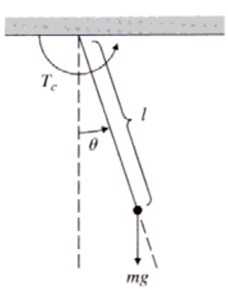
ME 343 – Control Systems

Lecture 03

August 28, 2009

Dynamic Model

MECHANICAL SYSTEM:



$T = I\alpha$ Newton's law

damping coefficient

$$I\alpha = -lmg\sin\theta - b\omega + T_c$$

$\omega = \dot{\theta}$ angular velocity

$\alpha = \dot{\omega} = \ddot{\theta}$ angular acceleration

$I = ml^2$ moment of inertia

$$\ddot{\theta} = -\frac{b}{ml^2}\dot{\theta} - \frac{g}{l}\sin\theta + \frac{T_c}{ml^2}$$

Which are the equilibrium points when T_c=0?

Stable

At equilibrium: $\ddot{\theta} = \dot{\theta} = 0 \Rightarrow 0 = -\frac{g}{l}\sin\theta \Rightarrow \theta = 0, \pi$

Unstable

Open loop simulations: pend_par.m, pendol01.mdl

Linearization

Dynamic System: $x^{(n)} = f\left(x^{(n-1)}, x^{(n-2)}, \ldots, x^{(1)}, x, u\right)$

$$0 = f\left(0,0,\ldots,0,x_o,u_o\right) \qquad \text{Equilibrium}$$

Denote $\delta x = x - x_o, \delta u = u - u_o$

$$\delta x^{(n)} = f\left(\delta x^{(n-1)}, \delta x^{(n-2)}, \ldots, \delta x^{(1)}, x_o + \delta x, u_o + \delta u\right)$$

Taylor Expansion

$$\delta x^{(n)} \approx \cancel{f\left(0,0,\ldots,0,x_o,u_o\right)} + \left.\frac{\partial f}{\partial x^{(n-1)}}\right|_{0,0,\ldots,0,x_o,u_o} \delta x^{(n-1)} + \left.\frac{\partial f}{\partial x^{(n-2)}}\right|_{0,0,\ldots,0,x_o,u_o} \delta x^{(n-2)} + \cdots$$
$$+ \left.\frac{\partial f}{\partial x^{(1)}}\right|_{0,0,\ldots,0,x_o,u_o} \delta x^{(1)} + \left.\frac{\partial f}{\partial x}\right|_{0,0,\ldots,0,x_o,u_o} \delta x + \left.\frac{\partial f}{\partial u}\right|_{x_o,u_o} \delta u$$

Linearization

Dynamic System: $\dot{x} = f\left(x,u\right) \qquad x \in R^n, u \in R^m$

$$0 = f\left(x_o,u_o\right) \qquad \text{Equilibrium}$$

Denote $\delta x = x - x_o, \delta u = u - u_o$

$$\delta \dot{x} = f\left(x_o + \delta x, u_o + \delta u\right)$$

Taylor Expansion

$$\delta \dot{x} \approx f\left(x_o,u_o\right) + \left.\frac{\partial f}{\partial x}\right|_{x_o,u_o} \delta x + \left.\frac{\partial f}{\partial u}\right|_{x_o,u_o} \delta u$$

$$F \equiv \left.\frac{\partial f}{\partial x}\right|_{x_o,u_o}, G \equiv \left.\frac{\partial f}{\partial u}\right|_{x_o,u_o} \quad \Rightarrow \quad \delta \dot{x} \approx F\delta x + G\delta u$$

Linearization

$$\delta\dot{x} \approx F\,\delta x + G\,\delta u$$

$$F \equiv \left.\frac{\partial f}{\partial x}\right|_{x_o,u_o} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x_o,u_o}, G \equiv \left.\frac{\partial f}{\partial u}\right|_{x_o,u_o} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{x_o,u_o}$$

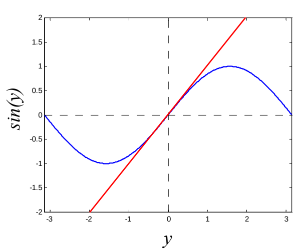
Linearization

What happens around $\theta=0$?

$$\theta = y \Rightarrow \qquad \ddot{y} = -\frac{b}{ml^2}\dot{y} - \frac{g}{l}\sin(y) + \frac{T_c}{ml^2}$$

By Taylor Expansion:

