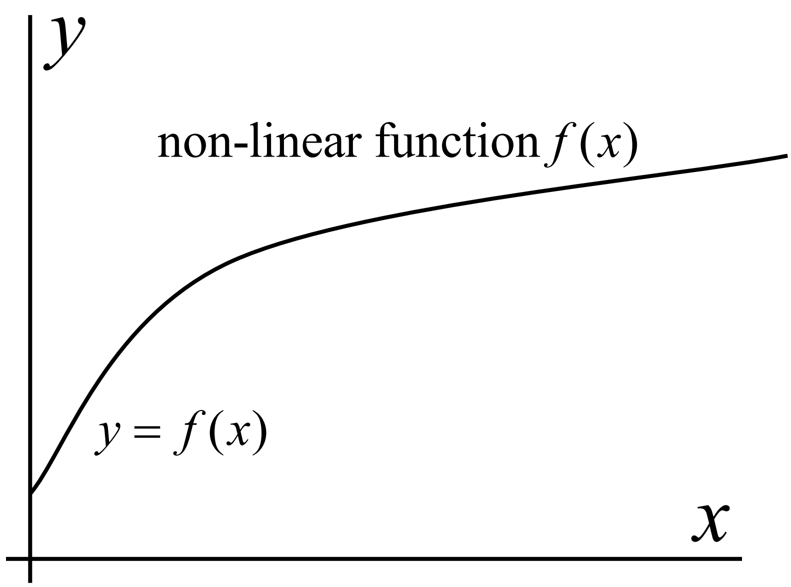
ME242 – MECHANICAL ENGINEERING SYSTEMS

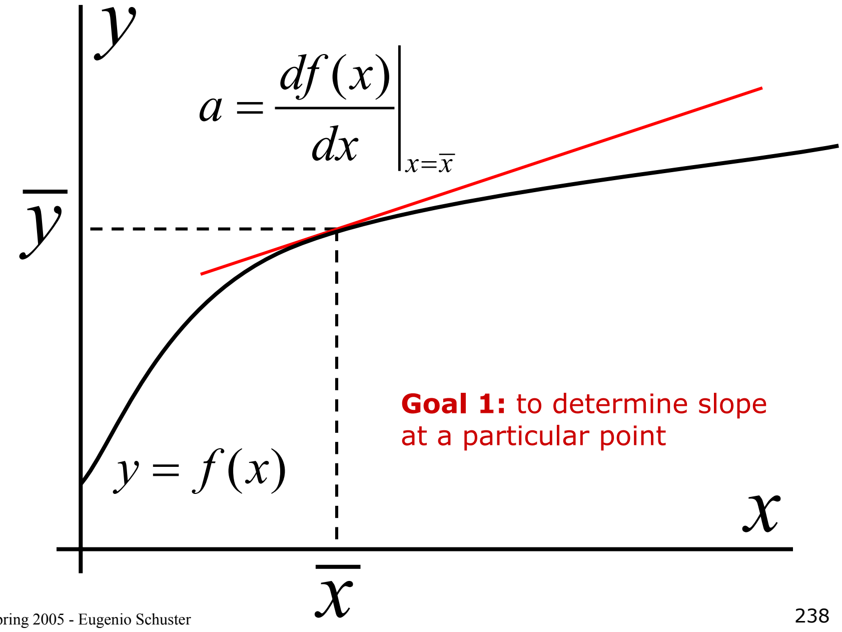
LECTURE 19:

