ME 343 – Control Systems

Lecture 21

October 12, 2009

Frequency Response

- We now know how to analyze and design systems via s-domain methods which yield dynamical information
  - The responses are described by the exponential modes
    - The modes are determined by the poles of the response Laplace Transform
- We next will look at describing system performance via frequency response methods
  - This guides us in specifying the system pole and zero positions

Sinusoidal Steady-State Response

Consider a stable transfer function with a sinusoidal input:

$$u(t) = A\cos(\omega t) \Leftrightarrow U(s) = \frac{A\omega}{s^2 + \omega^2}$$

The Laplace Transform of the response has poles

- Where the natural system modes lie
  - These are in the open left half plane Re(s)<0
- At the input modes s=+jω and s=-jω

$$Y(s) = G(s)U(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}\frac{A\omega}{(s^2+\omega^2)}$$

Only the response due to the poles on the imaginary axis remains after a sufficiently long time

This is the sinusoidal steady-state response

Sinusoidal Steady-State Response

- Input $u(t) = A\cos(\omega t + \phi) = A\cos\omega t\sin\phi - A\sin\omega t\cos\phi$
- Transform $U(s) = -A\cos\phi\frac{s}{s^2+\omega^2} + A\sin\phi\frac{\omega}{s^2+\omega^2}$
- Response Transform

$$Y(s) = G(s)U(s) = \underbrace{\frac{k}{s-j\omega} + \frac{k^*}{s+j\omega}}_{\text{forced response}} + \underbrace{\frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_N}{s-p_N}}_{\text{natural response}}$$

- Response Signal

$$y(t) = \underbrace{ke^{j\omega t} + k^*e^{-j\omega t}}_{\text{forced response}} + \underbrace{k_1e^{p_1t} + k_2e^{p_2t} + \cdots + k_Ne^{p_Nt}}_{\text{natural response}}$$

natural response $\xrightarrow{t\to\infty} 0$

- Sinusoidal Steady State Response

$$y_{SS}(t) = ke^{j\omega t} + k^*e^{-j\omega t}$$

Sinusoidal Steady-State Response

- Calculating the SSS response to $u(t) = A\cos(\omega t + \phi)$
  - Residue calculation

$$k = \lim_{s \to j\omega}[(s - j\omega)Y(s)] = \lim_{s \to j\omega}[(s - j\omega)G(s)U(s)]$$

$$= \lim_{s \to j\omega}\left[G(s)(s - j\omega)A\frac{s\cos\phi - \omega\sin\phi}{(s - j\omega)(s + j\omega)}\right] = G(j\omega)A\left[\frac{j\omega\cos\phi - \omega\sin\phi}{2j\omega}\right]$$

$$= AG(j\omega)\frac{1}{2}e^{j\phi} = \frac{1}{2}A|G(j\omega)|e^{j(\phi + \angle G(j\omega))}$$

  - Signal calculation

$$y_{ss}(t) = L^{-1}\left\{\frac{k}{s - j\omega} + \frac{k^*}{s + j\omega}\right\}$$

$$= |k|e^{j\angle K}e^{j\omega t} + |k|e^{-j\angle K}e^{-j\omega t}$$

$$= 2|k|\cos(\omega t + \angle K)$$

$$y_{ss}(t) = A|G(j\omega)|\cos(\omega t + \phi + \angle G(j\omega))$$

Sinusoidal Steady-State Response

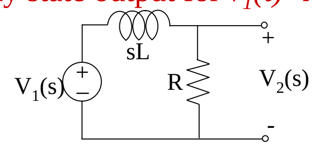
- Response to $u(t) = A\cos(\omega t + \phi)$
  is $y_{ss} = |G(j\omega)|A\cos(\omega t + \phi + \angle G(j\omega))$

  – Output frequency = input frequency
  – Output amplitude = input amplitude × $|G(j\omega)|$
  – Output phase = input phase + $\angle G(j\omega)$

- The Frequency Response of the transfer function *G(s)* is given by its evaluation as a function of a complex variable at $s=j\omega$
  - We speak of the amplitude response and of the phase response
    – They cannot independently be varied
      » Bode's relations of analytic function theory

Frequency Response

- Find the steady state output for $v_1(t) = A\cos(\omega t + \phi)$



- Compute the s-domain transfer function T(s)
  - Voltage divider $T(s) = \frac{R}{sL + R}$
- Compute the frequency response

$$|T(j\omega)| = \frac{R}{\sqrt{R^2 + (\omega L)^2}}, \qquad \angle T(j\omega) = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

- Compute the steady state output

$$v_{2SS}(t) = \frac{AR}{\sqrt{R^2 + (\omega L)^2}} \cos\left[\omega t + \phi - \tan^{-1}(\omega L / R)\right]$$

Bode Diagrams

- Log-log plot of mag(T), log-linear plot of arg(T) versus ω