ME242 – MECHANICAL ENGINEERING SYSTEMS

LECTURE 21:

- Phase Plane

PHASE PLANE – 1D

Given $\dot{x} = f(x)$ We are interested in the behavior of the trajectories

Stable Node:

$$\dot{x} = -x$$

Critical Point: $\bar{x} = 0$

Unstable Node:

$$\dot{x} = x^3$$

Critical Point: $\bar{x} = 0$

Saddle Point:

$$\dot{x} = x^2$$

Critical Point: $\bar{x} = 0$

PHASE PLANE – 1D

Examples: $\dot{x} = x(x-1)$

PHASE PLANE – 2D

Given $\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$ We are interested in the behavior of the trajectories

Local analysis around critical point $\rightarrow$ linearization

eigenvector eigenvalue

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} e^{\lambda_2 t}$$

EIGENVALUES AND EIGENVECTORS

Eigenvalues and Eigenvectors:

$$Av = \lambda v$$

Solution?:

$$Av = \lambda I v$$

$$(\lambda I - A)v = 0$$

A non-trivial solution requires

$$\det(\lambda I - A) = 0 \quad \Rightarrow \lambda$$

Known λ we can compute v from

$$(\lambda I - A)v = 0 \quad \Rightarrow v$$

LINEARIZATION – 2D PHASE PLANE

Linear System:

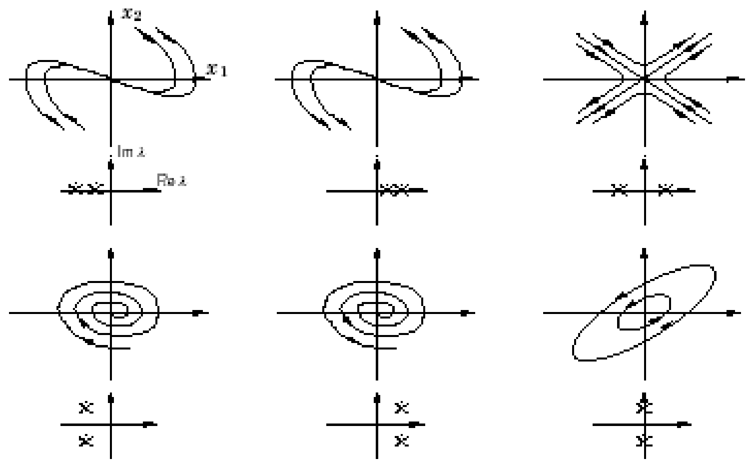
$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} e^{\lambda_2 t}$$

Classification of the equilibrium

If λ_1 and λ_2 are negative real	STABLE NODE
If λ_1 and λ_2 are positive real	UNSTABLE NODE
If λ_1 is positive real and λ_2 is negative real	SADDLE POINT
If λ_1 and λ_2 are imaginary conjugates	CENTER POINT
If λ_1 and λ_2 are complex conjugates	
Real part negative	STABLE FOCUS
Real part positive	UNSTABLE FOCUS

LINEARIZATION – 2D PHASE PLANE

STABLE NODE　　UNSTABLE NODE　　SADDLE POINT



STABLE FOCUS　　UNSTABLE FOCUS　　CENTER POINT

LINEARIZATION – 2D PHASE PLANE

Examples: $\dot{x} = Ax$

$$A = \begin{bmatrix} -1 & 1 \\ 1/4 & -1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 1/4 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 1 \\ -4 & -1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

2D PHASE PLANE

Examples: $$\dot{x}_1 = x_1(3 - x_1 - x_2)$$
$$\dot{x}_2 = x_2(x_1 - 1)$$