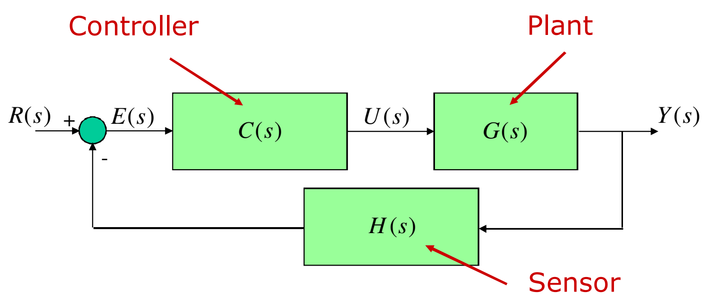
ME 343 – Control Systems

Lecture 16

September 28, 2009

Root Locus



$$C(s) = KD(s) \Rightarrow \quad \frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)H(s)} = \frac{C(s)G(s)}{1 + KL(s)}$$

Writing the loop gain as $KL(s)$ we are interested in tracking the closed-loop poles as "gain" K varies

Root Locus

$$\text{RL} = \text{zeros}\{1 + KL(s)\} = \text{roots}\{\text{den}(L) + K\text{num}(L)\}$$

when K varies from 0 to ∞ (positive Root Locus) or from 0 to -∞ (negative Root Locus)

$$1 + KL(s) = 0 \Leftrightarrow L(s) = -\frac{1}{K} \Leftrightarrow a(s) + Kb(s) = 0$$

Basic Properties:

- Number of branches = number of open-loop poles
- RL begins at open-loop poles

$$K = 0 \Rightarrow a(s) = 0$$

- RL ends at open-loop zeros or asymptotes

$$K = \infty \Rightarrow L(s) = 0 \Leftrightarrow \begin{cases} b(s) = 0 \\ s \to \infty \; (n - m > 0) \end{cases}$$

- RL symmetrical about Re-axis

Root Locus

Rule 1: The n branches of the locus start at the poles of $L(s)$ and m of these branches end on the zeros of $L(s)$.
n: order of the denominator of $L(s)$
m: order of the numerator of $L(s)$

Rule 2: The locus is on the real axis to the left of and odd number of poles and zeros.
In other words, an interval on the real axis belongs to the root locus if the total number of poles and zeros to the right is odd.
This rule comes from the phase condition!!!

Root Locus

Rule 3: As $K \to \infty$, m of the closed-loop poles approach the open-loop zeros, and n-m of them approach n-m asymptotes with angles

$$\phi_l = (2l+1)\frac{\pi}{n-m}, \qquad l = 0,1,\ldots,n-m-1$$

and centered at

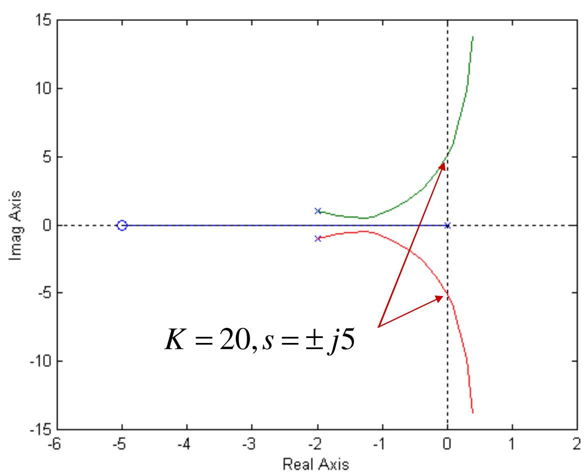
$$\alpha = \frac{b_1 - a_1}{n-m} = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m}, \qquad l = 0,1,\ldots,n-m-1$$

Root Locus

Rule 4: The locus crosses the $j\omega$ axis (looses stability) where the Routh criterion shows a transition from roots in the left half-plane to roots in the right-half plane.

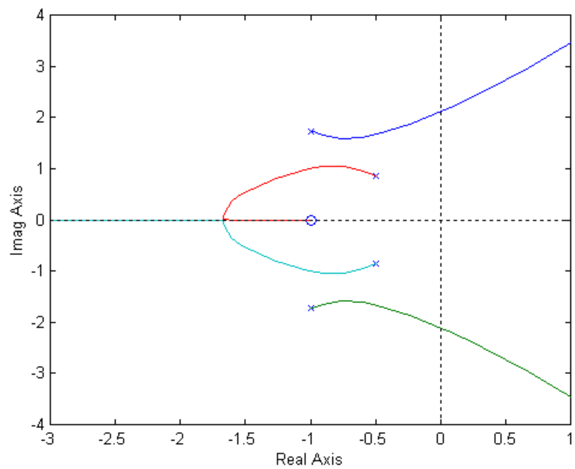
Example:

$$G(s) = \frac{s+5}{s(s^2+4s+5)}$$



Root Locus

Example: $$G(s)=\frac{s+1}{s^4+3s^3+7s^2+6s+4}$$



Root Locus

Design dangers revealed by the Root Locus:

• High relative degree: For $n\text{-}m\geq 3$ we have closed loop instability due to asymptotes.

$$G(s)=\frac{s+1}{s^4+3s^3+7s^2+6s+4}$$

• Nonminimum phase zeros: They attract closed loop poles into the RHP

$$G(s)=\frac{s-1}{s^2+s+1}$$

Note: Check code rootlocus.m

Root Locus

Viete's formula:

When the relative degree n-$m \geq 2$, the sum of the closed loop poles is constant

$$a_1 = -\sum \text{closed loop poles}$$

$$L(s) = \frac{b(s)}{a(s)} = \frac{s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$