ME242 – MECHANICAL ENGINEERING SYSTEMS

LECTURE 39:

- Vibrations   Appendix B

VIBRATIONS

Linear systems representation:

State Variables representation: $\dot{x} = Ax + Bu(t)$

First-order matrix differential equation

$\downarrow$ Vibrations Analysis

$$M\ddot{x} + C\dot{x} + Kx = F(t)$$

Second-order matrix differential equation

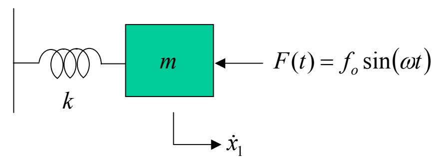
$C = 0$ Undamped ✓✓ $F(t) = 0$ Unforced ✓

$C \neq 0$ Damped $F(t) \neq 0$ Forced ✓

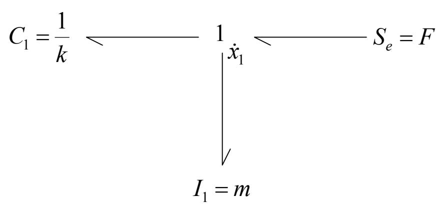
VIBRATIONS

Case Study:

System



Bondgraph



VIBRATIONS

Case Study:

Homogeneous solution

$$x_h(t) = c\sin(\omega_n t + \phi), \qquad \omega_n^2 = \frac{k}{m}$$

Particular solution

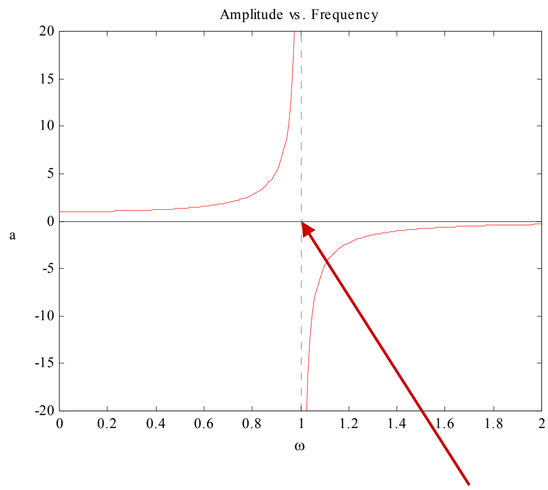
$$x_p(t) = a\sin(\omega t), \qquad a = \frac{f_o}{-m\omega^2 + k}$$

General solution

$$x(t) = x_h(t) + x_p(t)$$

VIBRATIONS

Case Study:



$$a = \frac{f_o}{-m\omega^2 + k}$$

Resonance

RESONANCE? WHAT RESONANCE?

Tacoma Narrows Bridge

At the time it opened for traffic in 1940, the Tacoma Narrows Bridge was the third longest suspension bridge in the world. It was promptly nicknamed "Galloping Gertie," due to its behavior in wind. Not only did the deck sway sideways, but vertical undulations also appeared in quite moderate winds. Drivers of cars reported that vehicles ahead of them would completely disappear and reappear from view several times as they crossed the bridge. Attempts were made to stabilize the structure with cables and hydraulic buffers, but they were unsuccessful. On November 7, 1940, only four months after it opened, the Tacoma Narrows Bridge collapsed in a wind of 42 mph -- even though the structure was designed to withstand winds of up to 120 mph.

Technical experts still disagree on the exact cause of the bridge's destruction, but most agree the collapse had something to do with a complex phenomenon called resonance: the same force that can cause a soprano's voice to shatter a glass.

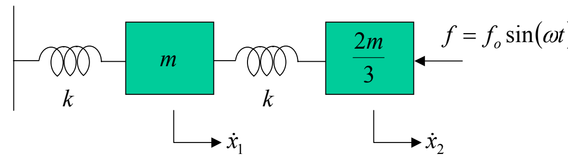
Today, wind tunnel testing of bridge designs is mandatory. As for the Tacoma Narrows bridge, reconstruction began in 1949. The new bridge is wider, has deep stiffening trusses under the roadway and a narrow gap down the middle -- all to dampen the effect of the wind.

