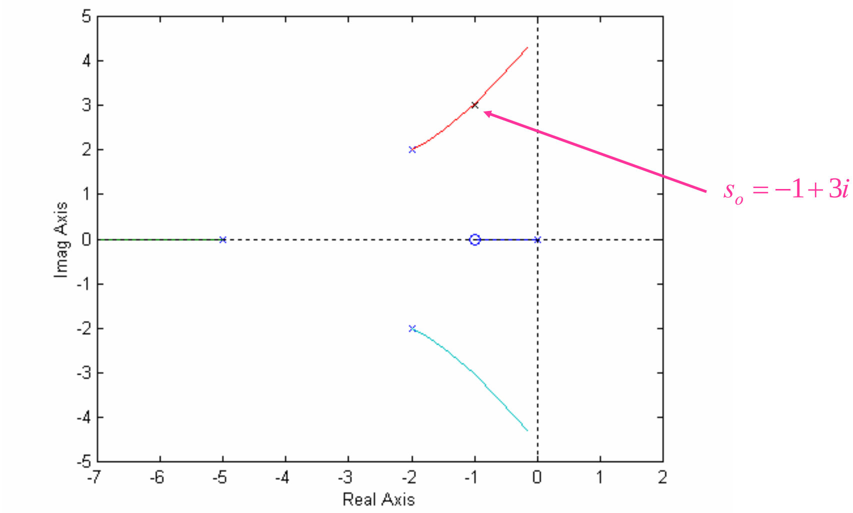
Root Locus by Phase Condition



$90^\circ - [108.43^\circ + 36.87^\circ + 45^\circ + 78.70^\circ] \approx -180^\circ \Rightarrow s_o = -1+3i$ belongs to the locus!

Note: Check code rlocus_phasecondition.m

Root Locus by Phase Condition



We need a systematic approach to plot the closed-loop poles as function of the gain $K \rightarrow$ ROOT LOCUS

Root Locus

$$\text{RL} = \text{zeros}\{1 + KL(s)\} = \text{roots}\{\text{den}(L) + K\text{num}(L)\}$$

when K varies from 0 to ∞ (positive Root Locus) or from 0 to -∞ (negative Root Locus)

$$1 + KL(s) = 0 \Leftrightarrow L(s) = -\frac{1}{K} \Leftrightarrow a(s) + Kb(s) = 0$$

Basic Properties:

- Number of branches = number of open-loop poles
- RL begins at open-loop poles

$$K = 0 \Rightarrow a(s) = 0$$

- RL ends at open-loop zeros or asymptotes

$$K = \infty \Rightarrow L(s) = 0 \Leftrightarrow \begin{cases} b(s) = 0 \\ s \rightarrow \infty \ (n - m > 0) \end{cases}$$

- RL symmetrical about Re-axis

Root Locus

Rule 1: The n branches of the locus start at the poles of $L(s)$ and m of these branches end on the zeros of $L(s)$.
n: order of the denominator of $L(s)$
m: order of the numerator of $L(s)$

Rule 2: The locus is on the real axis to the left of and odd number of poles and zeros.
In other words, an interval on the real axis belongs to the root locus if the total number of poles and zeros to the right is odd.
This rule comes from the phase condition!!!

Root Locus

Rule 3: As $K \to \infty$, m of the closed-loop poles approach the open-loop zeros, and $n\text{-}m$ of them approach $n\text{-}m$ asymptotes with angles

$$\phi_l = (2l+1)\frac{\pi}{n-m}, \qquad l = 0,1,\ldots,n-m-1$$

and centered at

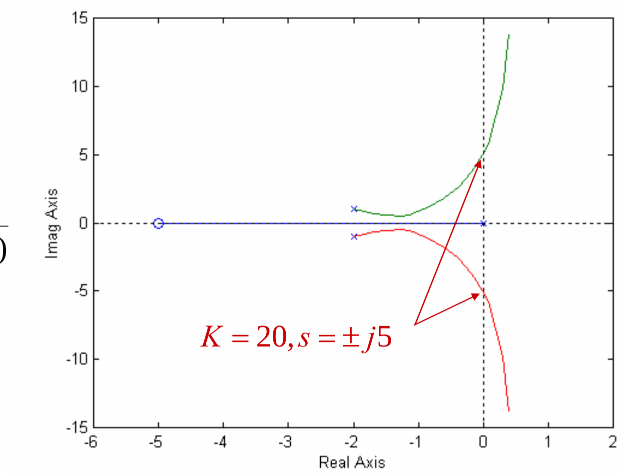
$$\alpha = \frac{b_1 - a_1}{n-m} = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m}, \qquad l = 0,1,\ldots,n-m-1$$

Root Locus

Rule 4: The locus crosses the $j\omega$ axis (looses stability) where the Routh criterion shows a transition from roots in the left half-plane to roots in the right-half plane.

Example:

$$G(s) = \frac{s+5}{s(s^2+4s+5)}$$



Root Locus

Design dangers revealed by the Root Locus:

• High relative degree: For n-$m \geq 3$ we have closed loop instability due to asymptotes.

$$G(s) = \frac{s+1}{s^4+3s^3+7s^2+6s+4}$$

• Nonminimum phase zeros: They attract closed loop poles into the RHP

$$G(s) = \frac{s-1}{s^2+s+1}$$

Note: Check code rootlocus.m

Root Locus

Viete's formula:

When the relative degree n-m≥2, the sum of the closed loop poles is constant

$$a_1 = -\sum \text{closed loop poles}$$

$$L(s) = \frac{b(s)}{a(s)} = \frac{s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

Root Locus- Magnitude and Phase Conditions

$$\text{RL} = \text{zeros}\{1 + KL(s)\} = \text{roots}\{\text{den}(L) + K\text{num}(L)\}$$

when K varies from 0 to ∞ (positive Root Locus) or from 0 to -∞ (negative Root Locus)

$$L(s) = K_p \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} = \left|K_p\right| \frac{r_1^z r_2^z \cdots r_m^z}{r_1^p r_2^p \cdots r_n^p} e^{i\left[\phi^{K_p} + \left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right) - \left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right)\right]}$$

$$K > 0 : L(s) = -\frac{1}{K} \Leftrightarrow \begin{array}{l} \left|L(s)\right| = \left|K_p\right| \dfrac{r_1^z r_2^z \cdots r_m^z}{r_1^p r_2^p \cdots r_n^p} = \dfrac{1}{K} \\ \angle L(s) = \phi^{K_p} + \left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right) - \left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right) = 180^\circ \end{array}$$

$$K < 0 : L(s) = -\frac{1}{K} \Leftrightarrow \begin{array}{l} \left|L(s)\right| = \left|K_p\right| \dfrac{r_1^z r_2^z \cdots r_m^z}{r_1^p r_2^p \cdots r_n^p} = -\dfrac{1}{K} \\ \angle L(s) = \phi^{K_p} + \left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right) - \left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right) = 0^\circ \end{array}$$

