ME242 – MECHANICAL ENGINEERING SYSTEMS

LECTURE 22:

- Numerical Methods

INTEGRATION OF ODE's

Given a scalar, first-order differential equation of the form

$$\frac{dy}{dt} = f(y,t) \qquad \text{with} \qquad y(t_o) = y_o$$

Goals:

- Advancement of the system in time by integration of ODE
- The quantity being integrated, f, is itself a function of the result of the integration, y.
- ODE's of higher order are generalization of this first order case

Method:

- We seek to approximate the solution $y(t)$ at timestep $t_{n+1}=t_n+h$ given t_o and the solution at the previously-computed timesteps t_1 to t_n

Note:

- ODE's of higher order are generalization of this first order case

INTEGRATION OF ODE's – EXPLICIT EULER

Considering the Taylor series expansion of $y(t)$ at $t_{n+1}=t_n+h$

$$y(t_{n+1}) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(t_n) + \frac{h^3}{6}y'''(t_n) + \cdots$$

Denoting the numerical approximation of $y(t_n)$ as y_n, and recalling that

$$y_n' = f(y_n, t_n)$$

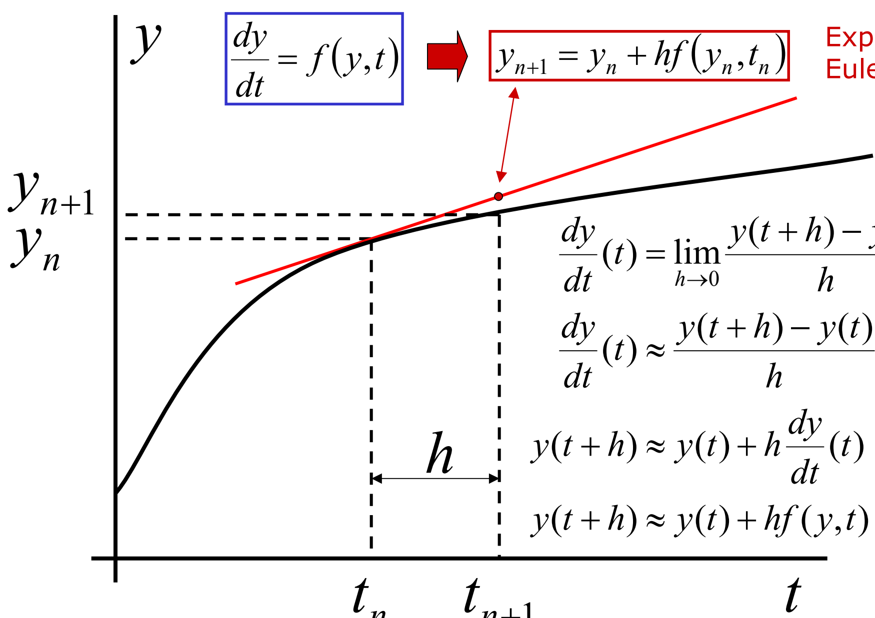
the time integration based on the first two terms of the series expansion is given by

$$y_{n+1} = y_n + hf(y_n, t_n)$$ Explicit Euler Method

Lecture22a.m

INTEGRATION OF ODE's – EXPLICIT EULER

$$\boxed{\frac{dy}{dt} = f(y,t)} \Rightarrow \boxed{y_{n+1} = y_n + hf(y_n, t_n)}$$ Explicit Euler



