ME 343 – Control Systems

Lecture 22

October 14, 2009

Frequency Response

$$u(t) = A\cos(\omega t + \phi) \longrightarrow \boxed{G(s)} \longrightarrow y_{ss} = |G(j\omega)| A\cos(\omega t + \phi + \angle G(j\omega))$$

Stable Transfer Function

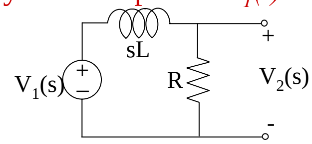
- After a transient, the output settles to a sinusoid with an amplitude magnified by $|G(j\omega)|$ and phase shifted by $\angle G(j\omega)$.

- Since all signals can be represented by sinusoids (Fourier series and transform), the quantities $|G(j\omega)|$ and $\angle G(j\omega)$ are extremely important.

- Bode developed methods for quickly finding $|G(j\omega)|$ and $\angle G(j\omega)$ for a given $G(s)$ and for using them in control design.

Frequency Response

- Find the steady state output for $v_1(t)=A\cos(\omega t+\phi)$



- Compute the s-domain transfer function T(s)
  - Voltage divider $T(s)=\dfrac{R}{sL+R}$
- Compute the frequency response

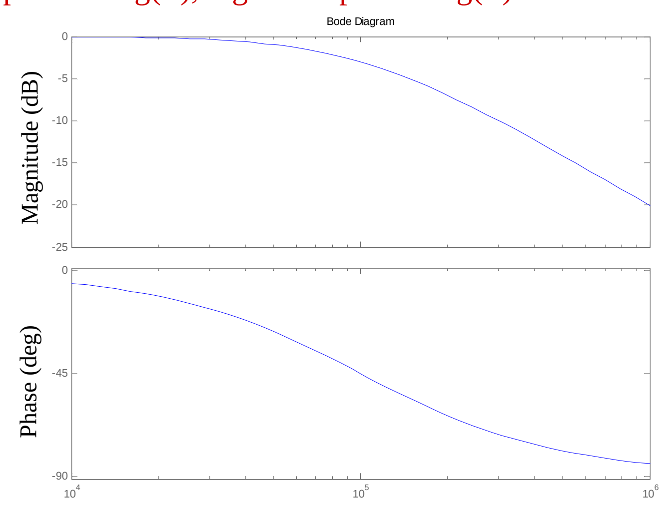
$$|T(j\omega)|=\frac{R}{\sqrt{R^2+(\omega L)^2}}, \qquad \angle T(j\omega)=-\tan^{-1}\left(\frac{\omega L}{R}\right)$$

- Compute the steady state output

$$v_{2SS}(t)=\frac{AR}{\sqrt{R^2+(\omega L)^2}}\cos\left[\omega t+\phi-\tan^{-1}(\omega L/R)\right]$$

Bode Diagrams

- Log-log plot of mag(T), log-linear plot of arg(T) versus ω



Bode Diagrams

$$G(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$$\Updownarrow$$

$$G(s) = |K|\frac{r_1^z r_2^z \cdots r_m^z}{r_1^p r_2^p \cdots r_n^p} e^{i\left[\phi^K + \left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right) - \left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right)\right]}$$

The magnitude and phase of G(s) when s=jω is given by:

Nonlinear in the magnitudes

$$|G(j\omega)| = |K|\frac{r_1^z r_2^z \cdots r_m^z}{r_1^p r_2^p \cdots r_n^p},$$

$$\angle G(j\omega) = \phi^K + \left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right) - \left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right)$$

Linear in the phases

Bode Diagrams

Why do we express $|G(j\omega)|$ in decibels?

$$|G(j\omega)|_{dB} = 20\log|G(j\omega)|$$

$$|G(j\omega)| = |K|\frac{r_1^z r_2^z \cdots r_m^z}{r_1^p r_2^p \cdots r_n^p} \Rightarrow |G(j\omega)|_{dB} = ?$$

By properties of the logarithm we can write:

$$20\log|G(s)| = 20\log|K| + \left(20\log r_1^z + 20\log r_m^z + \cdots + 20\log r_m^z\right) - \left(20\log r_1^p + 20\log r_2^p + \cdots + 20\log r_n^p\right)$$

