ME 343 – Control Systems

Lecture 25

October 21, 2009

Frequency Response

$$u(t) = A\cos(\omega t + \phi) \longrightarrow \boxed{G(s)} \longrightarrow y_{ss} = |G(j\omega)| A\cos(\omega t + \phi + \angle G(j\omega))$$

$G(s)$ → Stable Transfer Function

$$G(j\omega) = |G(j\omega)| e^{j\angle G(j\omega)}$$ BODE plots

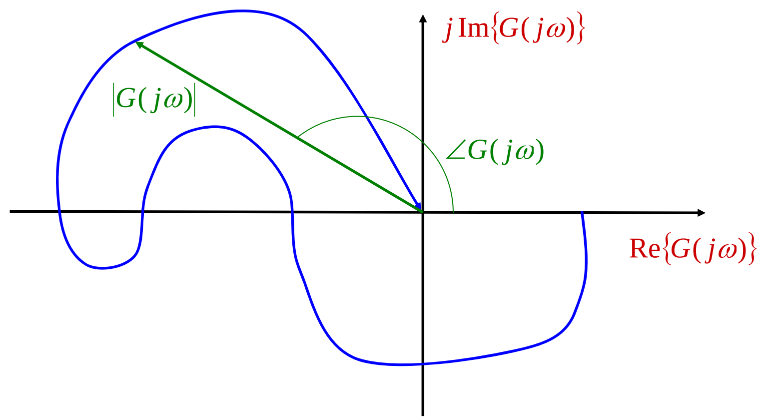
$$G(j\omega) = \mathrm{Re}\{G(j\omega)\} + j\,\mathrm{Im}\{G(j\omega)\}$$ NYQUIST plots

Nyquist Diagrams

$$G(j\omega) = \mathrm{Re}\{G(j\omega)\} + j\,\mathrm{Im}\{G(j\omega)\} = |G(j\omega)|e^{j\angle G(j\omega)}$$

How are the Bode and Nyquist plots related?

They are two way to represent the same information



Nyquist Diagrams

General procedure for sketching Nyquist Diagrams:

- Find $G(j0)$
- Find $G(j\infty)$
- Find ω^* such that $\mathrm{Re}\{G(j\omega^*)\}=0$; $\mathrm{Im}\{G(j\omega^*)\}$ is the intersection with the imaginary axis.
- Find ω^* such that $\mathrm{Im}\{G(j\omega^*)\}=0$; $\mathrm{Re}\{G(j\omega^*)\}$ is the intersection with the real axis.
- Connect the points

Nyquist Diagrams

Example: $$G(s)=\frac{1}{s(s+1)^2}$$

$$G(j\omega)=\frac{1}{j\omega(j\omega+1)^2}=\frac{1}{j\omega(j\omega+1)^2}\frac{(-j\omega)(1-j\omega)^2}{(-j\omega)(1-j\omega)^2}=\frac{-2\omega+j(\omega^2-1)}{\omega(\omega^2+1)^2}$$

1- $\omega\to 0: G(j\omega)=-2-j\infty$

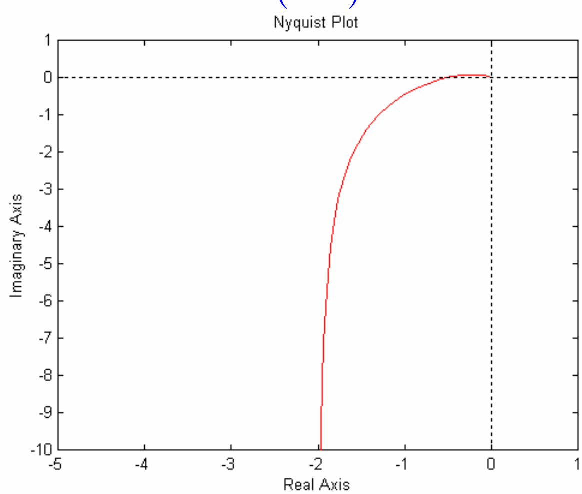
2- $\omega\to\infty: G(j\omega)\xrightarrow[\omega\to\infty]{} j\frac{1}{\omega^3}\xrightarrow[\omega\to\infty]{} 0$

3- $\mathrm{Re}\{G(j\omega)\}=0 \Leftrightarrow \omega=\infty$

4- $\mathrm{Im}\{G(j\omega)\}=0 \Leftrightarrow \omega=1, \omega=\infty \qquad \mathrm{Re}\{G(j1)\}=-\frac{1}{2}$