$$\frac{dy}{dt}(t) = \lim_{h\to 0}\frac{y(t+h)-y(t)}{h}$$

$$\frac{dy}{dt}(t) \approx \frac{y(t+h)-y(t)}{h}$$

$$y(t+h) \approx y(t) + h\frac{dy}{dt}(t)$$

$$y(t+h) \approx y(t) + hf(y,t)$$

INTEGRATION OF ODE's – EXPLICIT EULER

Consider the following scalar problem

$$\dot{y} = \lambda y \qquad \text{with} \qquad y(0) = y_o$$

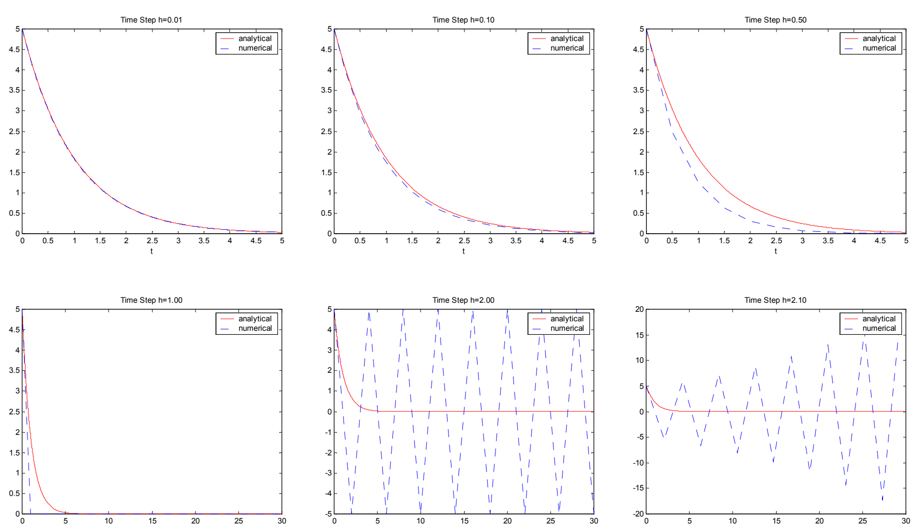
We note that the analytical solution to this problem is

$$y(t) = y_o e^{\lambda t}$$

We now use the Euler Method to find a numerical solution

Explicit Euler Method: $$y_{n+1} = y_n + h\lambda y_n$$

INTEGRATION OF ODE's – EXPLICIT EULER



INTEGRATION OF ODE's – IMPLICIT EULER

Considering the Taylor series expansion of $y(t)$ at $t_n=t_{n+1}-h$

$$y(t_n)=y(t_{n+1})-hy'(t_{n+1})+\frac{h^2}{2}y''(t_{n+1})-\frac{h^3}{6}y'''(t_{n+1})+\cdots$$

Denoting the numerical approximation of $y(t_n)$ as y_n, and recalling that

$$y'_{n+1}=f(y_{n+1},t_{n+1})$$

the time integration based on the first two terms of the series expansion is given by

$$y_{n+1}=y_n+hf(y_{n+1},t_{n+1})$$ Implicit Euler Method

Lecture22b.m

NOTE: Difficult to implement when f is nonlinear in $y(t)$

INTEGRATION OF ODE's – IMPLICIT EULER

Consider the following scalar problem

$$\dot{y}=\lambda y \qquad \text{with} \qquad y(0)=y_o$$

We note that the analytical solution to this problem is

$$y(t)=y_o e^{\lambda t}$$

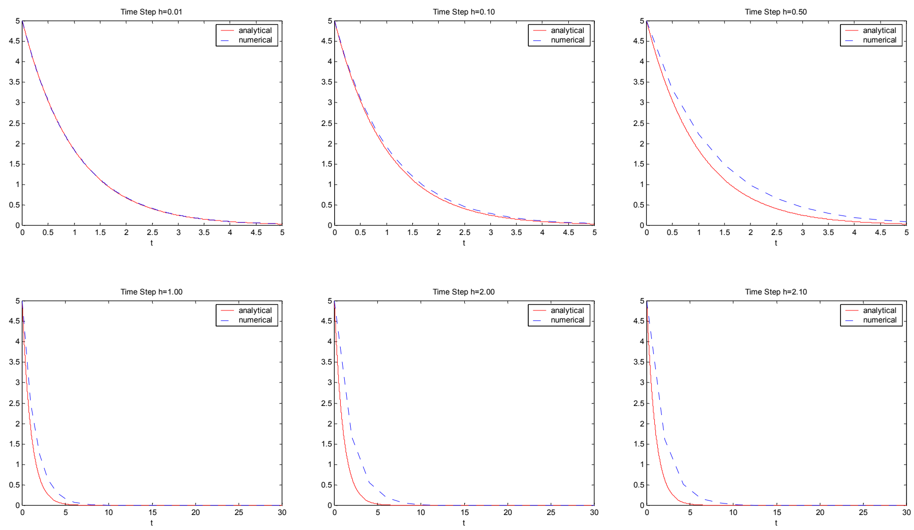
We now use the Euler Method to find a numerical solution