The magnitude and phase of G(s) when s=jω is given by:

Linear in the magnitudes (dB)

$$|G(s)|_{dB} = |K|_{dB} + \left(r_1^z\big|_{dB} + r_2^z\big|_{dB} + \cdots + r_m^z\big|_{dB}\right) - \left(r_1^p\big|_{dB} + r_2^p\big|_{dB} + \cdots + r_n^p\big|_{dB}\right)$$

$$\angle G(s) = \phi^K + \left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right) - \left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right)$$

Linear in the phases

Bode Diagrams

Why do we use a logarithmic scale? Let's have a look at our example:

$$T(s)=\frac{R}{sL+R}\Rightarrow T(j\omega)=\frac{R}{j\omega L+R}\Rightarrow |T(j\omega)|=\frac{R}{\sqrt{R^2+(\omega L)^2}}=\frac{1}{\sqrt{1+\left(\frac{\omega L}{R}\right)^2}}$$

Expressing the magnitude in dB:

$$|T(j\omega)|_{dB}=20\log 1-20\log\sqrt{1+\left(\frac{\omega L}{R}\right)^2}=-10\log\left[1+\left(\frac{\omega L}{R}\right)^2\right]$$

Asymptotic behavior:

$$\omega\to 0: |T(j\omega)|_{dB}\to 0$$

$$\omega\to\infty: |T(j\omega)|_{dB}\to -20\log\left(\frac{\omega}{R/L}\right)=20\log(R/L)-20\log\omega=\left.\frac{R}{L}\right|_{dB}-20\log\omega$$

LINEAR FUNCTION in logω!!! We plot $|G(j\omega)|_{dB}$ as a function of logω.

Bode Diagrams

Why do we use a logarithmic scale? Let's have a look at our example:

$$T(s)=\frac{R}{sL+R}\Rightarrow T(j\omega)=\frac{R}{j\omega L+R}=\frac{1}{j\omega\frac{L}{R}+1}$$

Expressing the phase:

$$\angle T(j\omega)=\angle\log 1-\angle\left(1+j\frac{\omega L}{R}\right)=-\tan^{-1}\left(\frac{\omega L}{R}\right)$$

Asymptotic behavior:

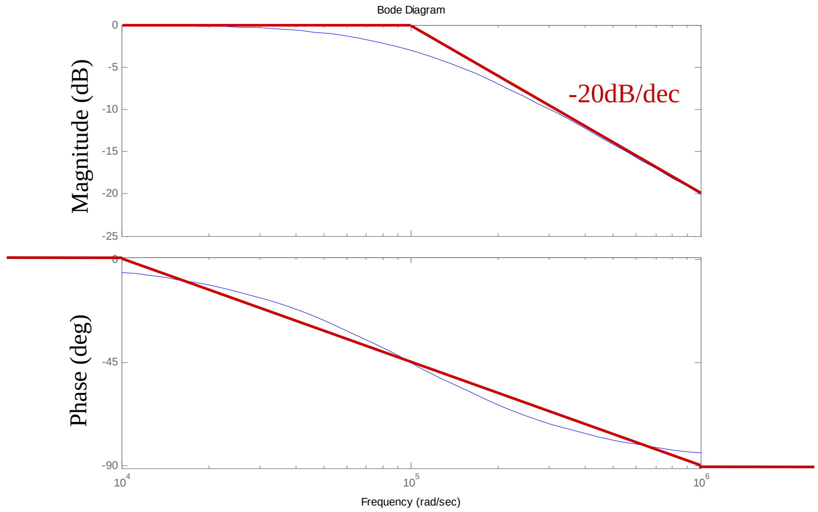
$$\angle G(j\omega)\xrightarrow[\omega\to 0]{}0^\circ \qquad \angle G(j\omega)\Big|_{\omega=\frac{R}{L}}=-45^\circ$$

$$\angle G(j\omega)\xrightarrow[\omega\to\infty]{}-90^\circ$$

LINEAR FUNCTION in logω!!! We plot $\angle G(j\omega)$ as a function of logω.

