ME242 – MECHANICAL ENGINEERING SYSTEMS

LECTURE 20:

- Linearization 5.4

LINEARIZATION – n^{th} ORDER ODEs

State Variable Representation:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \boldsymbol{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

LINEARIZATION – n^th ORDER ODEs

State Variable Representation:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$$

The equilibrium solution is given by

$$\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}} : \quad 0 = \boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$$

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} f_1(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n, \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m) \\ f_2(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n, \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m) \\ \vdots \\ f_n(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n, \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m) \end{bmatrix}$$

LINEARIZATION – n^th ORDER ODEs

We define the perturbation variables as

$$\boldsymbol{x} = \bar{\boldsymbol{x}} + \boldsymbol{x}^*$$

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{u}^*$$

Then we can write

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$$

as (by Taylor series expansion)

$$\frac{d\boldsymbol{x}}{dt} = \cancel{\frac{d\bar{\boldsymbol{x}}}{dt}} + \frac{d\boldsymbol{x}^*}{dt} = \cancel{\boldsymbol{f}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})} + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\substack{\boldsymbol{x}=\bar{\boldsymbol{x}} \\ \boldsymbol{u}=\bar{\boldsymbol{u}}}} \boldsymbol{x}^* + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{\substack{\boldsymbol{x}=\bar{\boldsymbol{x}} \\ \boldsymbol{u}=\bar{\boldsymbol{u}}}} \boldsymbol{u}^* + HOT$$

LINEARIZATION – n^{th} ORDER ODEs

After neglecting the HOT, we have

$$\frac{d\boldsymbol{x}^*}{dt} = A\boldsymbol{x}^* + B\boldsymbol{u}^*$$

$$A = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\substack{\boldsymbol{x}=\bar{\boldsymbol{x}}\\ \boldsymbol{u}=\bar{\boldsymbol{u}}}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\substack{\boldsymbol{x}=\bar{\boldsymbol{x}}\\ \boldsymbol{u}=\bar{\boldsymbol{u}}}} , \quad B = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{\substack{\boldsymbol{x}=\bar{\boldsymbol{x}}\\ \boldsymbol{u}=\bar{\boldsymbol{u}}}} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{\substack{\boldsymbol{x}=\bar{\boldsymbol{x}}\\ \boldsymbol{u}=\bar{\boldsymbol{u}}}}$$

LINEARIZATION – n^{th} ORDER ODEs

Examples:

LINEARIZATION – PHASE PLANE

State Variable Representation:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \underset{\text{Linearization around an equilibrium}}{\Rightarrow} \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The solution of the linearized equation can be written as:

eigenvector eigenvalue

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} e^{\lambda_2 t}$$

LINEARIZATION – PHASE PLANE

State Variable Representation:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \underset{\text{Linearization around an equilibrium}}{\Rightarrow} \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The solution of the linearized equation can be written as:

eigenvector eigenvalue

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} e^{\lambda_2 t}$$

LINEARIZATION – PHASE PLANE

Eigenvalues and Eigenvectors:

$$Av = \lambda v$$

Solution?:

$$Av = \lambda I v$$

$$(\lambda I - A)v = 0$$

A non-trivial solution requires

$$\det(\lambda I - A) = 0 \quad \Rightarrow \lambda$$

Known λ we can compute v from

$$(\lambda I - A)v = 0 \quad \Rightarrow v$$

LINEARIZATION – STABILITY

$$\frac{dx}{dt} = f(x) \underset{\text{Linearization around an equilibrium}}{\Rightarrow} \frac{dx^*}{dt} = Ax^* \underset{\text{Solution}}{\Rightarrow} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} e^{\lambda_2 t}$$

If the linearized system is strictly stable (i.e., if all eigenvalues of A are strictly in the left-half complex plane), then the equilibrium point is asymptotically stable (for the actual nonlinear system)

If the linearized system is unstable (i.e., if at least one eigenvalue of A is strictly in the right-half complex plane), then the equilibrium point is unstable (for the actual nonlinear system)

If the linearized system is marginally stable (i.e., if all eigenvalues of A are in the left-half complex plane, but at least one of them is on the imaginary axis), then one cannot conclude anything from the linear approximation (the equilibrium point may be stable, asymptotically stable or unstable for the actual nonlinear system)

LINEARIZATION – PHASE PLANE

Examples: