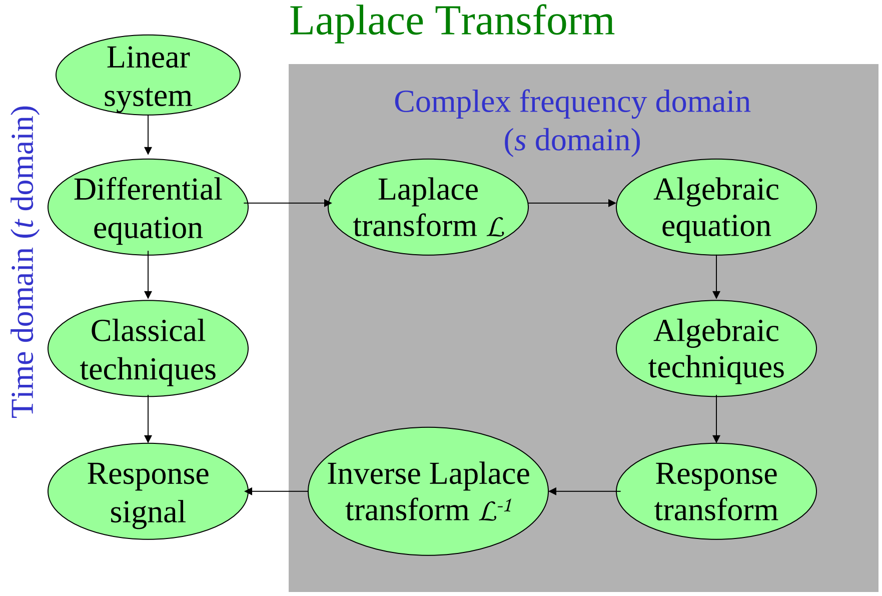
ME 343 – Control Systems

Lecture 05

September 2, 2009

Laplace Transform



The diagram commutes

Same answer whichever way you go

Laplace Transform Table

Signal	Waveform	Transform
impulse	$\delta(t)$	1
step	$u(t)$	$\frac{1}{s}$
ramp	$tu(t)$	$\frac{1}{s^2}$
exponential	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
damped ramp	$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$
sine	$\sin(\beta t)u(t)$	$\frac{\beta}{s^2+\beta^2}$
cosine	$\cos(\beta t)u(t)$	$\frac{s}{s^2+\beta^2}$
damped sine	$e^{-\alpha t}\sin(\beta t)u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
damped cosine	$e^{-\alpha t}\cos(\beta t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$

Solving LTI ODE's via Laplace Transform

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = b_m u^{(m)} + b_{m-1}u^{(m-1)} + \cdots + b_0 u$$

Initial Conditions: $y^{(n-1)}(0),\ldots,y(0),u^{(m-1)}(0),\ldots,u(0)$

Recall $\mathcal{L}\left\{\frac{d^k f(t)}{dt^k}\right\} = s^k F(s) - \sum_{j=0}^{k-1} f^{(k-1-j)}(0)s^j$

$$s^n Y(s) - \sum_{j=0}^{n-1} y^{(n-1-j)}(0)s^j + \sum_{i=0}^{n-1} a_i\left[s^i Y(s) - \sum_{j=0}^{i-1} y^{(i-1-j)}(0)s^j\right] = \sum_{i=0}^{m} b_i\left[s^i U(s) - \sum_{j=0}^{i-1} u^{(i-1-j)}(0)s^j\right]$$

$$Y(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}U(s) + \frac{\sum_{i=0}^{n-1} a_i \sum_{j=0}^{i-1} y^{(i-1-j)}(0)s^j - \sum_{i=0}^{m} b_i \sum_{j=0}^{i-1} u^{(i-1-j)}(0)s^j}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

For a given rational $U(s)$ we get $Y(s)=Q(s)/P(s)$

Computing Transfer Functions via Laplace Transform

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = b_{m-1}u^{(m-1)} + \cdots + b_0 u$$

Assume all <u>Initial Conditions Zero</u>:

$$\left(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right)Y(s) = \left(b_{m-1}s^{m-1} + \cdots + b_1 s + b_0\right)U(s)$$

Output Input

$$Y(s) = \frac{b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}U(s) = \frac{B(s)}{A(s)}U(s)$$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$= K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

Rational Functions

We shall mostly be dealing with TFs which are rational functions – ratios of polynomials in s

$$F(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$= K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

p_i are the poles and z_i are the zeros of the function

K is the scale factor or (sometimes) gain

A proper rational function has $n \geq m$

A strictly proper rational function has $n > m$

An improper rational function has $n < m$

Partial Fraction Expansion - Residues at Simple Poles

Functions of a complex variable with isolated, finite order poles have *residues* at the poles

$$F(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} = \frac{k_1}{(s-p_1)} + \frac{k_2}{(s-p_2)} + \cdots + \frac{k_n}{(s-p_n)}$$

$$(s-p_i)F(s) = \frac{k_1(s-p_i)}{(s-p_1)} + \frac{k_2(s-p_i)}{(s-p_2)} + \cdots + k_i + \cdots + \frac{k_n(s-p_i)}{(s-p_n)}$$

Residue at a simple pole: $k_i = \lim_{s\to p_i}(s-p_i)F(s)$

Partial Fraction Expansion - Residues at multiple poles

$$F(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)^r} = \frac{k_1}{(s-p_1)} + \frac{k_2}{(s-p_1)^2} + \cdots + \frac{k_r}{(s-p_1)^r}$$