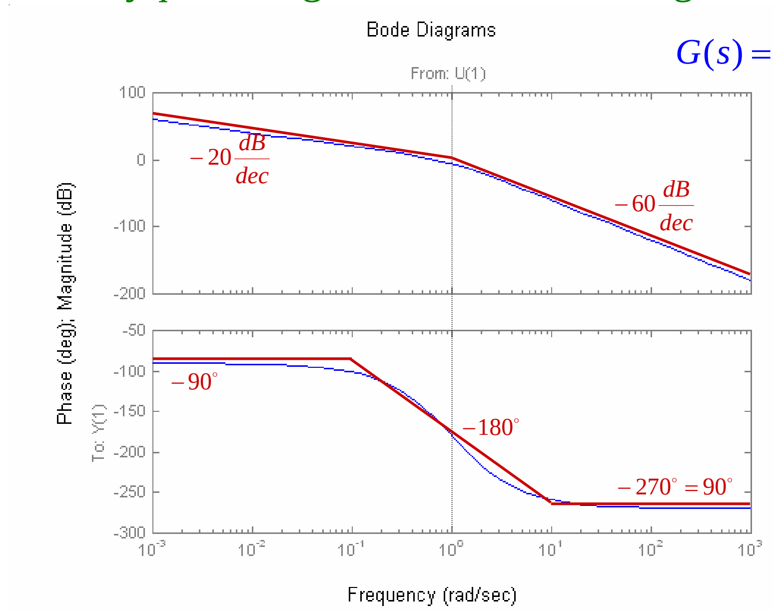
Nyquist Diagrams

Example: $$G(s)=\frac{1}{s(s+1)^2}$$

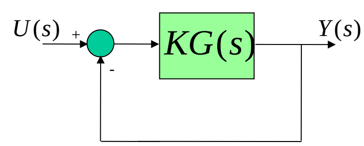


Nyquist Diagrams from Bode Diagrams

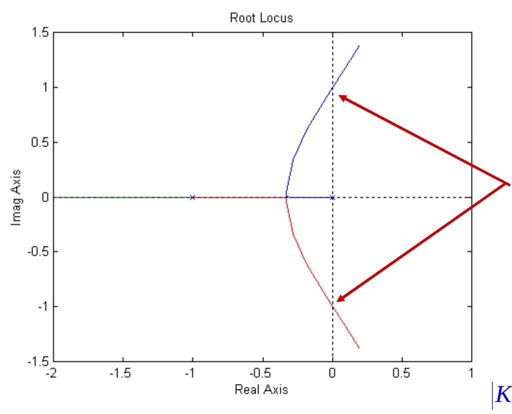
$$G(s) = \frac{1}{s(s+1)^2}$$



Neutral Stability



$$G(s) = \frac{1}{s(s+1)^2}$$



Root locus condition:

$$|KG(s)| = 1, \quad \angle G(s) = -180^{\circ}$$

At points of neutral stability RL condition hold for $s = j\omega$

$$|KG(j\omega)| = 1, \quad \angle G(j\omega) = -180^{\circ}$$

Stability: At $\angle G(j\omega) = -180^{\circ}$

$|KG(j\omega)| < 1$ If ↑K leads to instability

$|KG(j\omega)| > 1$ If ↓K leads to instability

Stability Margins

The GAIN MARGIN (GM) is the factor by which the gain can be raised before instability results.

$$|GM| < 1 \left(|GM|_{dB} < 0 \right) \Rightarrow$$ UNSTABLE SYSTEM

GM is equal to $1/|KG(j\omega)|$ $\left(-|KG(j\omega)|_{dB}\right)$ at the frequency where $\angle G(j\omega) = -180^\circ$.