Root Locus

Selecting K for desired closed loop poles on Root Locus:

If s_o belongs to the root locus, it must satisfies the characteristic equation for some value of K

$$L(s_o) = -\frac{1}{K}$$

Then we can obtain K as

$$K = -\frac{1}{L(s_o)}$$

$$K = \frac{1}{|L(s_o)|}$$

Root Locus – Phase lead compensator

Pure derivative control is not normally practical because of the amplification of the noise due to the differentiation and must be approximated:

$$D(s) = \frac{s+z}{s+p}, \qquad p > z$$

Phase lead COMPENSATOR

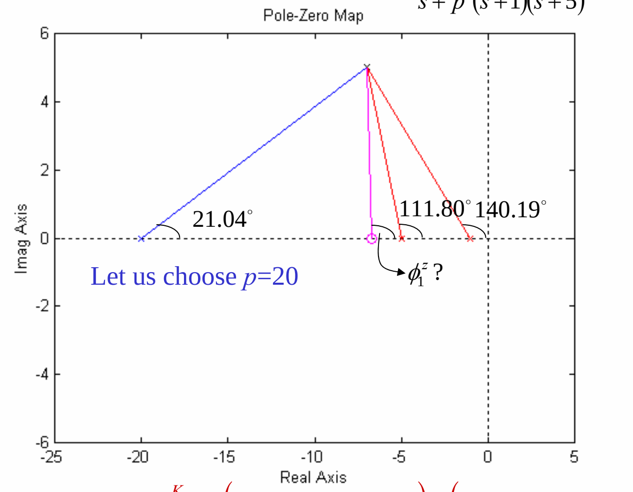
When we study frequency response we will understand why we call “Phase Lead” to this compensator.

$$L(s) = D(s)G(s) = \frac{s+z}{s+p}\frac{1}{(s+1)(s+5)}, \qquad p > z$$

How do we choose z and p to place the closed loop pole at s_o=-7+i5?

Root Locus – Phase lead compensator

Example: $L(s)=D(s)G(s)=\dfrac{s+z}{s+p}\dfrac{1}{(s+1)(s+5)},\quad p>z$



Phase lead COMPENSATOR

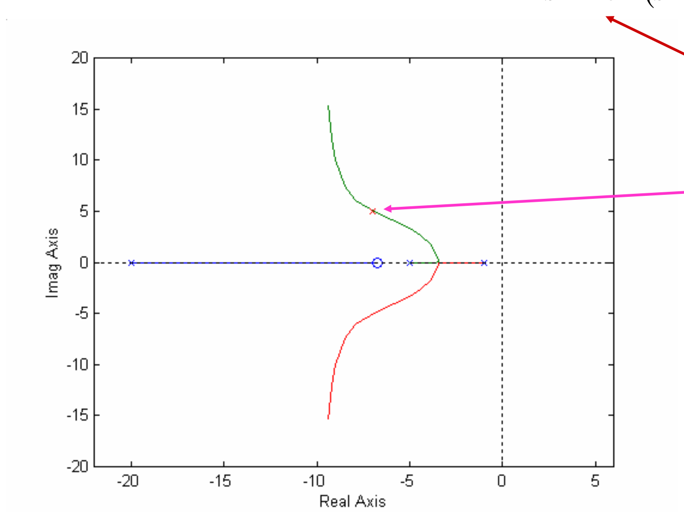
$$\angle L(s)=\phi^{K_p}+\left(\phi_1^z+\phi_2^z+\cdots+\phi_m^z\right)-\left(\phi_1^p+\phi_2^p+\cdots+\phi_n^p\right)=180^\circ$$

$$\phi_1^z=180^\circ+140.19^\circ+111.80^\circ+21.04^\circ=453.03^\circ=93.03^\circ \Rightarrow z=-6.735$$

Root Locus – Phase lead compensator

Example: $L(s)=D(s)G(s)=\dfrac{s+6.735}{s+20}\dfrac{1}{(s+1)(s+5)}$

Phase lead COMPENSATOR



$s_o=-7+i5$

$K=117$

Root Locus – Phase lead compensator

Selecting z and p is a trial an error procedure. In general:

- The zero is placed in the neighborhood of the closed-loop natural frequency, as determined by rise-time or settling time requirements.
- The poles is placed at a distance 5 to 20 times the value of the zero location. The pole is fast enough to avoid modifying the dominant pole behavior.

The exact position of the pole p is a compromise between:

- Noise suppression (we want a small value for p)
- Compensation effectiveness (we want large value for p)

Root Locus – Phase lag compensator

Example: $$L(s) = D(s)G(s) = \frac{s+6.735}{s+20}\frac{1}{(s+1)(s+5)}$$

$$K_p = \lim_{s\to 0} L(s) = \lim_{s\to 0} D(s)G(s) = \lim_{s\to 0}\frac{s+6.735}{s+20}\frac{1}{(s+1)(s+5)} = 6.735\times 10^{-2}$$

What can we do to increase K_p? Suppose we want $K_p=10$.

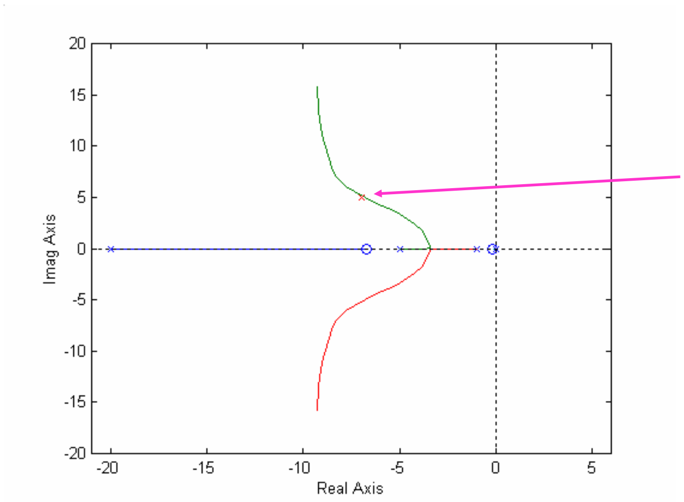
$$L(s) = D(s)G(s) = \frac{s+z}{s+p}\frac{s+6.735}{s+20}\frac{1}{(s+1)(s+5)}, \qquad p<z$$

Phase lag COMPENSATOR

We choose: $$\frac{z}{p} = \frac{1}{6.735}\times 10^{3} = 148.48$$

Root Locus – Phase lag compensator

Example: $L(s) = D(s)G(s) = \frac{s+0.14848}{s+0.001}\frac{s+6.735}{s+20}\frac{1}{(s+1)(s+5)}$



$s_o = -6.94 + i5.03$

$K = 18.31$

Root Locus – Phase lag compensator

Selecting z and p is a trial an error procedure. In general:

- The ratio zero/pole is chosen based on the error constant specification.
- We pick z and p small to avoid affecting the dominant dynamic of the system (to avoid modifying the part of the locus representing the dominant dynamics)
- Slow transient due to the small p is almost cancelled by an small z. The ratio zero/pole cannot be very big.

