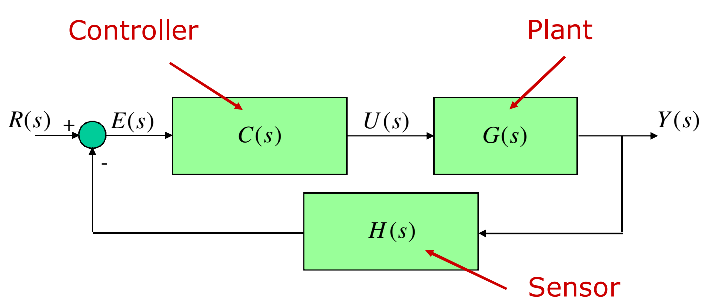
ME 343 – Control Systems

Lecture 18

October 2, 2009

Root Locus



$$C(s) = KD(s) \Rightarrow \quad \frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)H(s)} = \frac{C(s)G(s)}{1 + KL(s)}$$

Writing the loop gain as $KL(s)$ we are interested in tracking the closed-loop poles as "gain" K varies

Root Locus

$$\text{RL} = \text{zeros}\{1 + KL(s)\} = \text{roots}\{\text{den}(L) + K\text{num}(L)\}$$

when K varies from 0 to ∞ (positive Root Locus) or from 0 to -∞ (negative Root Locus)

$$1 + KL(s) = 0 \Leftrightarrow L(s) = -\frac{1}{K} \Leftrightarrow a(s) + Kb(s) = 0$$

Basic Properties:

- Number of branches = number of open-loop poles
- RL begins at open-loop poles

$$K = 0 \Rightarrow a(s) = 0$$

- RL ends at open-loop zeros or asymptotes

$$K = \infty \Rightarrow L(s) = 0 \Leftrightarrow \begin{cases} b(s) = 0 \\ s \to \infty \ (n - m > 0) \end{cases}$$

- RL symmetrical about Re-axis

Phase and Magnitude of a Transfer Function

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$G(s) = K \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

The factors K, $(s\text{-}z_j)$ and $(s\text{-}p_k)$ are complex numbers:

$$(s - z_j) = r_j^z e^{i\phi_j^z}, \qquad j = 1 \ldots m$$

$$(s - p_k) = r_k^p e^{i\phi_k^p}, \qquad k = 1 \cdots p$$

$$K = |K| e^{i\phi^K}$$

$$G(s) = |K| e^{i\phi^K} \frac{r_1^z e^{i\phi_1^z} r_2^z e^{i\phi_2^z} \cdots r_m^z e^{i\phi_m^z}}{r_1^p e^{i\phi_1^p} r_2^p e^{i\phi_2^p} \cdots r_n^p e^{i\phi_n^p}}$$

Phase and Magnitude of a Transfer Function

$$G(s) = |K| e^{i\phi^K} \frac{r_1^z e^{i\phi_1^z} r_2^z e^{i\phi_2^z} \cdots r_m^z e^{i\phi_m^z}}{r_1^p e^{i\phi_1^p} r_2^p e^{i\phi_2^p} \cdots r_n^p e^{i\phi_n^p}}$$

$$= |K| e^{i\phi^K} \frac{r_1^z r_2^z \cdots r_m^z e^{i\left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right)}}{r_1^p r_2^p \cdots r_n^p e^{i\left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right)}}$$

$$= |K| \frac{r_1^z r_2^z \cdots r_m^z}{r_1^p r_2^p \cdots r_n^p} e^{i\left[\phi^K + \left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right) - \left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right)\right]}$$

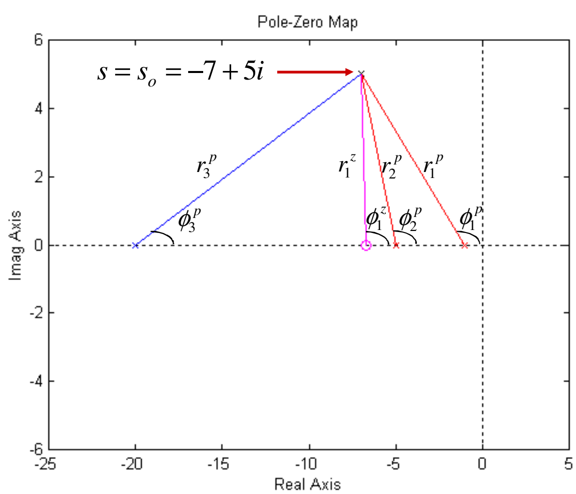
Now it is easy to give the phase and magnitude of the transfer function:

$$|G(s)| = |K| \frac{r_1^z r_2^z \cdots r_m^z}{r_1^p r_2^p \cdots r_n^p},$$

$$\angle G(s) = \phi^K + \left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right) - \left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right)$$