Bode Diagrams

- Log-log plot of mag(T), log-linear plot of arg(T) versus ω



Bode Diagrams

Decade: Any frequency range whose end points have a 10:1 ratio

A cutoff frequency occurs when the gain is reduced from its maximum passband value by a factor $1/\sqrt{2}$:

$$20\log\left(\frac{1}{\sqrt{2}}|T|_{MAX}\right) = 20\log|T|_{MAX} - 20\log\sqrt{2} \approx 20\log|T|_{MAX} - 3\text{dB}$$

Bandwith: frequency range spanned by the gain passband

Let’s have a look at our example:

$$|T(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega L}{R}\right)^2}} \Rightarrow \begin{cases} \omega = 0 & |T(j\omega)| = 1 \\ \omega = R/L & |T(j\omega)| = 1/\sqrt{2} \end{cases}$$

This is a low-pass filter!!! Passband gain=1, Cutoff frequency=R/L
The Bandwith is R/L!

General Transfer Function (Bode Diagrams)

$$G(j\omega)=K_o(j\omega)^m(j\omega\tau+1)^n\left[\left(\frac{j\omega}{\omega_n}\right)^2+2\zeta\frac{j\omega}{\omega_n}+1\right]^q$$

The magnitude (dB) (phase) is the sum of the magnitudes (dB) (phases) of each one of the terms. We learn how to plot each term, we learn how to plot the whole magnitude and phase Bode Plot.

Classes of terms:

1- $G(j\omega)=K_o$

2- $G(j\omega)=(j\omega)^m$

3- $G(j\omega)=(j\omega\tau+1)^n$

4- $G(j\omega)=\left[\left(\frac{j\omega}{\omega_n}\right)^2+2\zeta\frac{j\omega}{\omega_n}+1\right]^q$

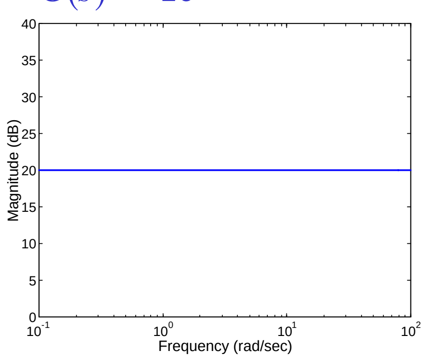
General Transfer Function: DC gain

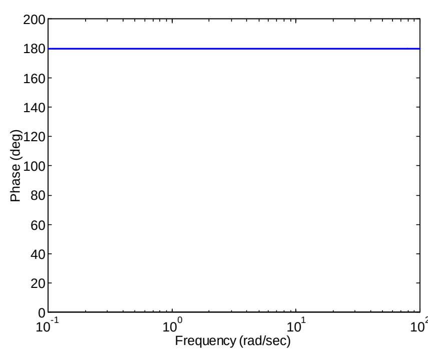
$$G(j\omega)=K_o$$

Magnitude and Phase: $|G(j\omega)|_{dB}=20\log|K_o|$

$$\angle G(j\omega)=\begin{cases}0 & \text{if } K_o>0\\ \pm\pi & \text{if } K_o<0\end{cases}$$