The exact position of z and p is a compromise between:

- Steady state error (we want a large value for z/p)
- The transient response (we want the pole p placed far from the origin)

Root Locus - Compensators

Phase lead compensator: $D(s) = \frac{s+z}{s+p}, \quad z < p$

Phase lag compensator: $D(s) = \frac{s+z}{s+p}, \quad z > p$

We will see why we call "phase lead" and "phase lag" to these compensators when we study frequency response

Frequency Response

- We now know how to analyze and design systems via s-domain methods which yield dynamical information
  - The responses are described by the exponential modes
    - The modes are determined by the poles of the response Laplace Transform
- We next will look at describing system performance via frequency response methods
  - This guides us in specifying the system pole and zero positions

Sinusoidal Steady-State Response

Consider a stable transfer function with a sinusoidal input:

$$u(t) = A\cos(\omega t) \Leftrightarrow U(s) = \frac{A\omega}{s^2 + \omega^2}$$

The Laplace Transform of the response has poles

- Where the natural system modes lie
  - These are in the open left half plane Re(s)<0
- At the input modes s=+jω and s=-jω

$$Y(s) = G(s)U(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}\frac{A\omega}{\left(s^2+\omega^2\right)}$$

Only the response due to the poles on the imaginary axis remains after a sufficiently long time

This is the sinusoidal steady-state response

Sinusoidal Steady-State Response

- Input $u(t) = A\cos(\omega t + \phi) = A\cos\omega t\sin\phi - A\sin\omega t\cos\phi$
- Transform $U(s) = -A\cos\phi\frac{s}{s^2+\omega^2} + A\sin\phi\frac{\omega}{s^2+\omega^2}$
- Response Transform

$$Y(s) = G(s)U(s) = \underbrace{\frac{k}{s-j\omega} + \frac{k^*}{s+j\omega}}_{\text{forced response}} + \underbrace{\frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_N}{s-p_N}}_{\text{natural response}}$$

- Response Signal

$$y(t) = \underbrace{ke^{j\omega t} + k^* e^{-j\omega t}}_{\text{forced response}} + \underbrace{k_1e^{p_1t} + k_2e^{p_2t} + \cdots + k_Ne^{p_Nt}}_{\text{natural response}} \xrightarrow{t\to\infty} 0$$

- Sinusoidal Steady State Response

$$y_{SS}(t) = ke^{j\omega t} + k^*e^{-j\omega t}$$

Sinusoidal Steady-State Response

- Calculating the SSS response to $u(t) = A\cos(\omega t + \phi)$
  - Residue calculation

$$k = \lim_{s\to j\omega}[(s-j\omega)Y(s)] = \lim_{s\to j\omega}[(s-j\omega)G(s)U(s)]$$

$$= \lim_{s\to j\omega}\left[G(s)(s-j\omega)A\frac{s\cos\phi-\omega\sin\phi}{(s-j\omega)(s+j\omega)}\right] = G(j\omega)A\left[\frac{j\omega\cos\phi-\omega\sin\phi}{2j\omega}\right]$$

$$= AG(j\omega)\frac{1}{2}e^{j\phi} = \frac{1}{2}A|G(j\omega)|e^{j(\phi+\angle G(j\omega))}$$

  - Signal calculation

$$y_{ss}(t) = L^{-1}\left\{\frac{k}{s-j\omega}+\frac{k^*}{s+j\omega}\right\}$$

$$= |k|e^{j\angle K}e^{j\omega t} + |k|e^{-j\angle K}e^{-j\omega t}$$

$$= 2|k|\cos(\omega t + \angle K)$$

$$y_{ss}(t) = A|G(j\omega)|\cos(\omega t + \phi + \angle G(j\omega))$$

Sinusoidal Steady-State Response

- Response to $u(t) = A\cos(\omega t + \phi)$
  is $y_{ss} = |G(j\omega)|A\cos(\omega t + \phi + \angle G(j\omega))$

  – Output frequency = input frequency
  – Output amplitude = input amplitude × $|G(j\omega)|$
  – Output phase = input phase + $\angle G(j\omega)$

- The Frequency Response of the transfer function *G(s)* is given by its evaluation as a function of a complex variable at $s=j\omega$
  - We speak of the amplitude response and of the phase response
    – They cannot independently be varied
      » Bode's relations of analytic function theory

Frequency Response

$$u(t) = A\cos(\omega t + \phi) \longrightarrow \boxed{G(s)} \longrightarrow y_{ss} = |G(j\omega)| A\cos(\omega t + \phi + \angle G(j\omega))$$

Stable Transfer Function

- After a transient, the output settles to a sinusoid with an amplitude magnified by $|G(j\omega)|$ and phase shifted by $\angle G(j\omega)$.

- Since all signals can be represented by sinusoids (Fourier series and transform), the quantities $|G(j\omega)|$ and $\angle G(j\omega)$ are extremely important.

- Bode developed methods for quickly finding $|G(j\omega)|$ and $\angle G(j\omega)$ for a given $G(s)$ and for using them in control design.

Bode Diagrams

$$G(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$$\Updownarrow$$

$$G(s) = |K|\frac{r_1^z r_2^z \cdots r_m^z}{r_1^p r_2^p \cdots r_n^p} e^{i\left[\phi^K + \left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right) - \left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right)\right]}$$

The magnitude and phase of G(s) when s=jω is given by:

Nonlinear in the magnitudes

$$|G(j\omega)| = |K|\frac{r_1^z r_2^z \cdots r_m^z}{r_1^p r_2^p \cdots r_n^p},$$

$$\angle G(j\omega) = \phi^K + \left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right) - \left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right)$$

Linear in the phases

Bode Diagrams

Why do we express $|G(j\omega)|$ in decibels?

$$|G(j\omega)|_{dB} = 20\log|G(j\omega)|$$

$$|G(j\omega)| = |K|\frac{r_1^z r_2^z \cdots r_m^z}{r_1^p r_2^p \cdots r_n^p} \Rightarrow |G(j\omega)|_{dB} = ?$$

By properties of the logarithm we can write:

$$20\log|G(s)| = 20\log|K| + \left(20\log r_1^z + 20\log r_m^z + \cdots + 20\log r_m^z\right) - \left(20\log r_1^p + 20\log r_2^p + \cdots + 20\log r_n^p\right)$$

The magnitude and phase of G(s) when s=jω is given by:

Linear in the magnitudes (dB)

$$|G(s)|_{dB} = |K|_{dB} + \left(r_1^z\big|_{dB} + r_2^z\big|_{dB} + \cdots + r_m^z\big|_{dB}\right) - \left(r_1^p\big|_{dB} + r_2^p\big|_{dB} + \cdots + r_n^p\big|_{dB}\right)$$

$$\angle G(s) = \phi^K + \left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right) - \left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right)$$

Linear in the phases

Bode Diagrams

Decade: Any frequency range whose end points have a 10:1 ratio

A cutoff frequency occurs when the gain is reduced from its maximum passband value by a factor $1/\sqrt{2}$:

$$20\log\left(\frac{1}{\sqrt{2}}|T|_{MAX}\right) = 20\log|T|_{MAX} - 20\log\sqrt{2} \approx 20\log|T|_{MAX} - 3\text{dB}$$

Bandwith: frequency range spanned by the gain passband

Let's have a look at our example:

$$|T(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega L}{R}\right)^2}} \Rightarrow \begin{cases} \omega = 0 & |T(j\omega)| = 1 \\ \omega = R/L & |T(j\omega)| = 1/\sqrt{2} \end{cases}$$

This is a low-pass filter!!! Passband gain=1, Cutoff frequency=R/L
The Bandwith is R/L!