- Linearization 5.4

LINEARIZATION – FUNCTION OF ONE VARIABLE

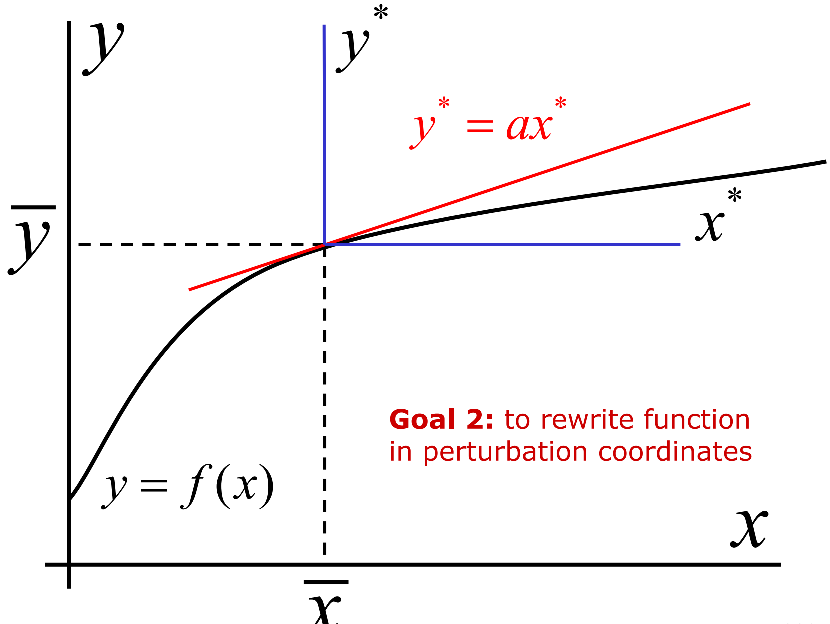


LINEARIZATION – FUNCTION OF ONE VARIABLE



Goal 1: to determine slope at a particular point

LINEARIZATION – FUNCTION OF ONE VARIABLE



Goal 2: to rewrite function in perturbation coordinates

LINEARIZATION – FUNCTION OF ONE VARIABLE

non-linear function $f(x)$ $\quad y = f(x)$

nominal value of x is $\bar{x}$

nominal value of y is $\bar{y}$

Nominal solution of $f(x)$ is $f(x)\big|_{x=\bar{x}} = f(\bar{x})$

Or $\quad \bar{y} = f(\bar{x})$

May be given $\bar{x}$ and have to solve for $\bar{y}$ or vice-versa

LINEARIZATION – FUNCTION OF ONE VARIABLE

$$y = f(x)$$

take Taylor series about nominal point

$$y = f(\bar{x}) + \left.\frac{df}{dx}\right|_{x=\bar{x}} (x - \bar{x}) + \left.\frac{d^2 f}{dx^2}\right|_{x=\bar{x}} \frac{(x - \bar{x})^2}{2} + \textit{higher order terms}$$

actual values

$$x = \bar{x} + x^*$$

$$y = \bar{y} + y^*$$

pertubation values

$$x^* = x - \bar{x}$$

$$y^* = y - \bar{y}$$

LINEARIZATION – FUNCTION OF ONE VARIABLE

Taylor series in perturbation coordinates,

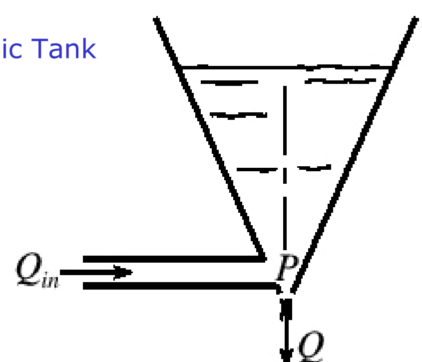
$$y - f(\bar{x}) = y - \bar{y} = y^* = \left.\frac{df}{dx}\right|_{x=\bar{x}} x^* + \cancel{\left.\frac{d^2 f}{dx^2}\right|_{x=\bar{x}} \frac{x^{*2}}{2}} + \cancel{\text{higher order terms}}$$

Keeping only the linear (first order) terms

$$y^* = a x^* \qquad \text{where} \qquad a = \left.\frac{df}{dx}\right|_{x=\bar{x}}$$

LINEARIZATION – FUNCTION OF ONE VARIABLE

Example: Conic Tank



Resistance

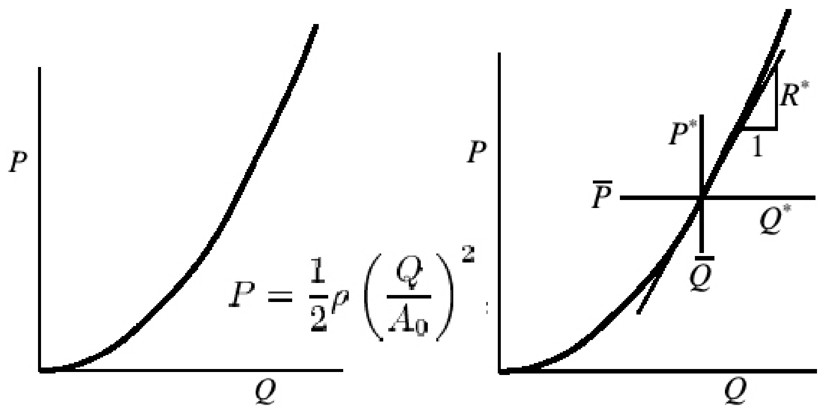
$$P = \frac{1}{2}\rho\left(\frac{Q}{A_0}\right)^2$$