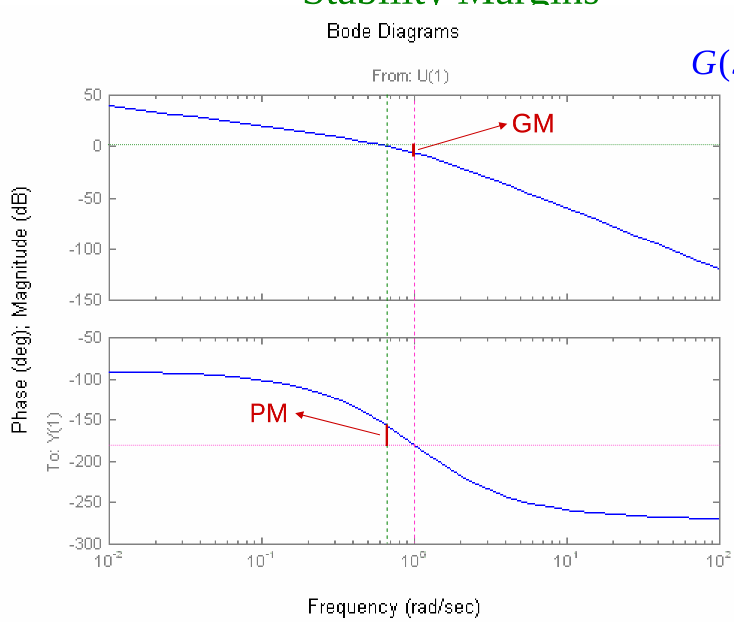
The PHASE MARGIN (PM) is the value by which the phase can be raised before instability results.

$$PM < 0 \Rightarrow$$ UNSTABLE SYSTEM

PM is the amount by which the phase of $G(j\omega)$ exceeds -180° when $|KG(j\omega)| = 1$ $\left(|KG(j\omega)|_{dB} = 0\right)$

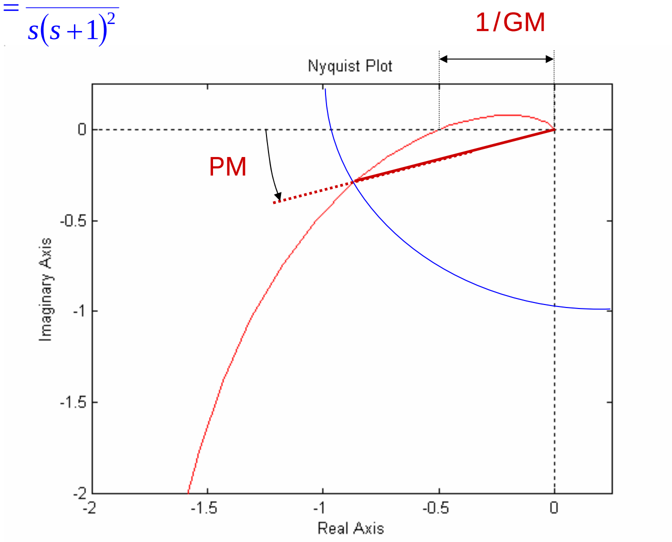
Stability Margins

$$G(s) = \frac{1}{s(s+1)^2}$$



Stability Margins

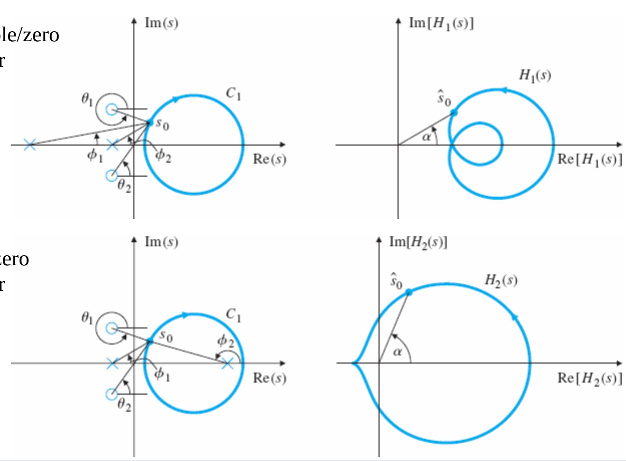
$$G(s) = \frac{1}{s(s+1)^2}$$



Nyquist Stability Criterion

Case 1: No pole/zero within contour

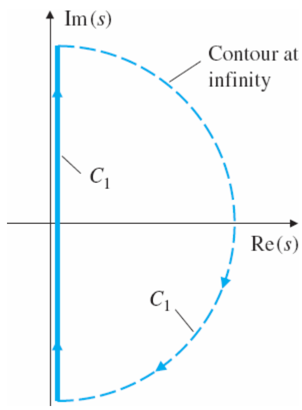
Case 2: Pole/zero within contour

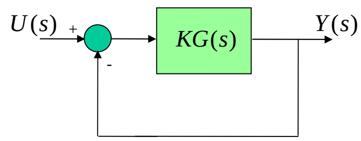


Argument Principle: A contour map of a complex function will encircle the origin Z-P times, where Z is the number of zeros and P is the number of poles of the function inside the contour.