General Transfer Function (Bode Diagrams)

$$G(j\omega) = K_o (j\omega)^m (j\omega\tau + 1)^n \left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta \frac{j\omega}{\omega_n} + 1 \right]^q$$

The magnitude (dB) (phase) is the sum of the magnitudes (dB) (phases) of each one of the terms. We learn how to plot each term, we learn how to plot the whole magnitude and phase Bode Plot.

Classes of terms:

1- $G(j\omega) = K_o$

2- $G(j\omega) = (j\omega)^m$

3- $G(j\omega) = (j\omega\tau + 1)^n$

4- $G(j\omega) = \left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta \frac{j\omega}{\omega_n} + 1 \right]^q$

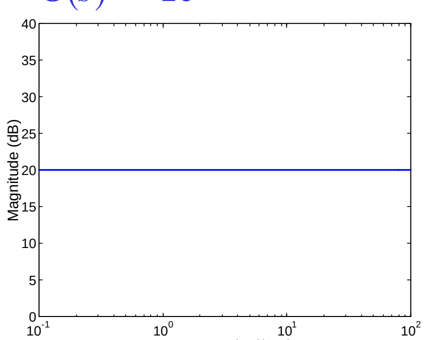
General Transfer Function: DC gain

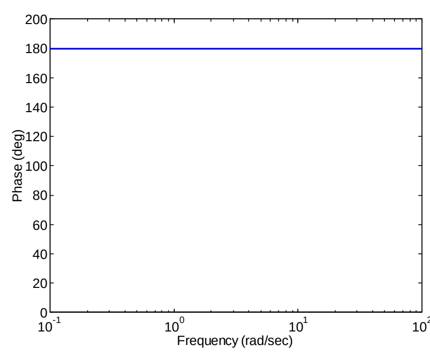
$$G(j\omega) = K_o$$

Magnitude and Phase: $|G(j\omega)|_{dB} = 20\log|K_o|$

$$\angle G(j\omega) = \begin{cases} 0 & \text{if } K_o > 0 \\ \pm\pi & \text{if } K_o < 0 \end{cases}$$

$G(s) = -10$



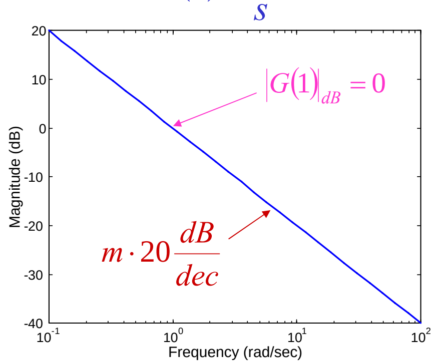


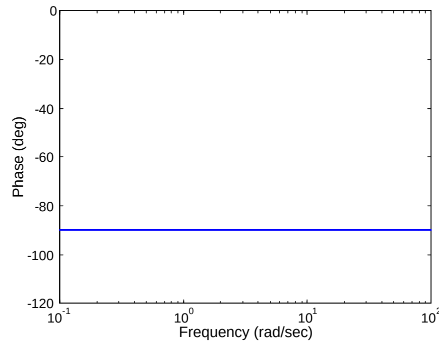
General Transfer Function: Poles/zeros at origin

$$G(j\omega) = (j\omega)^m$$

Magnitude and Phase: $\left|G(j\omega)\right|_{dB} = m \cdot 20\log\omega$

$$m = -1, G(s) = \frac{1}{s} \qquad \angle G(j\omega) = m\frac{\pi}{2}$$





General Transfer Function: Real poles/zeros

$$G(j\omega) = (j\omega\tau + 1)^n$$

Magnitude and Phase:

$$\left|G(j\omega)\right|_{dB} = n \cdot 10\log(\omega^2\tau^2 + 1)$$

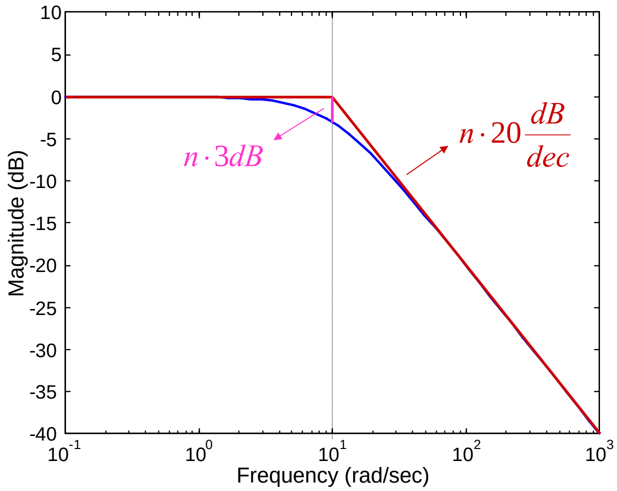
$$\angle G(j\omega) = n\tan^{-1}(\omega\tau)$$

Asymptotic behavior:

$$\left|G(j\omega)\right|_{dB} \xrightarrow[\omega<<1/\tau]{} 0 \qquad \angle G(j\omega) \xrightarrow[\omega<<1/\tau]{} 0^\circ$$

$$\left|G(j\omega)\right|_{dB} \xrightarrow[\omega>>1/\tau]{} n \cdot \tau\big|_{dB} + n \cdot 20\log\omega \qquad \angle G(j\omega) \xrightarrow[\omega>>1/\tau]{} n \cdot 90^\circ$$