Implicit Euler Method: $y_{n+1}=y_n+h\lambda y_{n+1}$

$$\Downarrow$$

$$y_{n+1}=\frac{1}{1-h\lambda}y_n$$

INTEGRATION OF ODE's – IMPLICIT EULER



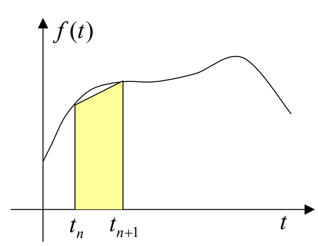
INTEGRATION OF ODE's – CRANK-NICHOLSON

The formal solution of $\frac{dy}{dt} = f(y,t)$

Over the interval $[t_n, t_{n+1}]$ is given by

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(y,t)dt$$

Approximating the integral with the trapezoidal rule



$$\int_{t_n}^{t_{n+1}} f(y,t)dt \approx h\frac{f(y_n,t_n)+f(y_{n+1},t_{n+1})}{2}$$

We obtain a new integration method:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(y_n,t_n)+f(y_{n+1},t_{n+1})\right]$$

Crank-Nicholson Method

Lecture22c.m

INTEGRATION OF ODE's – CRANK-NICHOLSON

Consider the following scalar problem

$$\dot{y} = \lambda y \qquad \text{with} \qquad y(0) = y_o$$

We note that the analytical solution to this problem is

$$y(t) = y_o e^{\lambda t}$$

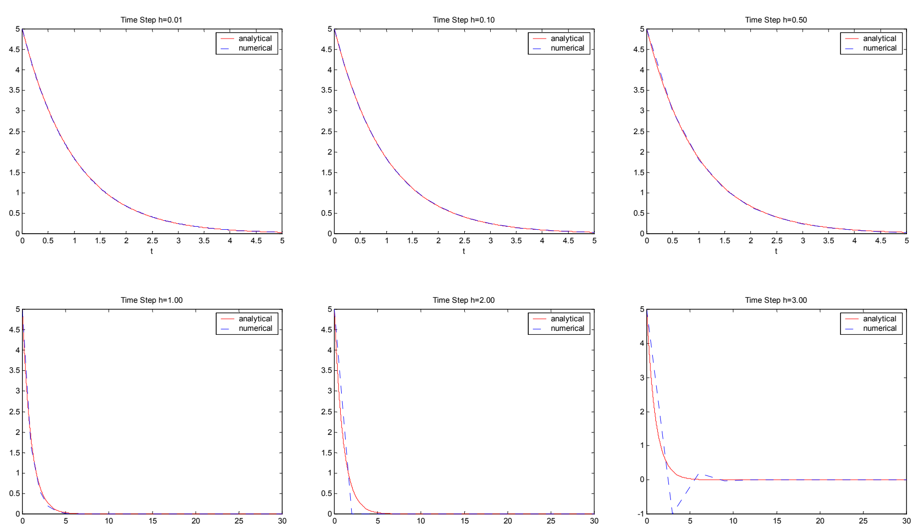
We now use the Euler Method to find a numerical solution

Crank-Nicholson Method: $y_{n+1} = y_n + \frac{h}{2}\lambda\left[y_n + y_{n+1}\right]$