Source: British Columbia Institute of Technology

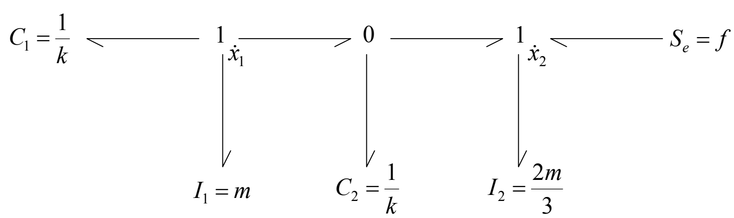
VIBRATIONS

Case Study:

System



Bondgraph



VIBRATIONS

Case Study:

State variable representation

$$\dot{q}_1 = \frac{p_1}{I_1}$$

$$\dot{p}_1 = -\frac{q_1}{C_1} - \frac{q_2}{C_2}$$

$$\dot{q}_2 = \frac{p_1}{I_1} - \frac{p_2}{I_2}$$

$$\dot{p}_2 = \frac{q_2}{C_2} + f$$

$$\longrightarrow$$

$$q_1 = x_1$$

$$q_2 = x_1 - x_2$$

$$p_1 = I_1 v_1$$

$$p_2 = I_2 v_2$$

$$C = \frac{1}{k}, I_1 = m, I_2 = \frac{2}{3} m$$

$$\dot{x}_1 = v_1$$

$$\dot{v}_1 = \frac{1}{m}\left[-kx_1 - k(x_1 - x_2)\right]$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_2 = \frac{1}{\frac{2}{3} m}\left[k(x_1 - x_2) + f\right]$$

VIBRATIONS

Case Study:

State variable representation

$$\dot{x}_1 = v_1$$
$$\dot{v}_1 = \frac{1}{m}\left[-kx_1 - k(x_1 - x_2)\right]$$
$$\dot{x}_2 = v_2$$
$$\dot{v}_2 = \frac{1}{\frac{2}{3}m}\left[k(x_1 - x_2) + f\right]$$

$$x = \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} \Rightarrow \dot{x} = Ax + Bu,$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2k/m & 0 & k/m & 0 \\ 0 & 0 & 0 & 1 \\ 3k/(2m) & 0 & -3k/(2m) & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3/(2m) \end{bmatrix}, u = f$$

VIBRATIONS

Case Study:

Vibration representation

$$\dot{x}_1 = v_1$$
$$\dot{v}_1 = \frac{1}{m}\left[-kx_1 - k(x_1 - x_2)\right]$$
$$\dot{x}_2 = v_2$$
$$\dot{v}_2 = \frac{1}{\frac{2}{3}m}\left[k(x_1 - x_2) + f\right]$$

$$\longrightarrow$$

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$
$$\frac{2}{3}m\ddot{x}_2 - kx_1 + kx_2 = f$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow M\ddot{x} + Kx = F, \quad M = \begin{bmatrix} m & 0 \\ 0 & 2m/3 \end{bmatrix}, K = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}, F = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

VIBRATIONS

Modal Motions:

$$M\ddot{x} + Kx = 0$$

Homogeneous Equation

Stiffness matrix

Inertia matrix

$$x = x_0 \sin(\omega t + \phi) \Rightarrow \left(-\omega^2 M + K\right) x_0 = 0$$

$$\Updownarrow$$

$$A = M^{-1}K, \quad \left(A - \omega^2 I\right) x_0 = 0$$

Eigenvalue Problem!