General Transfer Function: Real poles/zeros



$$n = -1, \tau = 1/10$$

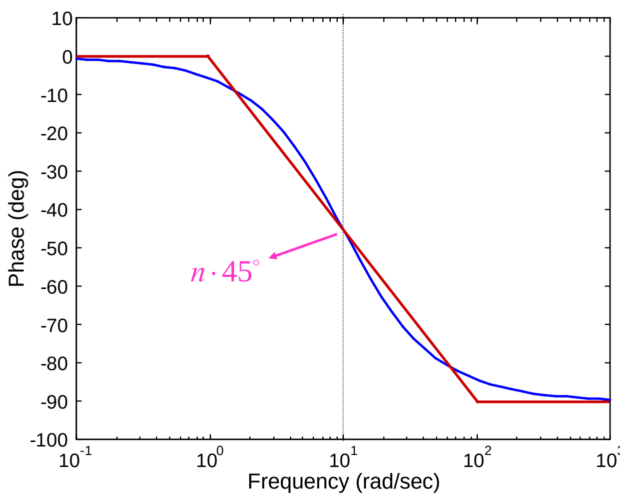
$$G(s) = \frac{1}{\frac{s}{10} + 1}$$

$$\left|G(j0)\right|_{dB} = 0dB$$

$$\left|G(j1/\tau)\right|_{dB} = n \cdot 3dB$$

$$\left|G(\infty)\right|_{dB} = \mathrm{sgn}(n)\infty dB$$

General Transfer Function: Real poles/zeros



$$n = -1, \tau = 1/10$$

$$G(s) = \frac{1}{\frac{s}{10} + 1}$$

$$\angle G(j0) = 0^\circ$$

$$\angle G(j1/\tau) = n \cdot 45^\circ$$

$$\angle G(j\infty) = n \cdot 90^\circ$$

General Transfer Function: Complex poles/zeros

$$G(j\omega)=\left[\left(\frac{j\omega}{\omega_n}\right)^2+2\zeta\frac{j\omega}{\omega_n}+1\right]^q$$

Magnitude and Phase:

$$\left|G(j\omega)\right|_{dB}=q\cdot 10\log\left[\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2\right]$$

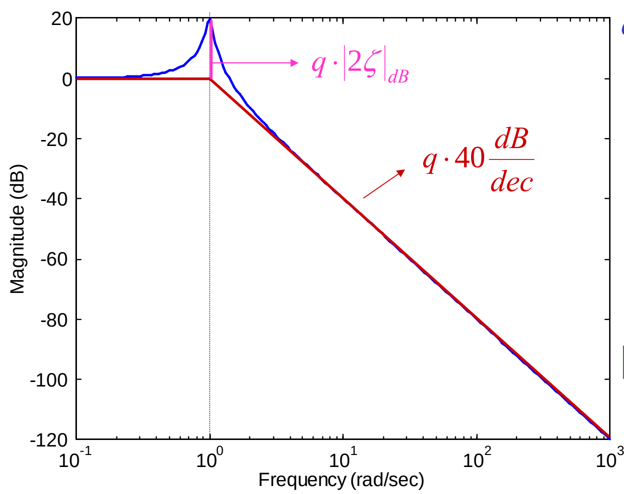
$$\angle G(j\omega)=q\cdot\tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}\right)$$

Asymptotic behavior:

$$\left|G(j\omega)\right|_{dB}\xrightarrow[\omega<<\omega_n]{}0 \qquad \angle G(j\omega)\xrightarrow[\omega<<\omega_n]{}0^\circ$$

$$\left|G(j\omega)\right|_{dB}\xrightarrow[\omega>>\omega_n]{}-2q\cdot\left.\omega_n\right|_{dB}+q\cdot 40\log\omega \qquad \angle G(j\omega)\xrightarrow[\omega>>\omega_n]{}q\cdot 180^\circ$$

General Transfer Function: Complex poles/zeros



$$q=-1,\omega_n=1,\zeta=0.05$$

$$G(s)=\frac{1}{s^2+0.1s+1}$$

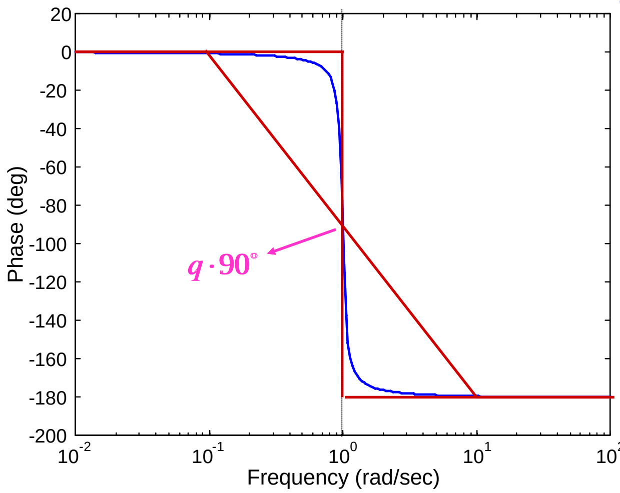
$$\left|G(j0)\right|_{dB}=0dB$$

$$\left|G(j\omega_n)\right|_{dB}=q\cdot\left(3dB+\left.\zeta\right|_{dB}\right)$$

$$\left|G(j\infty)\right|_{dB}=\mathrm{sgn}(q)\infty dB$$

$$\left|G(j\omega)\right|_{dB}^{MAX}=\left|G(j\omega_r)\right|_{dB}=q\cdot\left(2\zeta\sqrt{1-\zeta^2}\right)_{dB}\Leftrightarrow\omega=\omega_r=\frac{\omega_n}{\sqrt{1-\zeta^2}}$$

General Transfer Function: Complex poles/zeros



$$q=-1, \omega_n=1, \zeta=0.05$$

$$G(s)=\frac{1}{s^2+0.1s+1}$$

$$\angle G(j0)=0^\circ$$
$$\angle G(j1/\tau)=q\cdot 90^\circ$$
$$\angle G(j\infty)=q\cdot 180^\circ$$

Frequency Response: Poles/Zeros in the RHP

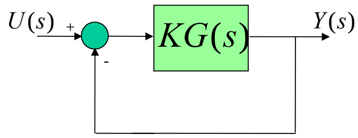
- Same $|G(j\omega)|$.
- The effect on $\angle G(j\omega)$ is opposite than the stable case.

An unstable pole behaves like a stable zero
An “unstable” zero behaves like a “stable” pole

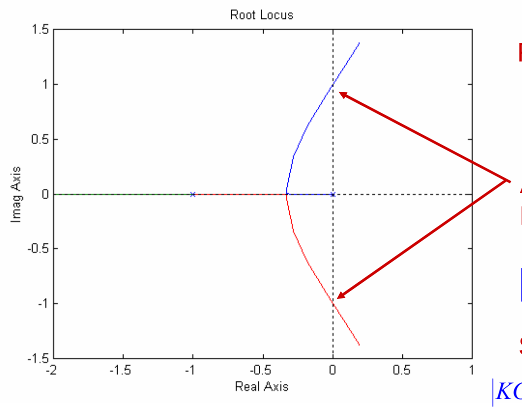
Example: $$G(s)=\frac{1}{s-2}$$

This frequency response cannot be found experimentally but can be computed and used for control design.