Residue at a multiple pole: $k_j = \frac{1}{(r-j)!}\lim_{s\to p_i}\frac{d^{r-j}}{ds^{r-j}}\left[(s-p_i)^r F(s)\right], \quad j = 1\cdots r$

Example: $\frac{2s^2+5s}{(s+1)^3} = \frac{k_1}{s+1} + \frac{k_2}{(s+1)^2} + \frac{k_3}{(s+1)^3}$

$\lim_{s\to -3}\frac{(s+1)^3(2s^2+5s)}{(s+1)^3} = -3 \quad \boxed{k_3}$

$\lim_{s\to -1}\frac{d}{ds}\left[\frac{(s+1)^3(2s^2+5s)}{(s+1)^3}\right] = 1 \quad \boxed{k_2}$

$\frac{1}{2!}\lim_{s\to -1}\frac{d^2}{ds^2}\left[\frac{(s+1)^3(2s^2+5s)}{(s+1)^3}\right] = 2 \quad \boxed{k_1}$

$$L^{-1}\left(\frac{2s^2+5s}{(s+1)^3}\right) = L^{-1}\left(\frac{2}{s+1} + \frac{1}{(s+1)^2} - \frac{3}{(s+1)^3}\right) = e^{-t}\left(2+t-3t^2\right)u(t)$$

Partial Fraction Expansion - Residues at Complex Poles

Compute residues at the poles $\lim_{s \to a} (s-a)F(s)$

Bundle complex conjugate pole pairs into second-order terms if you want … but you will need to be careful!

$$(s-\alpha-j\beta)(s-\alpha+j\beta) = \left[s^2 - 2\alpha s + \left(\alpha^2+\beta^2\right)\right]$$

Inverse Laplace Transform is a sum of complex exponentials. But the answer will be <u>real</u>.

Inverting Laplace Transforms in Practice

We have a table of inverse LTs

Write F(s) as a partial fraction expansion

$$\begin{aligned}F(s) &= \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \\ &= K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \\ &= \frac{\alpha_1}{(s-p_1)} + \frac{\alpha_2}{(s-p_2)} + \frac{\alpha_{31}}{(s-p_3)} + \frac{\alpha_{32}}{(s-p_3)^2} + \frac{\alpha_{33}}{(s-p_3)^3} + \ldots + \frac{\alpha_q}{(s-p_q)}\end{aligned}$$

Now appeal to linearity to invert via the table

- Surprise!
- Nastiness: computing the partial fraction expansion is best done by calculating the residues

Inverse Laplace Transform

Exercise: Find the Inverse Laplace transform of

$$F(s)=\frac{20(s+3)}{(s+1)(s^2+2s+5)} \longrightarrow F(s)=\frac{k_1}{s+1}+\frac{k_2}{s+1-j2}+\frac{k_2^*}{s+1+j2}$$

$$k_1=\lim_{s\to -1}(s+1)F(s)=\left.\frac{20(s+3)}{s^2+2s+5}\right|_{s=-1}=10$$

$$k_2=\lim_{s\to -1+2j}(s+1-2j)F(s)=\left.\frac{20(s+3)}{(s+1)(s+1+2j)}\right|_{s=-1+2j}=-5-5j=5\sqrt{2}e^{j\frac{5}{4}\pi}$$

$$f(t)=\left[10e^{-t}+5\sqrt{2}e^{(-1+j2)t+j\frac{5}{4}\pi}+5\sqrt{2}e^{(-1-j2)t-j\frac{5}{4}\pi}\right]u(t)$$

$$=\left[10e^{-t}+10\sqrt{2}e^{-t}\cos(2t+\frac{5\pi}{4})\right]u(t)$$

Inverse Laplace Transform

Exercise: Find $v(t)$

$$\frac{dv(t)}{dt}+6v(t)=4u(t)$$

$$v(0-)=-3$$

$$V(s)=\frac{4}{s(s+6)}-\frac{3}{s+6}$$

$$v(t)=\frac{2}{3}u(t)-\frac{11}{3}e^{-6t}u(t)$$

Exercise: Find $v(t)$

$$\frac{d^2v(t)}{dt^2}+4\frac{dv(t)}{dt}+3v(t)=5e^{-2t}$$

$$v(0-)=-2, v'(0-)=2$$

$$V(s)=\frac{5}{(s+1)(s+2)(s+3)}-\frac{2}{s+1}$$

$$v(t)=\left(\frac{1}{2}e^{-t}-5e^{-2t}+\frac{5}{2}e^{-3t}\right)u(t)$$

What about $v(t)$?

Not Strictly Proper Laplace Transforms

Find the inverse LT of $F(s)=\dfrac{s^3+6s^2+12s+8}{s^2+4s+3}$

Convert to polynomial plus strictly proper rational function

Use polynomial division

$$
\begin{aligned}
F(s) &= s+2+\frac{s+2}{s^2+4s+3} \\
&= s+2+\frac{0.5}{s+1}+\frac{0.5}{s+3}
\end{aligned}
$$

Invert as normal

$$f(t)=\left[\frac{d\delta(t)}{dt}+2\delta(t)+0.5e^{-t}+0.5e^{-3t}\right]u(t)$$