Nyquist Stability Criterion

Let us consider this contour and closed-loop system





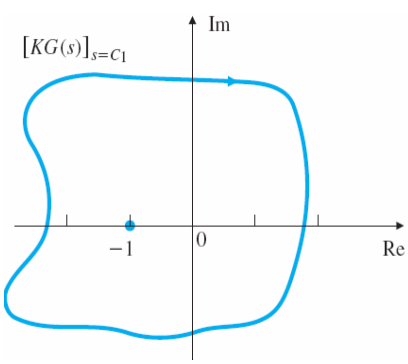
The closed-loop poles are the solutions (roots) of:

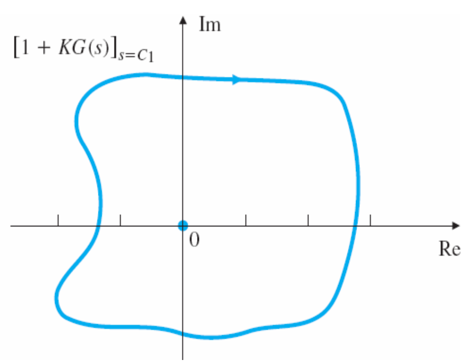
$$1 + KG(s) = 0$$

The evaluation of $H(s)$ will encircle the origin only if $H(s)$ has a RHP zero or pole

Nyquist Stability Criterion

Let us apply the argument principle to the function $H(s) = 1 + KG(s)$.





If the plot of $1+KG(s)$ encircles the origin, the plot of $KG(s)$ encircles -1 on the real axis.

Nyquist Stability Criterion

By writing

$$1+KG(s)=1+K\frac{b(s)}{a(s)}=\frac{a(s)+Kb(s)}{a(s)}$$

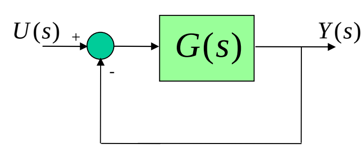
we can conclude that the poles of $1+KG(s)$ are also the poles of $G(s)$. Assuming no pole of $G(s)$ in the RHP, an encirclement of the point -1 by $KG(s)$ indicates a zero of $1+KG(s)$ in the RHP, and thus an unstable pole of the closed-loop system.

A clockwise contour of C1 enclosing a zero of $1+KG(s)$ will result in $KG(s)$ encircling the -1 point in the clockwise direction.
A clockwise contour of C1 enclosing a pole of $1+KG(s)$ will result in $KG(s)$ encircling the -1 point in the counterclockwise direction.

The net number of clockwise encirclements of the point -1, N, equals the number of zeros (closed-loop poles) in the RHP, Z, minus the number of poles (open-loop poles) in the RHP, P:

$$N=Z-P$$

Nyquist Stability Criterion



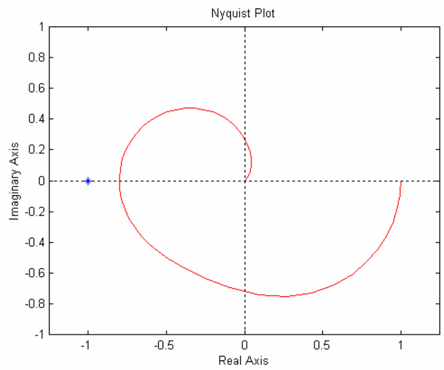
When is this transfer function Stable?

NYQUIST: The closed loop is asymptotically stable if the number of counterclockwise encirclements (N negative) of the point (-1+j0) by the Nyquist curve of $G(j\omega)$ is equal to the number of poles of $G(s)$ with positive real parts (unstable poles) (P).

Corollary: If the open-loop system $G(s)$ is stable ($P=0$), then the closed-loop system is also stable provided $G(s)$ makes no encirclement of the point (-1+j0) ($N=0$) .

Nyquist Stability Criterion

$$G(s) = \frac{1}{s^4 + 2s^3 + 3s^2 + 3s + 1}$$



$$G(s) = \frac{1}{s^4 + 5s^3 + 3s^2 + 3s + 1}$$