VIBRATIONS

Modal Motions:

$$\left(-\omega^2 M + K\right) x_0 = 0$$

Characteristic equation:

$$\Delta \equiv \left|-\omega^2 M + K\right| = 0 \longrightarrow \omega_i^2$$

$$\left(-\omega_i^2 M + K\right) x_{0i} = 0 \longrightarrow x_{0i}$$

$$\boxed{x_h(t) = \sum_{i=1}^{n} c_i x_{0i} \sin(\omega_i t + \phi_i)}$$

VIBRATIONS

Case Study:

$$\boxed{x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow M\ddot{x} + Kx = 0, \quad M = \begin{bmatrix} m & 0 \\ 0 & 2m/3 \end{bmatrix}, K = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}}$$

Characteristic equation:

$$\Delta \equiv \left| -\omega^2 M + K \right| = \begin{vmatrix} -m\omega^2 + 2k & -k \\ -k & -\frac{2}{3}m\omega^2 + k \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{array}{ll} \omega_1^2 = 3\frac{k}{m}, & x_{01} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ \omega_2^2 = \frac{1}{2}\frac{k}{m}, & x_{02} = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \end{array}$$

$$\boxed{x_h(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \sin\left(\sqrt{3\frac{k}{m}}t + \phi_1\right) + c_2 \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \sin\left(\sqrt{\frac{1}{2}\frac{k}{m}}t + \phi_2\right)}$$

VIBRATIONS

Modal Motions:

$$M\ddot{x} + Kx = F$$

Decoupling

$$x = Py, \quad P = \begin{bmatrix} x_{01} & x_{02} & \cdots & x_{0n} \end{bmatrix} \longrightarrow MP\ddot{y} + KPy = F$$

$$P'MP\ddot{y} + P'KPy = P'F$$

$$\boxed{x_{0i}^{'}Mx_{0j} = 0, \quad x_{0i}^{'}Kx_{0j} = 0}$$

$$M_D = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & M_n \end{bmatrix}, K_D = \begin{bmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K_n \end{bmatrix}, F_D = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \quad \begin{array}{c} M_D\ddot{y} + K_D y = F_D \\ M_i\ddot{y}_i + K_i y_i = f_i \end{array}$$

VIBRATIONS

Case Study:

$$x=\begin{bmatrix}x_1\\x_2\end{bmatrix}\Rightarrow M\ddot{x}+Kx=F,\quad M=\begin{bmatrix}m & 0\\0 & 2m/3\end{bmatrix},K=\begin{bmatrix}2k & -k\\-k & k\end{bmatrix},F=\begin{bmatrix}0\\f\end{bmatrix}$$

Transformation matrix:

$$x_{01}=\begin{bmatrix}-1\\1\end{bmatrix},x_{02}=\begin{bmatrix}2/3\\1\end{bmatrix}\Rightarrow P=\begin{bmatrix}2/3 & -1\\1 & 1\end{bmatrix}\quad y=Px\Rightarrow\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}\frac{2}{3}x_1-x_2\\x_1+x_2\end{bmatrix}\Leftrightarrow\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}\frac{3}{5}(y_1+y_2)\\-\frac{3}{5}\left(y_1-\frac{2}{3}y_2\right)\end{bmatrix}$$

$$M_D=P'MP=\begin{bmatrix}\frac{10}{9}m & 0\\0 & \frac{5}{3}m\end{bmatrix},K_D=P'KP=\begin{bmatrix}\frac{5}{9}k & 0\\0 & 5k\end{bmatrix},F_D=\begin{bmatrix}f\\f\end{bmatrix}\qquad\Rightarrow M_D\ddot{y}+K_Dy=F_D$$

$$\frac{10}{9}m\ddot{y}_1+\frac{5}{9}ky_1=f$$

$$\frac{5}{3}m\ddot{y}_2+5ky_2=f$$

VIBRATIONS

Case Study:

$$x=\begin{bmatrix}x_1\\x_2\end{bmatrix}\Rightarrow M\ddot{x}+Kx=F,\quad M=\begin{bmatrix}m & 0\\0 & 2m/3\end{bmatrix},K=\begin{bmatrix}2k & -k\\-k & k\end{bmatrix},F=\begin{bmatrix}0\\f\end{bmatrix}$$

$$\frac{10}{9}m\ddot{y}_1+\frac{5}{9}ky_1=f \qquad\longrightarrow\qquad y_{1p}=\frac{f_o}{-\frac{10}{9}m\omega^2+\frac{5}{9}k}\sin(\omega t)=a_1\sin(\omega t)$$