Neutral Stability



$$G(s) = \frac{1}{s(s+1)^2}$$



Root locus condition:

$$|KG(s)| = 1, \quad \angle G(s) = -180^\circ$$

At points of neutral stability
RL condition hold for $s = j\omega$

$$|KG(j\omega)| = 1, \quad \angle G(j\omega) = -180^\circ$$

Stability: At $\angle G(j\omega) = -180^\circ$

$|KG(j\omega)| < 1$ If ↑K leads to instability

$|KG(j\omega)| > 1$ If ↓K leads to instability

Stability Margins

The GAIN MARGIN (GM) is the factor by which the gain can be raised before instability results.

$$|GM| < 1 \left(|GM|_{dB} < 0 \right) \Rightarrow$$ UNSTABLE SYSTEM

GM is equal to $1/|KG(j\omega)|$ $\left(-|KG(j\omega)|_{dB}\right)$ at the frequency where $\angle G(j\omega) = -180^\circ$.

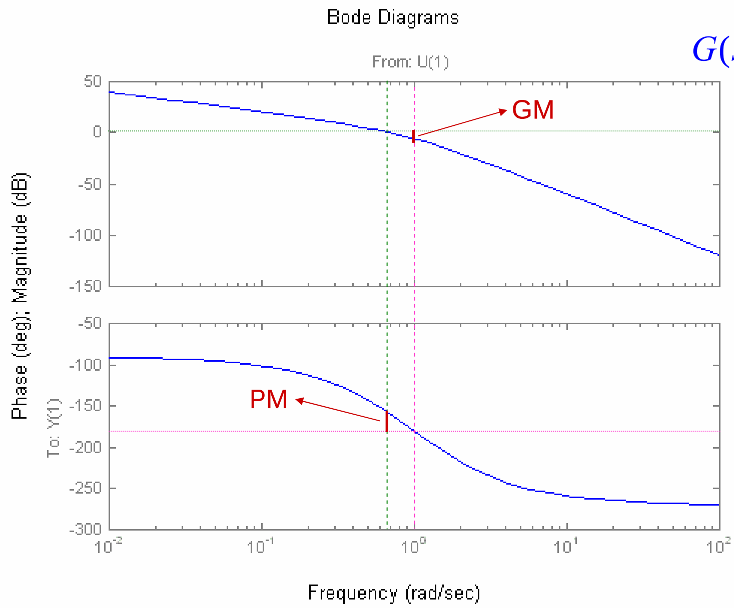
The PHASE MARGIN (PM) is the value by which the phase can be raised before instability results.

$$PM < 0 \Rightarrow$$ UNSTABLE SYSTEM

PM is the amount by which the phase of $G(j\omega)$ exceeds -180° when $|KG(j\omega)| = 1$ $\left(|KG(j\omega)|_{dB} = 0\right)$

Stability Margins

$$G(s) = \frac{1}{s(s+1)^2}$$



Frequency Response

$$u(t) = A\cos(\omega t + \phi) \longrightarrow \boxed{G(s)} \longrightarrow y_{ss} = |G(j\omega)| A\cos(\omega t + \phi + \angle G(j\omega))$$

Stable Transfer Function

$$G(j\omega) = |G(j\omega)| e^{j\angle G(j\omega)}$$ BODE plots

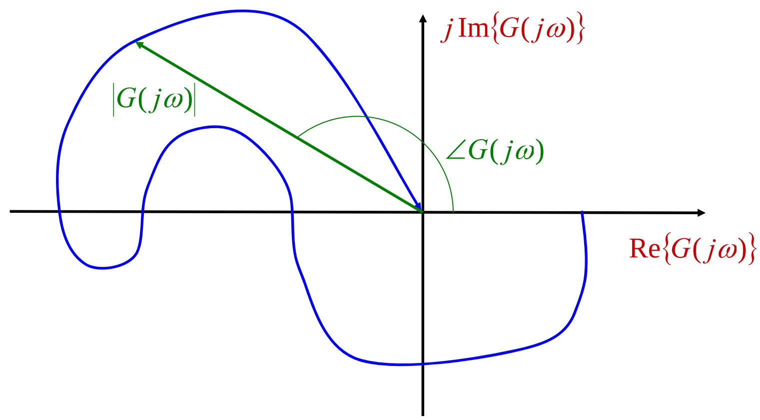
$$G(j\omega) = \mathrm{Re}\{G(j\omega)\} + j\,\mathrm{Im}\{G(j\omega)\}$$ NYQUIST plots

Nyquist Diagrams

$$G(j\omega) = \mathrm{Re}\{G(j\omega)\} + j\,\mathrm{Im}\{G(j\omega)\} = |G(j\omega)|e^{j\angle G(j\omega)}$$

How are the Bode and Nyquist plots related?

They are two way to represent the same information

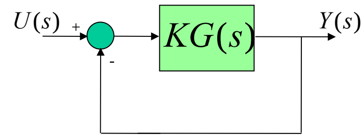


Nyquist Diagrams

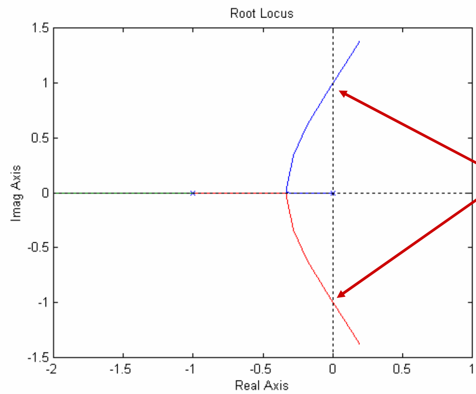
General procedure for sketching Nyquist Diagrams:

- Find $G(j0)$
- Find $G(j\infty)$
- Find ω^* such that $\mathrm{Re}\{G(j\omega^*)\}=0$; $\mathrm{Im}\{G(j\omega^*)\}$ is the intersection with the imaginary axis.
- Find ω^* such that $\mathrm{Im}\{G(j\omega^*)\}=0$; $\mathrm{Re}\{G(j\omega^*)\}$ is the intersection with the real axis.
- Connect the points

Neutral Stability



$$G(s) = \frac{1}{s(s+1)^2}$$



Root locus condition:

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At points of neutral stability RL condition hold for $s = j\omega$

$|KG(j\omega)| = 1, \quad \angle G(j\omega) = -180^\circ$

Stability: At $\angle G(j\omega) = -180^\circ$

$|KG(j\omega)| < 1$ If ↑K leads to instability

$|KG(j\omega)| > 1$ If ↓K leads to instability

Stability Margins

The GAIN MARGIN (GM) is the factor by which the gain can be raised before instability results.

$$|GM| < 1 \left(|GM|_{dB} < 0 \right) \Rightarrow \text{UNSTABLE SYSTEM}$$

GM is equal to $1/|KG(j\omega)|$ $\left(-|KG(j\omega)|_{dB} \right)$ at the frequency where $\angle G(j\omega) = -180^\circ$.

