ME 343 – Control Systems

Lecture 04

August 31, 2009

Laplace Transform

Function $f(t)$ of time

Piecewise continuous and exponential order $|f(t)| < Ke^{bt}$

$$F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt \qquad \mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\alpha - j\infty}^{\alpha + j\infty} F(s)e^{st}ds$$

0- limit is used to capture transients and discontinuities at $t=0$

s is a complex variable ($\sigma + j\omega$)

- There is a need to worry about regions of convergence of the integral

Units of s are sec^{-1}=Hz

- A frequency

If $f(t)$ is volts (amps) then $F(s)$ is volt-seconds (amp-seconds)

Laplace Transform Examples

Step function – unit Heavyside Function

After Oliver Heavyside (1850-1925) $u(t)=\begin{cases}0, \text{ for } t<0\\ 1, \text{ for } t\geq 0\end{cases}$

$$F(s)=\int_{0-}^{\infty}u(t)e^{-st}dt=\int_{0-}^{\infty}e^{-st}dt=-\left.\frac{e^{-st}}{s}\right|_0^{\infty}=-\left.\frac{e^{-(\sigma+j\omega)t}}{\sigma+j\omega}\right|_0^{\infty}=\frac{1}{s}\ \text{ if }\ \sigma>0$$

Exponential function

After Oliver Exponential (1176 BC- 1066 BC)

$$F(s)=\int_0^{\infty}e^{-\alpha t}e^{-st}dt=\int_0^{\infty}e^{-(s+\alpha)t}dt=-\left.\frac{e^{-(s+\alpha)t}}{s+\alpha}\right|_0^{\infty}=\frac{1}{s+\alpha}\ \text{ if }\ \sigma>\alpha$$

Delta (impulse) function $\delta(t)$

$$F(s)=\int_{0-}^{\infty}\delta(t)e^{-st}dt=1\ \text{ for all } s$$

Laplace Transform Table

Signal	Waveform	Transform
impulse	$\delta(t)$	1
step	$u(t)$	$\frac{1}{s}$
ramp	$tu(t)$	$\frac{1}{s^2}$
exponential	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
damped ramp	$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$
sine	$\sin(\beta t)u(t)$	$\frac{\beta}{s^2+\beta^2}$
cosine	$\cos(\beta t)u(t)$	$\frac{s}{s^2+\beta^2}$
damped sine	$e^{-\alpha t}\sin(\beta t)u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
damped cosine	$e^{-\alpha t}\cos(\beta t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$

Laplace Transform Properties

Linearity: (absolutely critical property)

$$\mathcal{L}\{Af_1(t)+Bf_2(t)\}=A\mathcal{L}\{f_1(t)\}+B\mathcal{L}\{f_2(t)\}=AF_1(s)+BF_2(s)$$

Integration property: $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}=\frac{F(s)}{s}$

Differentiation property: $\mathcal{L}\left\{\frac{df(t)}{dt}\right\}=sF(s)-f(0-)$

$$\mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\}=s^2F(s)-sf(0-)-f'(0-)$$

$$\mathcal{L}\left\{\frac{d^mf(t)}{dt^m}\right\}=s^mF(s)-s^{m-1}f(0-)-s^{m-2}f'(0-)-\cdots-f^{(m)}(0-)$$

Laplace Transform Properties

Translation properties:

s-domain translation: $\mathcal{L}\{e^{-\alpha t}f(t)\}=F(s+\alpha)$

t-domain translation: $\mathcal{L}\{f(t-a)u(t-a)\}=e^{-as}F(s)$ for $a>0$

Initial Value Property: $\lim_{t\to 0+} f(t)=\lim_{s\to\infty} sF(s)$

Final Value Property: $\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$

If all poles of $F(s)$ are in the LHP

Laplace Transform Properties

Time Scaling: $\mathcal{L}\{f(at)\} = \frac{1}{|a|}F(\frac{s}{a})$

Multiplication by time: $\mathcal{L}\{tf(t)\} = -\frac{dF(s)}{ds}$

Convolution: $\mathcal{L}\{\int_0^t f(\tau)g(t-\tau)d\tau\} = F(s)G(s)$

Time product: $\mathcal{L}\{f(t)g(t)\} = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} F(s)G(s-\lambda)d\lambda$