$G(s)=-10$



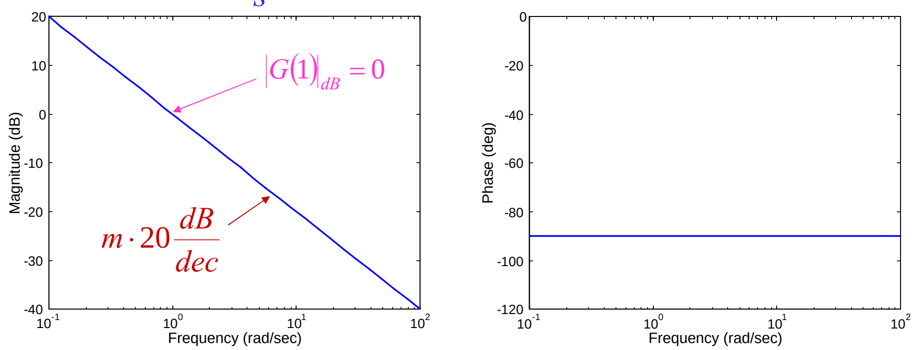


General Transfer Function: Poles/zeros at origin

$$G(j\omega) = (j\omega)^m$$

Magnitude and Phase: $\left|G(j\omega)\right|_{dB} = m \cdot 20 \log \omega$

$m = -1, G(s) = \dfrac{1}{s}$ $\qquad \angle G(j\omega) = m\dfrac{\pi}{2}$



General Transfer Function: Real poles/zeros

$$G(j\omega) = (j\omega\tau + 1)^n$$

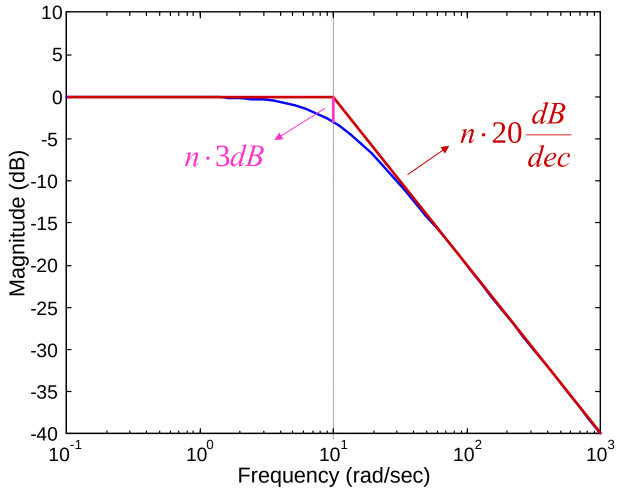
Magnitude and Phase:

$$\left|G(j\omega)\right|_{dB} = n \cdot 10 \log\left(\omega^2\tau^2 + 1\right)$$
$$\angle G(j\omega) = n \tan^{-1}(\omega\tau)$$

Asymptotic behavior:

$$\left|G(j\omega)\right|_{dB} \xrightarrow[\omega << 1/\tau]{} 0 \qquad \angle G(j\omega) \xrightarrow[\omega << 1/\tau]{} 0^\circ$$
$$\left|G(j\omega)\right|_{dB} \xrightarrow[\omega >> 1/\tau]{} n \cdot \left.\tau\right|_{dB} + n \cdot 20 \log \omega \qquad \angle G(j\omega) \xrightarrow[\omega >> 1/\tau]{} n \cdot 90^\circ$$

General Transfer Function: Real poles/zeros



$n = -1, \tau = 1/10$

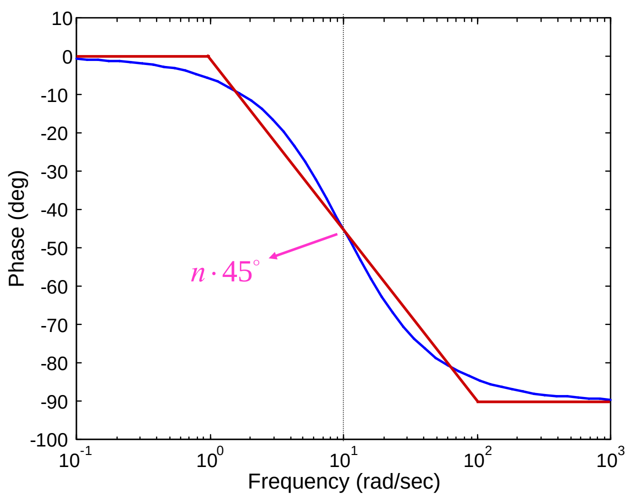
$$G(s) = \frac{1}{\frac{s}{10}+1}$$

$$\left|G(j0)\right|_{dB} = 0dB$$

$$\left|G(j1/\tau)\right|_{dB} = n \cdot 3dB$$

$$\left|G(\infty)\right|_{dB} = \mathrm{sgn}(n)\infty dB$$

General Transfer Function: Real poles/zeros



$n = -1, \tau = 1/10$

$$G(s) = \frac{1}{\frac{s}{10}+1}$$

$$\angle G(j0) = 0^\circ$$

$$\angle G(j1/\tau) = n \cdot 45^\circ$$

$$\angle G(j\infty) = n \cdot 90^\circ$$

General Transfer Function: Complex poles/zeros

$$G(j\omega)=\left[\left(\frac{j\omega}{\omega_n}\right)^2+2\zeta\frac{j\omega}{\omega_n}+1\right]^q$$