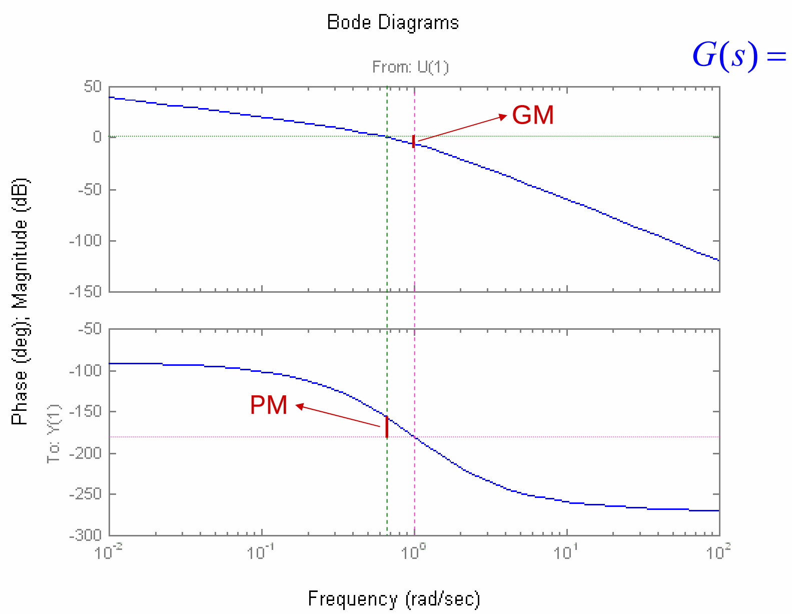
The PHASE MARGIN (PM) is the value by which the phase can be raised before instability results.

$$PM < 0 \Rightarrow \text{UNSTABLE SYSTEM}$$

PM is the amount by which the phase of $G(j\omega)$ exceeds -180° when $|KG(j\omega)| = 1$ $\left(|KG(j\omega)|_{dB} = 0 \right)$

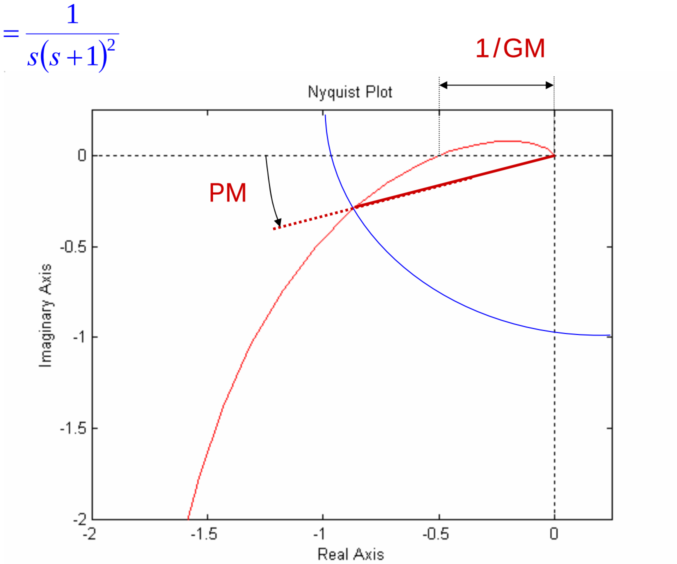
Stability Margins

$$G(s) = \frac{1}{s(s+1)^2}$$



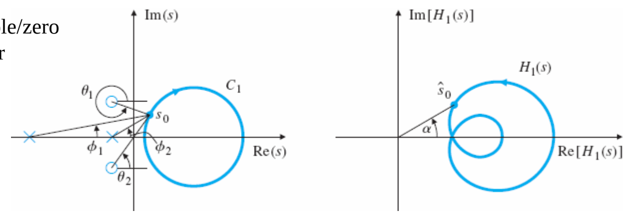
Stability Margins

$$G(s) = \frac{1}{s(s+1)^2}$$

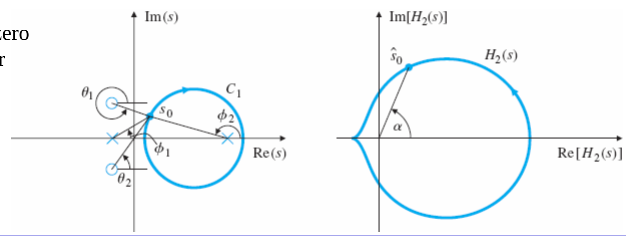


Nyquist Stability Criterion

Case 1: No pole/zero within contour



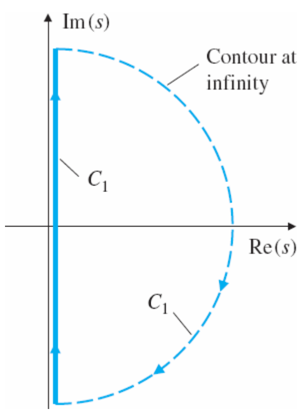
Case 2: Pole/zero within contour

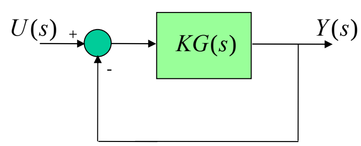


Argument Principle: A contour map of a complex function will encircle the origin Z-P times, where Z is the number of zeros and P is the number of poles of the function inside the contour.

Nyquist Stability Criterion

Let us consider this contour and closed-loop system





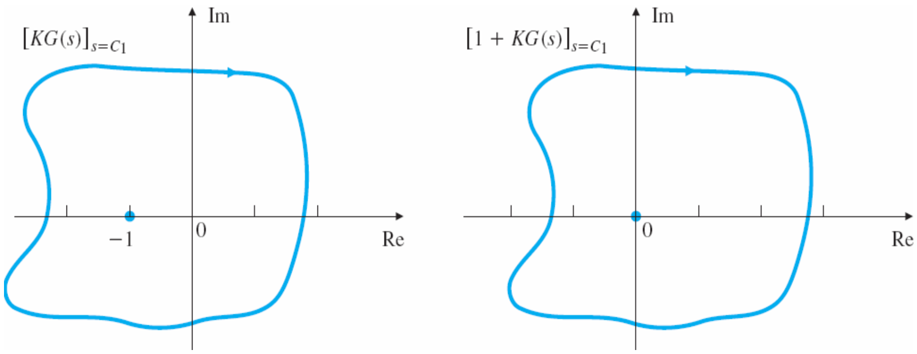
The closed-loop poles are the solutions (roots) of:

$$1 + KG(s) = 0$$

The evaluation of $H(s)$ will encircle the origin only if $H(s)$ has a RHP zero or pole

Nyquist Stability Criterion

Let us apply the argument principle to the function $H(s) = 1+KG(s)$.



If the plot of $1+KG(s)$ encircles the origin, the plot of $KG(s)$ encircles -1 on the real axis.

