ME242 – MECHANICAL ENGINEERING SYSTEMS

LECTURE 40:

- Vibrations — Appendix B

VIBRATIONS

Linear systems representation:

State Variables representation: $\dot{x} = Ax + Bu(t)$

First-order matrix differential equation

Vibrations Analysis

$$M\ddot{x} + C\dot{x} + Kx = F(t)$$

Second-order matrix differential equation

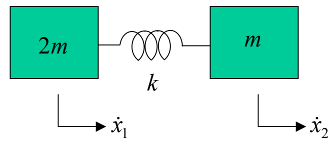
$C = 0$ Undamped $\qquad F(t) = 0$ Unforced

$C \neq 0$ Damped $\qquad F(t) \neq 0$ Forced

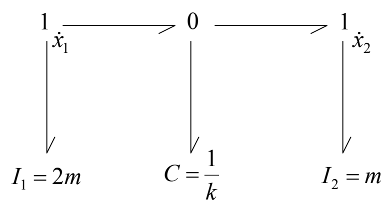
VIBRATIONS – SEMIDEFINITE SYSTEMS

Case Study:

System



Bondgraph



VIBRATIONS

Case Study:

State variable representation

$$\dot{q} = \frac{p_1}{I_1} - \frac{p_2}{I_2}$$
$$\dot{p}_1 = -\frac{q}{C}$$
$$\dot{p}_2 = \frac{q}{C}$$

$$\xrightarrow{\quad\quad\quad}$$

$$q = x_1 - x_2$$
$$p_1 = I_1 v_1$$
$$p_2 = I_2 v_2$$
$$C = \frac{1}{k}, I_1 = 2m, I_2 = m$$

$$\dot{x}_1 = v_1$$
$$\dot{v}_1 = -\frac{k}{2m}(x_1 - x_2)$$
$$\dot{x}_2 = v_2$$
$$\dot{v}_2 = \frac{k}{m}(x_1 - x_2)$$

VIBRATIONS

Case Study:

Vibration representation

$$\dot{x}_1 = v_1$$
$$\dot{v}_1 = -\frac{k}{2m}(x_1 - x_2)$$
$$\dot{x}_2 = v_2$$
$$\dot{v}_2 = \frac{k}{m}(x_1 - x_2)$$

$$\longrightarrow$$

$$2m\ddot{x}_1 + kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 - kx_1 + kx_2 = 0$$

$$\boxed{x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow M\ddot{x} + Kx = 0, \quad M = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}, K = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}$$

VIBRATIONS

Modal Motions:

$$M\ddot{x} + Kx = 0$$

Stiffness matrix

Inertia matrix

Homogeneous Equation

$$x = x_0 \sin(\omega t + \phi) \Rightarrow \left(-\omega^2 M + K\right)x_0 = 0$$
$$\Updownarrow$$
$$A = M^{-1}K, \quad \left(A - \omega^2 I\right)x_0 = 0$$

Eigenvalue Problem!

VIBRATIONS

Modal Motions:

$$\left(-\omega^2 M + K\right)x_0 = 0$$

Characteristic equation:

$$\Delta \equiv \left|-\omega^2 M + K\right| = 0 \longrightarrow \omega_i^2$$

$$\left(-\omega_i^2 M + K\right)x_{0i} = 0 \longrightarrow x_{0i}$$

$$\boxed{x_h(t) = \sum_{i=1}^{n} c_i x_{0i} \sin\left(\omega_i t + \phi_i\right)}$$

VIBRATIONS

Case Study:

$$\boxed{x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow M\ddot{x} + Kx = 0, \quad M = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}, K = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}$$

Characteristic equation:

$$\Delta \equiv \left|-\omega^2 M + K\right| = \begin{vmatrix} -2m\omega^2 + k & -k \\ -k & m\omega^2 + k \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{matrix} \omega_1^2 = 0\dfrac{k}{m}, & x_{01} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \omega_2^2 = \dfrac{3}{2}\dfrac{k}{m}, & x_{02} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \end{matrix}$$

$$\boxed{x_h(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\left(c_{11} + c_{12}t\right) + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \sin\left(\sqrt{\frac{3}{2}\frac{k}{m}}\,t + \phi_2\right)}$$

VIBRATIONS

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$\Big\downarrow$ Vibrations Analysis

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$C = 0$ Undamped ✓✓ $F(t) = 0$ Unforced ✓✓

$C \neq 0$ Damped ✓ $F(t) \neq 0$ Forced ✓✓

VIBRATIONS – MODAL DAMPING

Modal Motions:

$$M\ddot{x} + C\dot{x} + Kx = F$$

Decoupling (modal matrix P based on zero damping)

$$x = Py, \quad P = \begin{bmatrix} x_{01} & x_{02} & \cdots & x_{0n} \end{bmatrix} \longrightarrow MP\ddot{y} + CP\dot{y} + KPy = F$$

$$P'MP\ddot{y} + P'CP\dot{y} + P'KPy = P'F$$

$$M_i\ddot{y}_i + K_i y_i = f_i \qquad \boxed{x_{0i}^{'}Mx_{0j} = 0, \quad x_{0i}^{'}Kx_{0j} = 0}$$

$$M_D = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & M_n \end{bmatrix}, K_D = \begin{bmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K_n \end{bmatrix}, F_D = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \qquad M_D\ddot{y} + C_{ND}\dot{y} + K_D y = F_D$$

C_{ND} is no diagonal

VIBRATIONS – MODAL DAMPING

Special Case: Proportional Damping

$$C = \alpha M + \beta K$$

$$x = Py, \quad P = \begin{bmatrix} x_{01} & x_{02} & \cdots & x_{0n} \end{bmatrix} \longrightarrow MP\ddot{y} + CP\dot{y} + KPy = F$$

$$P'MP\ddot{y} + P'CP\dot{y} + P'KPy = P'F$$

$$\boxed{x_{0i}^{'}Mx_{0j} = 0, \quad x_{0i}^{'}Kx_{0j} = 0}$$

$$C_D = \begin{bmatrix} \alpha M_1 + \beta K_1 & 0 & \cdots & 0 \\ 0 & \alpha M_2 + \beta K_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \alpha M_n + \beta K_n \end{bmatrix}$$

$$M_D\ddot{y} + C_D\dot{y} + K_D y = F_D$$

$$M_i\ddot{y}_i + (\alpha M_i + \beta K_i)\dot{y}_i + K_i y_i = f_i$$