Magnitude and Phase:

$$|G(j\omega)|_{dB}=q\cdot 10\log\left[\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2\right]$$

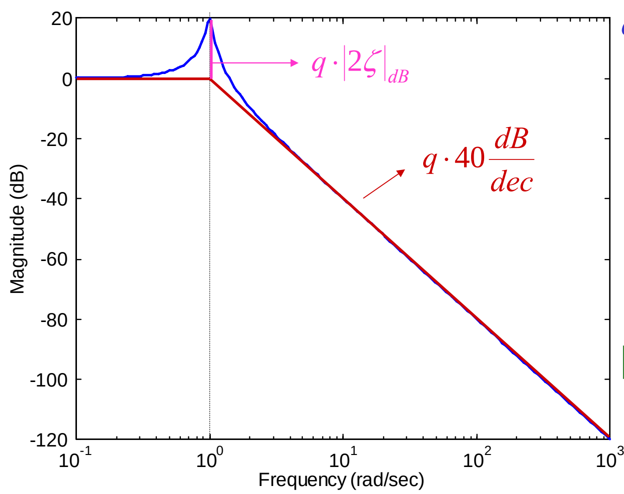
$$\angle G(j\omega)=q\cdot\tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}\right)$$

Asymptotic behavior:

$$|G(j\omega)|_{dB}\xrightarrow[\omega<<\omega_n]{}0 \qquad \angle G(j\omega)\xrightarrow[\omega<<\omega_n]{}0^\circ$$

$$|G(j\omega)|_{dB}\xrightarrow[\omega>>\omega_n]{}-2q\cdot\omega_n|_{dB}+q\cdot 40\log\omega \qquad \angle G(j\omega)\xrightarrow[\omega>>\omega_n]{}q\cdot 180^\circ$$

General Transfer Function: Complex poles/zeros



$q=-1,\omega_n=1,\zeta=0.05$

$$G(s)=\frac{1}{s^2+0.1s+1}$$

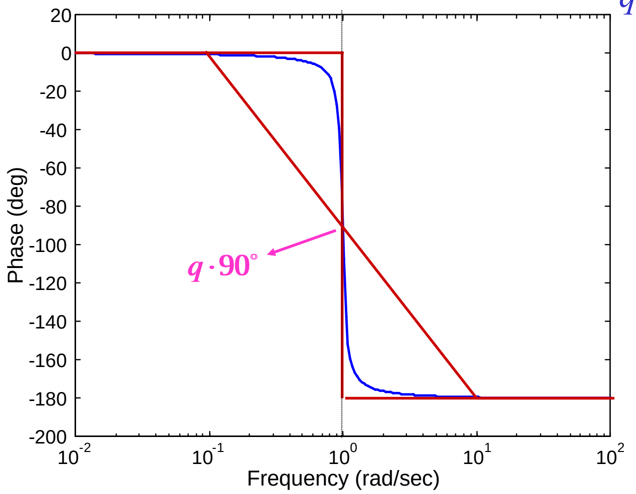
$$|G(j0)|_{dB}=0dB$$

$$|G(j\omega_n)|_{dB}=q\cdot\left(3dB+\zeta|_{dB}\right)$$

$$|G(j\infty)|_{dB}=\mathrm{sgn}(q)\infty dB$$

$$|G(j\omega)|_{dB}^{MAX}=|G(j\omega_r)|_{dB}=q\cdot\left(2\zeta\sqrt{1-\zeta^2}\right)_{dB}\Leftrightarrow\omega=\omega_r=\frac{\omega_n}{\sqrt{1-\zeta^2}}$$

General Transfer Function: Complex poles/zeros



$q = -1, \omega_n = 1, \zeta = 0.05$

$$G(s) = \frac{1}{s^2 + 0.1s + 1}$$

$$\angle G(j0) = 0^{\circ}$$

$$\angle G(j1/\tau) = q \cdot 90^{\circ}$$

$$\angle G(j\infty) = q \cdot 180^{\circ}$$