Laplace Transform

Exercise: Find the Laplace transform of the following waveform

$$f(t) = [2 + 2\sin(2t) - 2\cos(2t)]u(t) \qquad F(s) = \frac{4(s+2)}{s(s^2+4)}$$

Exercise: Find the Laplace transform of the following waveform

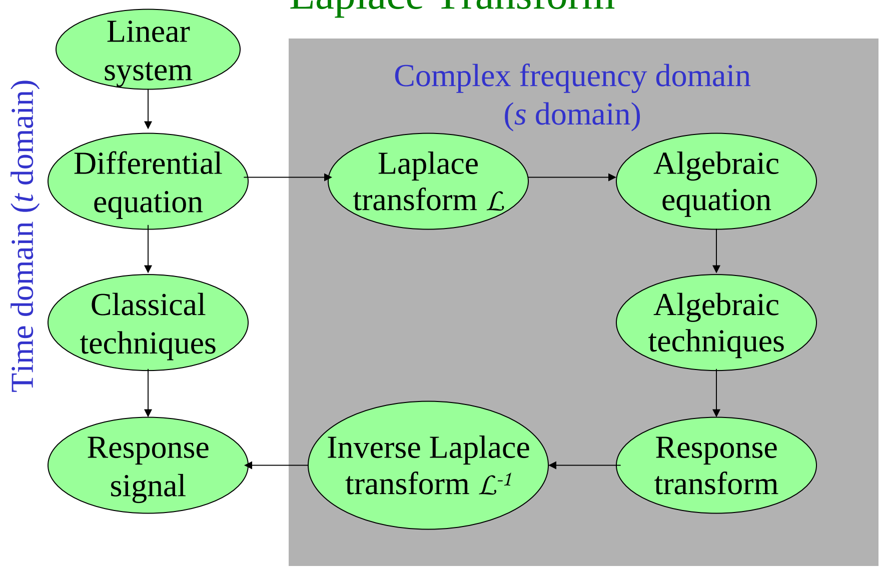
$$f(t) = e^{-4t}u(t) + 5\int_0^t \sin(4x)dx \qquad F(s) = \frac{s^3 + 36s + 80}{s(s+4)(s^2+16)}$$

$$f(t) = 5e^{-40t}u(t) + \frac{d\left[5te^{-40t}\right]}{dt}u(t) \qquad F(s) = \frac{10s + 200}{(s+40)^2}$$

Exercise: Find the Laplace transform of the following waveform

$$f(t) = Au(t) - 2Au(t-T) + Au(t-2T) \qquad F(s) = \frac{A(1-e^{-Ts})^2}{s}$$

Laplace Transform



The diagram commutes

Same answer whichever way you go

Solving LTI ODE's via Laplace Transform

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = b_m u^{(m)} + b_{m-1}u^{(m-1)} + \cdots + b_0 u$$

Initial Conditions: $y^{(n-1)}(0),\ldots,y(0),u^{(m-1)}(0),\ldots,u(0)$

Recall $\mathcal{L}\left\{\dfrac{d^k f(t)}{dt^k}\right\} = s^k F(s) - \sum_{j=0}^{k-1} f^{(k-1-j)}(0)s^j$

$$s^n Y(s) - \sum_{j=0}^{n-1} y^{(n-1-j)}(0)s^j + \sum_{i=0}^{n-1} a_i \left[s^i Y(s) - \sum_{j=0}^{i-1} y^{(i-1-j)}(0)s^j \right] = \sum_{i=0}^{m} b_i \left[s^i U(s) - \sum_{j=0}^{i-1} u^{(i-1-j)}(0)s^j \right]$$

$$Y(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} U(s) + \frac{\sum_{i=0}^{n-1} a_i \sum_{j=0}^{i-1} y^{(i-1-j)}(0)s^j - \sum_{i=0}^{m} b_i \sum_{j=0}^{i-1} u^{(i-1-j)}(0)s^j}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

For a given rational $U(s)$ we get $Y(s)=Q(s)/P(s)$

Laplace Transform

Exercise: Find the Laplace transform *V(s)*

$$\frac{dv(t)}{dt} + 6v(t) = 4u(t)$$
$$v(0-) = -3$$

$$V(s) = \frac{4}{s(s+6)} - \frac{3}{s+6}$$

Exercise: Find the Laplace transform *V(s)*

$$\frac{d^2v(t)}{dt^2} + 4\frac{dv(t)}{dt} + 3v(t) = 5e^{-2t}$$
$$v(0-) = -2, v'(0-) = 2$$

$$V(s) = \frac{5}{(s+1)(s+2)(s+3)} - \frac{2}{s+1}$$

What about *v(t)*?