ME242 – MECHANICAL ENGINEERING SYSTEMS

LECTURE 38:

- Vibrations — Appendix B

VIBRATIONS

Linear systems representation:

State Variables representation: $\dot{x} = Ax + Bu(t)$

First-order matrix differential equation

Vibrations Analysis

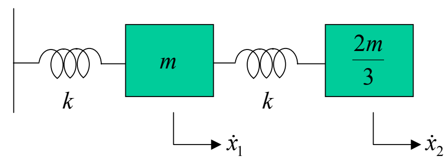
$$M\ddot{x} + C\dot{x} + Kx = F(t)$$

Second-order matrix differential equation

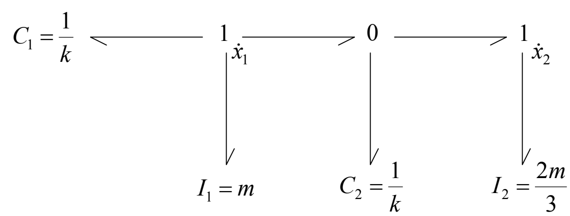
VIBRATIONS

Case Study:

System



Bondgraph



VIBRATIONS

Case Study:

State variable representation

$$\dot{q}_1 = \frac{p_1}{I_1}$$

$$\dot{p}_1 = -\frac{q_1}{C_1} - \frac{q_2}{C_2}$$

$$\dot{q}_2 = \frac{p_1}{I_1} - \frac{p_2}{I_2}$$

$$\dot{p}_2 = \frac{q_2}{C_2}$$

$$\longrightarrow$$

$$q_1 = x_1$$

$$q_2 = x_1 - x_2$$

$$p_1 = I_1 v_1$$

$$p_2 = I_2 v_2$$

$$C = \frac{1}{k}, I_1 = m, I_2 = \frac{2}{3}m$$

$$\dot{x}_1 = v_1$$

$$\dot{v}_1 = \frac{1}{m}\left[-kx_1 - k(x_1 - x_2)\right]$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_2 = \frac{1}{{}^{2}\!/_{3}\, m} k(x_1 - x_2)$$

VIBRATIONS

Case Study:

State variable representation

$$\dot{x}_1 = v_1$$
$$\dot{v}_1 = \frac{1}{m}\left[-kx_1 - k(x_1 - x_2)\right]$$
$$\dot{x}_2 = v_2$$
$$\dot{v}_2 = \frac{1}{\frac{2}{3}m}k(x_1 - x_2)$$

$$\boxed{x = \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} \Rightarrow \dot{x} = Ax, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2k/m & 0 & k/m & 0 \\ 0 & 0 & 0 & 1 \\ 3k/2m & 0 & -3k/2m & 0 \end{bmatrix}}$$

VIBRATIONS

Case Study:

Vibration representation

$$\dot{x}_1 = v_1$$
$$\dot{v}_1 = \frac{1}{m}\left[-kx_1 - k(x_1 - x_2)\right]$$
$$\dot{x}_2 = v_2$$
$$\dot{v}_2 = \frac{1}{\frac{2}{3}m}k(x_1 - x_2)$$

$$\longrightarrow$$

$$m\ddot{x}_1 = -2kx_1 + kx_2$$
$$\frac{2}{3}m\ddot{x}_2 = kx_1 - kx_2$$

$$\boxed{x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow M\ddot{x} + Kx = 0, \quad M = \begin{bmatrix} m & 0 \\ 0 & 2m/3 \end{bmatrix}, K = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}}$$

VIBRATIONS

Case Study:

Vibration representation

$$\dot{x}_1 = v_1$$
$$\dot{v}_1 = \frac{1}{m}\left[-kx_1 - k(x_1 - x_2)\right]$$
$$\dot{x}_2 = v_2$$
$$\dot{v}_2 = \frac{1}{2/3\, m} k(x_1 - x_2)$$

$$\longrightarrow$$

$$m\ddot{x}_1 = -2kx_1 + kx_2$$
$$\frac{2}{3} m\ddot{x}_2 = kx_1 - kx_2$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow M\ddot{x} + Kx = 0, \quad M = \begin{bmatrix} m & 0 \\ 0 & 2m/3 \end{bmatrix}, K = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}$$

VIBRATIONS

Modal Motions:

$$M\ddot{x} + Kx = 0$$

Stiffness matrix

Inertia matrix

$$x = x_0 \sin(\omega t + \phi) \Rightarrow \left(-\omega^2 M + K\right)x_0 = 0$$
$$\Updownarrow$$
$$A = M^{-1}K, \quad \left(A - \omega^2 I\right)x_0 = 0$$

Eigenvalue Problem!