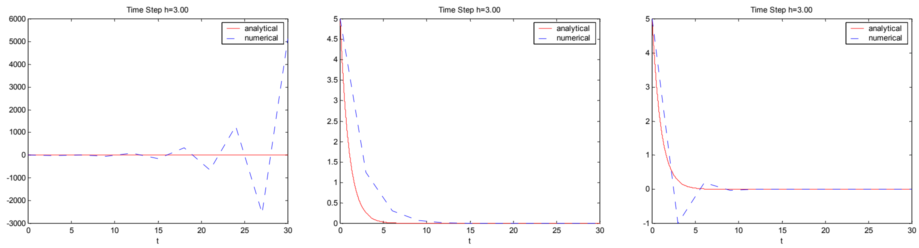
$$\Downarrow$$

$$y_{n+1} = \frac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}} y_n = \frac{2+h\lambda}{2-h\lambda} y_n$$

INTEGRATION OF ODE's – CRANK-NICHOLSON



INTEGRATION OF ODE's - STABILITY



EXPLICIT EULER   IMPLICIT EULER   CRANK-NICHOLSON

INTEGRATION OF ODE's - STABILITY

Consider the following scalar problem

$$\dot{y} = \lambda y \qquad \text{with} \qquad y(0) = y_o$$

Example: Explicit Euler Method $\quad y_{n+1} = y_n + h\lambda y_n = (1 + h\lambda) y_n$

Thus, assuming h constant, the solution at time step n is

$$y_n = (1 + h\lambda)^n y_o \equiv \sigma^n y_o \Rightarrow \sigma = 1 + h\lambda$$

For large n, the numerical solution remains stable iff

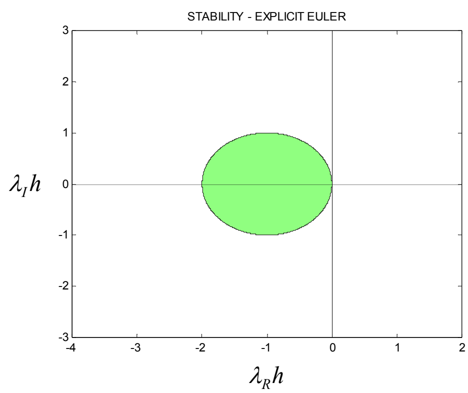
$$|\sigma| \leq 1 \Rightarrow (1 + h\lambda_R)^2 + (h\lambda_I)^2 \leq 1 \qquad \text{assuming } \lambda = \lambda_R + i\lambda_I$$

INTEGRATION OF ODE's - STABILITY

Consider the following scalar problem

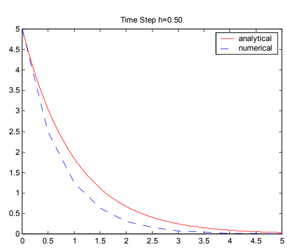
$$\dot{y} = \lambda y \qquad \text{with} \qquad y(0) = y_o$$

Explicit Euler Method – Stability Region:

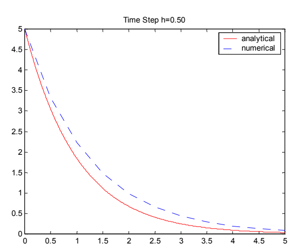


Stability.m

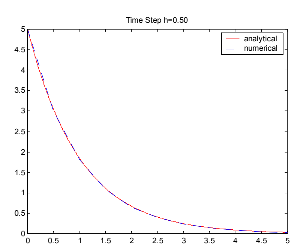
INTEGRATION OF ODE's - ACCURACY



EXPLICIT EULER



IMPLICIT EULER



CRANK-NICHOLSON

INTEGRATION OF ODE's - ACCURACY

Consider the following scalar problem

$$\dot{y} = \lambda y \qquad \text{with} \qquad y(0) = y_o$$

Exact Solution:

$$y(t_n) = y_o e^{\lambda t_n} = y_o \left(e^{\lambda h}\right)^n = y_o \left(1 + \lambda h + \frac{\lambda^2 h^2}{2} + \frac{\lambda^3 h^3}{6}\right)^n$$

Example: Explicit Euler Method:

$$y_{n+1} = y_n + h\lambda y_n = (1 + h\lambda) y_n$$

$$y_n = y_o (1 + h\lambda)^n$$

By comparison, the leading error is proportional to h^2

EXPLICIT EULER is first-order accurate