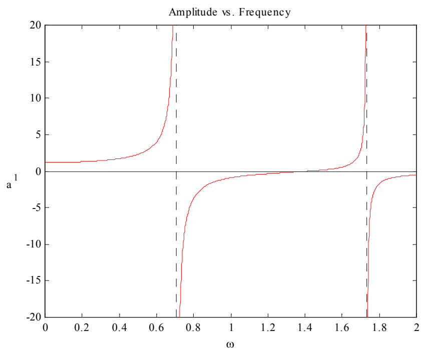
$$\frac{5}{3}m\ddot{y}_2+5ky_2=f \qquad\longrightarrow\qquad y_{2p}=\frac{f_o}{-\frac{5}{3}m\omega^2+5k}\sin(\omega t)=a_2\sin(\omega t)$$

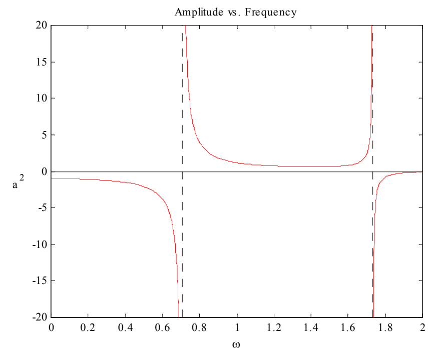
$$x_{1p}=\frac{3}{5}(y_{1p}+y_{2p}) \qquad\longrightarrow\qquad x_{1p}=\frac{3}{5}\left(\frac{1}{-\frac{10}{9}m\omega^2+\frac{5}{9}k}+\frac{1}{-\frac{5}{3}m\omega^2+5k}\right)f_o\sin(\omega t)$$

$$x_{2p}=-\frac{3}{5}\left(y_{1p}-\frac{2}{3}y_{2p}\right) \qquad\longrightarrow\qquad x_{2p}=-\frac{3}{5}\left(\frac{1}{-\frac{10}{9}m\omega^2+\frac{5}{9}k}-\frac{2}{3}\frac{1}{-\frac{5}{3}m\omega^2+5k}\right)f_o\sin(\omega t)$$

VIBRATIONS

Case Study:





$$a_1 = \frac{3}{5}\left(\frac{1}{-\frac{10}{9}m\omega^2 + \frac{5}{9}k} + \frac{1}{-\frac{5}{3}m\omega^2 + 5k}\right)f_o \qquad a_2 = -\frac{3}{5}\left(\frac{1}{-\frac{10}{9}m\omega^2 + \frac{5}{9}k} - \frac{2}{3}\frac{1}{-\frac{5}{3}m\omega^2 + 5k}\right)f_o$$

VIBRATIONS

Case Study:

$$\boxed{x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow M\ddot{x} + Kx = F, \quad M = \begin{bmatrix} m & 0 \\ 0 & 2m/3 \end{bmatrix}, K = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}, F = \begin{bmatrix} 0 \\ f \end{bmatrix}}$$

$$x(t) = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}\sin\left(\sqrt{3\frac{k}{m}}t + \phi_1\right) + c_2\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}\sin\left(\sqrt{\frac{1}{2}\frac{k}{m}}t + \phi_2\right) + \begin{bmatrix} \frac{3}{5}\left(\frac{1}{-\frac{10}{9}m\omega^2 + \frac{5}{9}k} + \frac{1}{-\frac{5}{3}m\omega^2 + 5k}\right) \\ -\frac{3}{5}\left(\frac{1}{-\frac{10}{9}m\omega^2 + \frac{5}{9}k} - \frac{2}{3}\frac{1}{-\frac{5}{3}m\omega^2 + 5k}\right) \end{bmatrix}\sin(\omega t)$$