VIBRATIONS

Modal Motions:

$$\left(-\omega^2 M + K\right)x_0 = 0$$

Characteristic equation:

$$\Delta \equiv \left|-\omega^2 M + K\right| = 0 \longrightarrow \omega_i^2$$

$$\left(-\omega_i^2 M + K\right)x_{0i} = 0 \longrightarrow x_{0i}$$

$$\boxed{x(t) = \sum_{i=1}^{n} c_i x_{0i} \sin(\omega_i t + \phi_i)}$$

VIBRATIONS

Case Study:

$$\boxed{x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow M\ddot{x} + Kx = 0, \quad M = \begin{bmatrix} m & 0 \\ 0 & 2m/3 \end{bmatrix}, K = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}}$$

Characteristic equation:

$$\Delta \equiv \left|-\omega^2 M + K\right| = \begin{vmatrix} -m\omega^2 + 2k & -k \\ -k & -\frac{2}{3}m\omega^2 + k \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{matrix} \omega_1^2 = 3\frac{k}{m}, & x_{01} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ \omega_2^2 = \frac{1}{2}\frac{k}{m}, & x_{02} = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} \end{matrix}$$

$$\boxed{x(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \sin\left(\sqrt{3\frac{k}{m}}t + \phi_1\right) + c_2 \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} \sin\left(\sqrt{\frac{1}{2}\frac{k}{m}}t + \phi_2\right)}$$

VIBRATIONS

Modal Motions:

$$\left(-\omega^2 M + K\right)x_0 = 0$$

Properties of eigenvectors

$$\omega_i^2 M x_{0i} = K x_{0i} \Rightarrow \omega_i^2 x_{0j}' M x_{0i} = x_{0j}' K x_{0i} \Rightarrow \omega_i^2 x_{0i}' M x_{0j} = x_{0i}' K x_{0j}$$

$$\omega_j^2 M x_{0j} = K x_{0j} \Rightarrow \qquad \omega_j^2 x_{0i}' M x_{0j} = x_{0i}' K x_{0j}$$

$$\left(\omega_i^2 - \omega_j^2\right) x_{0i}' M x_{0j} = 0$$

$$\left(\frac{1}{\omega_i^2} - \frac{1}{\omega_j^2}\right) x_{0i}' K x_{0j} = 0$$

$$\Downarrow$$

$$x_{0i}' M x_{0j} = 0, \quad x_{0i}' K x_{0j} = 0$$

VIBRATIONS

Modal Motions:

$$M\ddot{x} + Kx = 0$$

Decoupling

$$x = Py, \qquad P = \begin{bmatrix} x_{01} & x_{02} & \cdots & x_{0n} \end{bmatrix} \longrightarrow M P\ddot{y} + KPy = 0$$

$$P' M P\ddot{y} + P' K P y = 0$$

$$x_{0i}' M x_{0j} = 0, \quad x_{0i}' K x_{0j} = 0$$

$$M_D = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & M_n \end{bmatrix}, K_D = \begin{bmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & K_n \end{bmatrix}$$

$$M_D \ddot{y} + K_D y = 0$$

$$M_i \ddot{y}_i + K_i y_i = 0$$

VIBRATIONS

Case Study:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow M\ddot{x} + Kx = 0, \quad M = \begin{bmatrix} m & 0 \\ 0 & 2m/3 \end{bmatrix}, K = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}$$

Transformation matrix:

$$x_{01} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, x_{02} = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} \Rightarrow P = \begin{bmatrix} -1 & 1 \\ 1 & 3/2 \end{bmatrix} \quad y = Px \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ x_1 + \frac{3}{2}x_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{5}\left(\frac{3}{2}y_1 - y_2\right) \\ \frac{2}{5}(y_1 + y_2) \end{bmatrix}$$

$$M_D = P'MP = \begin{bmatrix} \frac{5}{3}m & 0 \\ 0 & \frac{5}{2}m \end{bmatrix}, K_D = P'KP = \begin{bmatrix} 5k & 0 \\ 0 & \frac{5}{4}k \end{bmatrix} \quad \Rightarrow M_D\ddot{y} + K_D y = 0$$

$$\frac{5}{3}m\ddot{y}_1 + 5ky_1 = 0$$

$$\frac{5}{2}m\ddot{y}_2 + \frac{5}{4}ky_2 = 0$$