Nyquist Stability Criterion

By writing

$$1+KG(s) = 1+K\frac{b(s)}{a(s)} = \frac{a(s)+Kb(s)}{a(s)}$$

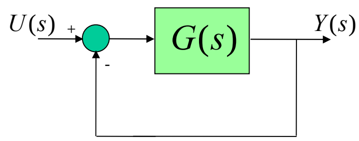
we can conclude that the poles of $1+KG(s)$ are also the poles of $G(s)$. Assuming no pole of $G(s)$ in the RHP, an encirclement of the point -1 by $KG(s)$ indicates a zero of $1+KG(s)$ in the RHP, and thus an unstable pole of the closed-loop system.

A clockwise contour of C1 enclosing a zero of $1+KG(s)$ will result in $KG(s)$ encircling the -1 point in the clockwise direction.
A clockwise contour of C1 enclosing a pole of $1+KG(s)$ will result in $KG(s)$ encircling the -1 point in the counterclockwise direction.

The net number of clockwise encirclements of the point -1, N, equals the number of zeros (closed-loop poles) in the RHP, Z, minus the number of poles (open-loop poles) in the RHP, P:

$$N = Z - P$$

Nyquist Stability Criterion



When is this transfer function Stable?

NYQUIST: The closed loop is asymptotically stable if the number of counterclockwise encirclements (N negative) of the point (-1+j0) by the Nyquist curve of $G(j\omega)$ is equal to the number of poles of $G(s)$ with positive real parts (unstable poles) (P).

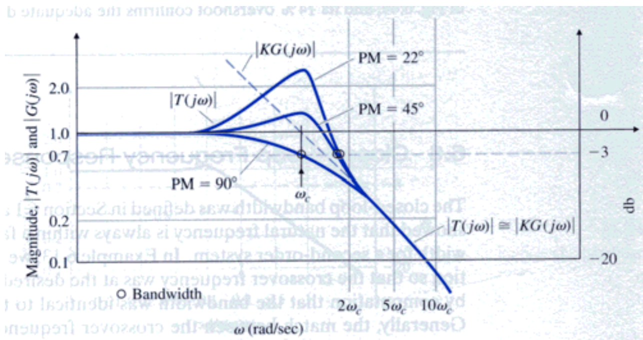
Corollary: If the open-loop system $G(s)$ is stable ($P=0$), then the closed-loop system is also stable provided $G(s)$ makes no encirclement of the point (-1+j0) ($N=0$) .

Specifications in the Frequency Domain

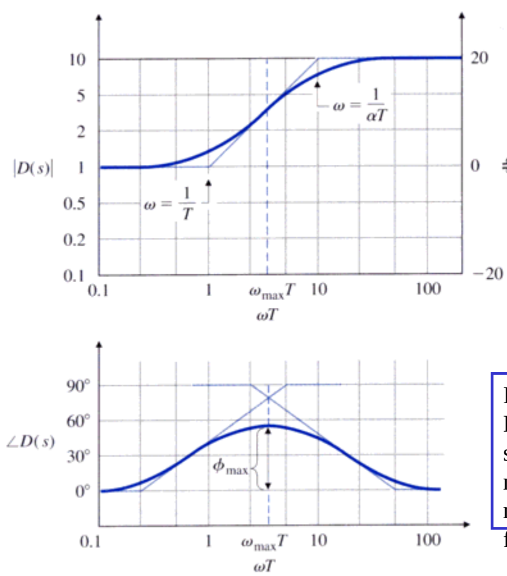
1. The crossover frequency ω_c, which determines bandwith ω_{BW}, rise time t_r and settling time t_s.
2. The phase margin PM, which determines the damping coefficient ζ and the overshoot M_p.
3. The low-frequency gain, which determines the steady-state error characteristics.

Specifications in the Frequency Domain

The crossover frequency: $\omega_c \le \omega_{BW} \le 2\omega_c$



Frequency Response – Phase Lead Compensators



$$D(s) = \frac{Ts+1}{\alpha Ts+1}, \qquad \alpha < 1$$

$$\alpha = \frac{1-\sin\phi_{MAX}}{1+\sin\phi_{MAX}}$$

$$\log\omega_{MAX} = \frac{1}{2}\left[\log\left(\frac{1}{T}\right) + \log\left(\frac{1}{\alpha T}\right)\right]$$

It is a high-pass filter and approximates PD control. It is used whenever substantial improvement in damping is needed. It tends to increase the speed of response of a system for a fixed low-frequency gain.

Frequency Response – Phase Lead Compensators

1. Determine the open-loop gain K to satisfy error or bandwidth requirements:
   - To meet error requirement, pick K to satisfy error constants (K_p, K_v, K_a) so that e_{ss} specification is met.
   - To meet bandwidth requirement, pick K so that the open-loop crossover frequency is a factor of two below the desired closed-loop bandwidth.

2. Determine the needed phase lead → α based on the PM specification. $\alpha = \dfrac{1-\sin\phi_{MAX}}{1+\sin\phi_{MAX}}$

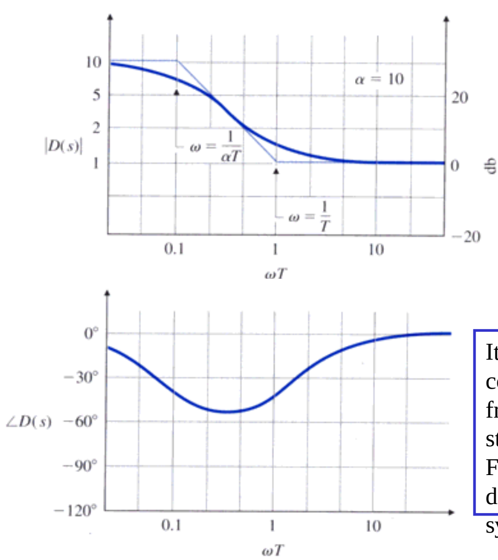
3. Pick ω_{MAX} to be at the crossover frequency.

4. Determine the zero and pole of the compensator.
   $z=1/T=\omega_{MAX}\,\alpha^{1/2}$ $\quad$ $p=1/\alpha T=\omega_{MAX}\,\alpha^{1/2}$

5. Draw the compensated frequency response and check PM.

6. Iterate on the design. Add additional compensator if needed.

Frequency Response – Phase Lag Compensators



$$D(s) = \alpha\frac{Ts+1}{\alpha Ts+1}, \qquad \alpha > 1$$

It is a low-pass filter and approximates PI control. It is used to increase the low frequency gain of the system and improve steady state response for fixed bandwidth. For a fixed low-frequency gain, it will decrease the speed of response of the system.

Frequency Response – Phase Lag Compensators

1. Determine the open-loop gain K that will meet the PM requirement without compensation.

2. Draw the Bode plot of the uncompensated system with crossover frequency from step 1 and evaluate the low-frequency gain.

3. Determine α to meet the low frequency gain error requirement.

4. Choose the corner frequency $\omega=1/T$ (the zero of the compensator) to be one decade below the new crossover frequency ω_c.

5. The other corner frequency (the pole of the compensator) is then $\omega=1/\alpha T$.

6. Iterate on the design