Phase and Magnitude of a Transfer Function

Example: $$G(s) = \frac{(s + 6.735)}{(s+1)(s+5)(s+20)}$$



$$|G(s)| = \frac{r_1^z}{r_1^p r_2^p r_3^p}$$

$$\angle G(s) = \phi_1^z - \left(\phi_1^p + \phi_2^p + \phi_3^p\right)$$

Root Locus- Magnitude and Phase Conditions

$$\text{RL} = \text{zeros}\{1 + KL(s)\} = \text{roots}\{\text{den}(L) + K\text{num}(L)\}$$

when K varies from 0 to ∞ (positive Root Locus) or from 0 to -∞ (negative Root Locus)

$$L(s) = K_p \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} = |K_p| \frac{r_1^z r_2^z \cdots r_m^z}{r_1^p r_2^p \cdots r_n^p} e^{i\left[\phi^{K_p} + \left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right) - \left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right)\right]}$$

$$K > 0 : L(s) = -\frac{1}{K} \Leftrightarrow \begin{array}{l} |L(s)| = |K_p| \dfrac{r_1^z r_2^z \cdots r_m^z}{r_1^p r_2^p \cdots r_n^p} = \dfrac{1}{K} \\ \angle L(s) = \phi^{K_p} + \left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right) - \left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right) = 180^\circ \end{array}$$

$$K < 0 : L(s) = -\frac{1}{K} \Leftrightarrow \begin{array}{l} |L(s)| = |K_p| \dfrac{r_1^z r_2^z \cdots r_m^z}{r_1^p r_2^p \cdots r_n^p} = -\dfrac{1}{K} \\ \angle L(s) = \phi^{K_p} + \left(\phi_1^z + \phi_2^z + \cdots + \phi_m^z\right) - \left(\phi_1^p + \phi_2^p + \cdots + \phi_n^p\right) = 0^\circ \end{array}$$

Root Locus

Selecting K for desired closed loop poles on Root Locus:

If s_o belongs to the root locus, it must satisfies the characteristic equation for some value of K

$$L(s_o) = -\frac{1}{K}$$

Then we can obtain K as

$$K = -\frac{1}{L(s_o)}$$

$$K = \frac{1}{|L(s_o)|}$$

Root Locus

Example: $L(s) = G(s) = \dfrac{1}{(s+1)(s+5)}$

$$s_o = -3 + i4 \Rightarrow K = \frac{1}{|L(s_o)|} = |s_o + 1||s_o + 5| = |-3 + i4 + 1||-3 + i4 + 5|$$

$$= \sqrt{(-2)^2 + 4^2}\sqrt{(2)^2 + 4^2} = 20$$

Using MATLAB:

```
sys=tf(1,poly([-1 -5]))
so=-3+4i
[K,POLES]=rlocfind(sys,so)
```


Root Locus

Example: $L(s) = G(s) = \dfrac{1}{(s+1)(s+5)}$

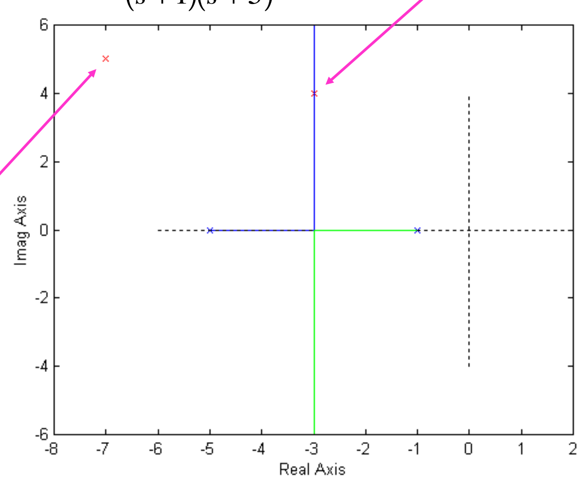
$s_o = -3 + i4$

$s_o = -7 + i5$

$\Downarrow$

$$K = \frac{1}{|L(s_o)|} = 42.06$$

$s_o = -7 + i5$



When we use the absolute value formula we are assuming that the point belongs to the Root Locus!