ME 343 – Control Systems

Lecture 09

September 11, 2009

Stability

$$\frac{Y(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$\frac{Y(s)}{R(s)} = K \frac{(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

$$\frac{Y(s)}{R(s)} = \frac{k_1}{(s - p_1)} + \frac{k_2}{(s - p_2)} + \cdots + \frac{k_n}{(s - p_n)}$$

Impulse response:

$$R(s) = 1 \Rightarrow Y(s) = \frac{k_1}{(s - p_1)} + \frac{k_2}{(s - p_2)} + \cdots + \frac{k_n}{(s - p_n)}$$

$$y(t) = k_1 e^{p_1 t} + k_2 e^{p_2 t} + \cdots + k_n e^{p_n t}$$

Stability

$$y(t) = k_1 e^{p_1 t} + k_2 e^{p_2 t} + \cdots + k_n e^{p_n t}$$

We want: $e^{p_i t} \xrightarrow[t\to\infty]{} 0 \quad \forall i = 1\ldots n$

Definition: A system is asymptotically stable (a.s.) if

$$\mathrm{Re}\{p_i\} < 0 \quad \forall i$$

Characteristic polynomial: $a(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$

Characteristic equation: $a(s) = 0$

Stability

Necessary condition for asymptotical stability (a.s.):

$$a_i > 0 \quad \forall i$$

Use this as the first test!

If any a_i<0, the the system is UNSTABLE!

Example: $s^2 + s - 2 = 0$

$(s+2)(s-1) = 0$

Routh's Criterion

Necessary and sufficient condition

Do not have to find the roots p_i!

Routh's Array:

$$
\begin{array}{lllll}
s^n & 1 & a_2 & a_4 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & \\
s^{n-3} & c_1 & c_2 & c_3 & \\
s^{n-4} & d_1 & d_2 & & \\
\vdots & & & & \\
s^0 & a_n & & &
\end{array}
$$

a_n Depends on whether n is even or odd

$$b_1 = \frac{a_1 a_2 - a_3}{a_1}, \quad b_2 = \frac{a_1 a_4 - a_5}{a_1}, \quad b_3 = \frac{a_1 a_6 - a_7}{a_1} \quad \cdots$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}, \quad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}, \quad \cdots$$

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}, \quad d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}, \quad \cdots$$

$$\vdots \qquad \vdots$$

Routh's Criterion

How to remember this?

Routh's Array:

$$
\begin{array}{lllll}
s^n & m_{11} & m_{12} & m_{13} & \cdots \\
s^{n-1} & m_{21} & m_{22} & m_{23} & \cdots \\
s^{n-2} & m_{31} & m_{32} & m_{33} & \cdots \\
s^{n-3} & m_{41} & m_{42} & m_{43} & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}
$$

$$m_{1,j} = a_{2j-2},$$

$$m_{2,j} = a_{2j-1},$$

$$m_{i,j} = -\frac{\begin{vmatrix} m_{i-2,1} & m_{i-2,j+1} \\ m_{i-1,1} & m_{i-1,j+1} \end{vmatrix}}{m_{i-1,1}}, \forall i \geq 3$$

Routh's Criterion

The criterion:

- The system is **asymptotically stable** if and only if all the elements in the first column of the Routh's array are positive

- The number of roots with positive real parts is equal to the number of sign changes in the first column of the Routh array

Routh's Criterion

Example 1: $s^2 + a_1 s + a_2 = 0$

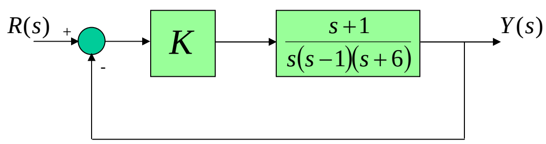
Example 2: $s^3 + a_1 s^2 + a_2 s + a_3 = 0$

Example 3: $s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4 = 0$

Example 4: $s^3 + 5s^2 + (k-6)s + k = 0$

Routh's Criterion

Example: Determine the range of K over which the system is stable



Routh's Criterion

Special Case I: Zero in the first column
We replace the zero with a small positive constant $\varepsilon>0$ and proceed as before. We then apply the stability criterion by taking the limit as $\varepsilon\to 0$

Example: $s^4 + 2s^3 + 4s^2 + 8s + 10 = 0$

Routh's Criterion

Special Case II: Entire row is zero

This indicates that there are complex conjugate pairs. If the ith row is zero, we form an auxiliary equation from the previous nonzero row:

$$a_1(s) = \beta_1 s^{i+1} + \beta_2 s^{i-1} + \beta_3 s^{i-3} + \cdots$$

Where β_i are the coefficients of the $(i+1)$th row in the array. We then replace the ith row by the coefficients of the derivative of the auxiliary polynomial.

Example: $s^5 + 2s^4 + 4s^3 + 8s^2 + 10s + 20 = 0$