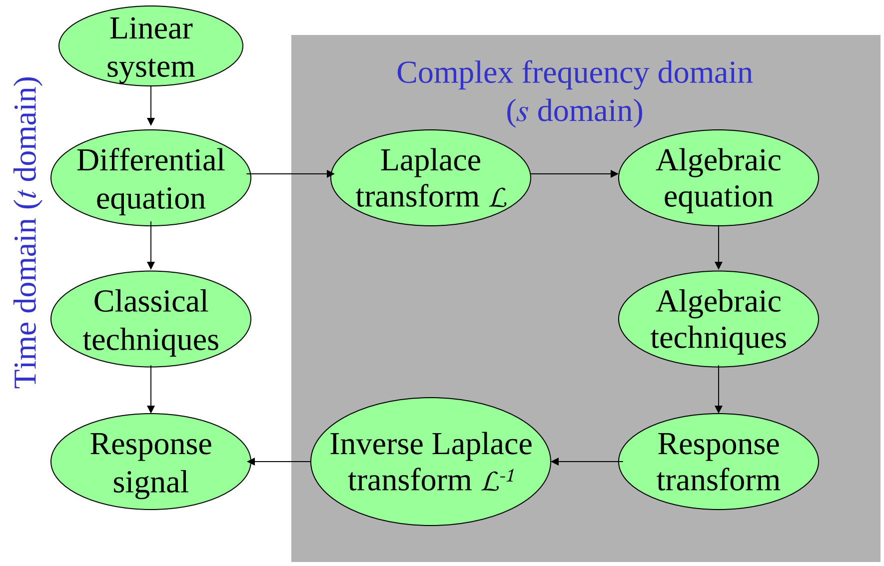
$$\sin(y) = y + h.o.t. \Rightarrow \sin(y) \approx y$$



Linearized Equation:

$$\ddot{y} = -\frac{b}{ml^2}\dot{y} - \frac{g}{l}y + \frac{T_c}{ml^2}$$

Laplace Transform



Transfer function

$$u \equiv T_c \Rightarrow \quad \ddot{y} = -\frac{b}{ml^2}\dot{y} - \frac{g}{l}y + \frac{T_c}{ml^2}$$

$$\ddot{y} + \frac{b}{ml^2}\dot{y} + \frac{g}{l}y = \frac{u}{ml^2}$$

Laplace Transform

$$\mathcal{L}\left\{\frac{d^m f(t)}{dt^m}\right\} = s^m F(s),\ U(s) = \mathcal{L}\{u\},\ Y(s) = \mathcal{L}\{y\}$$

Transfer Function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\frac{1}{ml^2}}{s^2 + \frac{b}{ml^2}s + \frac{g}{l}}$$

Characteristic Equation

Solution of the ODE

$$T_c = 0 \Rightarrow \ddot{y} + \frac{b}{ml^2}\dot{y} + \frac{g}{l}y = 0$$ What is the solutions $y(t)$?

Characteristic Equation

$$\lambda^2 + \frac{b}{ml^2}\lambda + \frac{g}{l} = 0 \Rightarrow \lambda_{1,2} = \frac{-\frac{b}{ml^2} \pm \sqrt{\left(\frac{b}{ml^2}\right)^2 - 4\frac{g}{l}}}{2}$$

$$G(s) = \frac{1/ml^2}{s^2 + \frac{b}{ml^2}s + \frac{g}{l}} \qquad y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

The dynamics of the system is given by the roots of the denominator (poles) of the trasfer function

real(λ_1,λ_2)<0 $\Rightarrow$ STABLE SYSTEM

We use feedback control for PERFORMANCE

Linearization

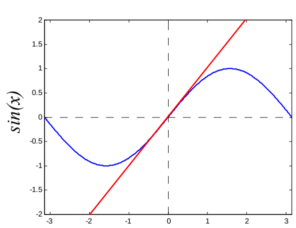
What happens around $\theta=\pi$?

$$\theta = \pi + x \Rightarrow \quad \ddot{x} = -\frac{b}{ml^2}\dot{x} - \frac{g}{l}\sin(\pi + x) + \frac{T_c}{ml^2}$$

$$\ddot{x} = -\frac{b}{ml^2}\dot{x} + \frac{g}{l}\sin(x) + \frac{T_c}{ml^2}$$

By Taylor Expansion:

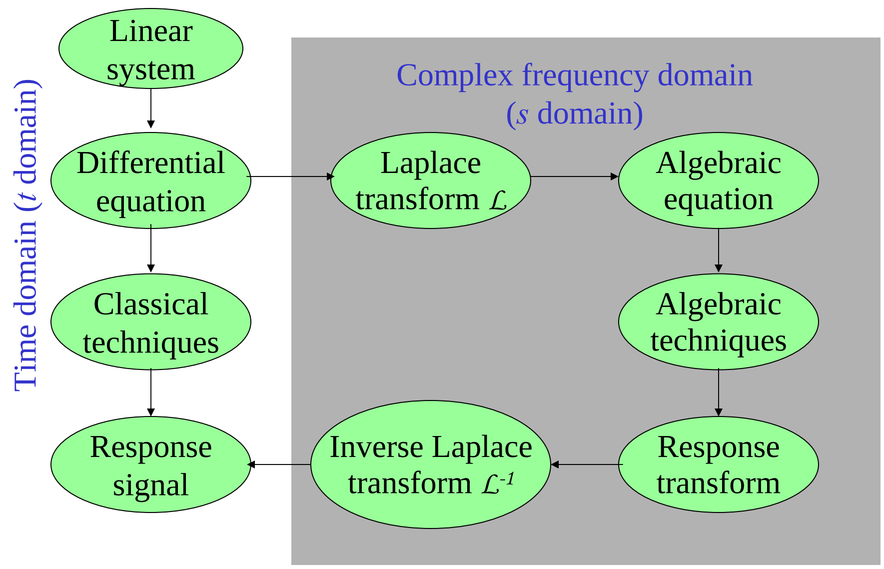
$$\sin(x) = x + h.o.t. \Rightarrow \sin(x) \approx x$$



Linearized Equation:

$$\ddot{x} = -\frac{b}{ml^2}\dot{x} + \frac{g}{l}x + \frac{T_c}{ml^2}$$

Laplace Transform



Transfer Function

$$u \equiv T_c \Rightarrow \quad \ddot{x} = -\frac{b}{ml^2}\dot{x} + \frac{g}{l}x + \frac{T_c}{ml^2}$$

$$\ddot{x} + \frac{b}{ml^2}\dot{x} - \frac{g}{l}x = \frac{u}{ml^2}$$

Laplace Transform

$$\mathcal{L}\left\{\frac{d^m f(t)}{dt^m}\right\} = s^m F(s), \; U(s) = \mathcal{L}\{u\}, \; X(s) = \mathcal{L}\{x\}$$

Transfer Function

$$G(s) = \frac{X(s)}{U(s)} = \frac{\frac{1}{ml^2}}{s^2 + \frac{b}{ml^2}s - \frac{g}{l}}$$

Characteristic Equation

Solution of the ODE

$T_c = 0 \Rightarrow \ddot{x} + \frac{b}{ml^2}\dot{x} - \frac{g}{l}y = 0$ What is the solutions $y(t)$?

Characteristic Equation

$$\lambda^2 + \frac{b}{ml^2}\lambda - \frac{g}{l} = 0 \Rightarrow \lambda_{1,2} = \frac{-\frac{b}{ml^2} \pm \sqrt{\left(\frac{b}{ml^2}\right)^2 + 4\frac{g}{l}}}{2}$$

$$G(s) = \frac{1/ml^2}{s^2 + \frac{b}{ml^2}s - \frac{g}{l}} \qquad x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

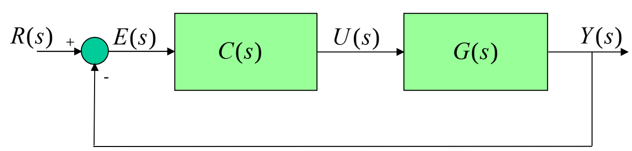
The dynamics of the system is given by the roots of the denominator (poles) of the trasfer function

real(λ_1)>0 or real(λ_2)>0 $\Rightarrow$ UNSTABLE SYSTEM!!!

We use feedback control for STABILITY AND PERFORMANCE

Closed-loop Control

$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)},$$
$$\frac{E(s)}{R(s)} = \frac{1}{1 + C(s)G(s)}.$$



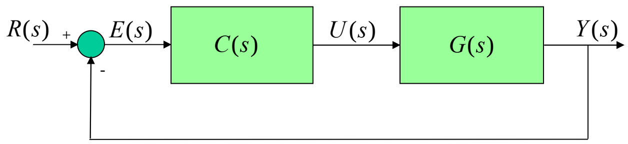
PID: Proportional – Integral – Derivative

$$C(s) = \frac{U(s)}{E(s)} = K_p + \frac{K_I}{s} + K_D s$$

$$u(t) = K_p e(t) + K_I \int_0^d e(\tau)d\tau + K_D \frac{de(t)}{dt}$$

Closed loop simulations: pid.m

Closed-loop Control



$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1+C(s)G(s)} = \frac{(1/ml^2)(K_I + K_P s + K_D s^2)}{s^3 + \dfrac{b+K_D}{ml^2}s^2 + \left(\dfrac{g}{l} + \dfrac{K_p}{ml^2}\right)s + \dfrac{K_I}{ml^2}}$$

We can place the poles at the desired location to obtain the desired dynamics

CLASSICAL CONTROL (ME 343)

Closed loop simulations: pend_par.m, pid.m
pendclPID01.mdl