Compliance

$$V = \frac{\pi \tan^2 \alpha}{3}\left(\frac{P}{\rho g}\right)^3$$

LINEARIZATION – FUNCTION OF ONE VARIABLE

Linearization of Resistance, $e = e(f)$



LINEARIZATION – FUNCTION OF ONE VARIABLE

$$P = \frac{1}{2}\rho\left(\frac{Q}{A_o}\right)^2$$

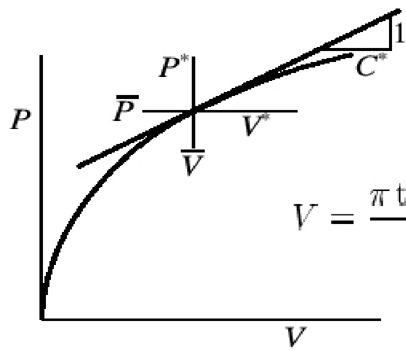
$$R^* = \left.\frac{dP}{dQ}\right|_{Q=\overline{Q}} = \frac{\rho\overline{Q}}{A_o^2} = \frac{\sqrt{2\rho\overline{P}}}{A_o} = \frac{2\rho^{3/2}}{A_o}\left(\frac{3\overline{V}}{\pi\tan^2\alpha}\right)^{1/6}$$

alternatively use

$$Q = A_0\sqrt{2P/\rho} \quad \Longrightarrow \quad \left.\frac{dQ}{dP}\right|_{P=\overline{P}} = 1/R^*$$

LINEARIZATION – FUNCTION OF ONE VARIABLE

Linearization of Compliance, $e = e(q)$



$$V = \frac{\pi \tan^2 \alpha}{3} \left(\frac{P}{\rho g} \right)^3$$

LINEARIZATION – FUNCTION OF ONE VARIABLE

$$V = \frac{\pi \tan^2 \alpha}{3} \left(\frac{P}{\rho g} \right)^3 \Rightarrow P = \rho g \left(\frac{3V}{\pi \tan^2 \alpha} \right)^{1/3}$$

$$\frac{1}{C^*} = \left. \frac{dP}{dV} \right|_{V=\overline{V}} = \frac{\rho g}{3} \left(\frac{3}{\overline{V}^2 \pi \tan^2 \alpha} \right)^{1/3}$$

alternatively use

$$V = \frac{\pi \tan^2 \alpha}{3} \left(\frac{P}{\rho g} \right)^3 \quad \Rightarrow \quad \left. \frac{dV}{dP} \right|_{P=\overline{P}} = C^*$$

LINEARIZATION – FUNCTION OF TWO VARIABLES

$$y = f(x, u)$$

Taylor's expansion (only first order terms):

$$y \simeq \overline{y} + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\overline{x}\\u=\overline{u}}} (x - \overline{x}) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\overline{x}\\u=\overline{u}}} (u - \overline{u})$$

Defining: $x^* = x - \overline{x}, \quad y^* = y - \overline{y}, \quad u^* = u - \overline{u}$

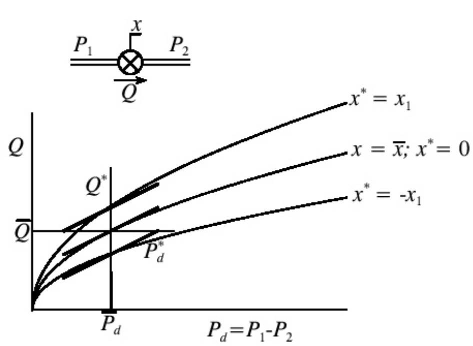
We can write:

$$y^* \simeq c_x x^* + c_u u^*; \qquad c_x = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\overline{x}\\u=\overline{u}}}; \qquad c_u = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\overline{x}\\y=\overline{y}}}$$

LINEARIZATION – FUNCTION OF TWO VARIABLES

Example: Adjustable hydraulic valve

$$Q = Q(P_d, x) \qquad P_d = \frac{\rho}{2}\left(\frac{Q}{c_D A}\right)^2 \qquad A = A_0\left(\frac{x}{x_0}\right)^{1.5}$$



LINEARIZATION – FUNCTION OF TWO VARIABLES

$$y = f(x,u) \quad \Rightarrow \quad y^* \simeq c_x x^* + c_u u^*; \qquad c_x = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}}; \qquad c_u = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\bar{x} \\ y=\bar{y}}}$$

$$Q = Q(P_d, x) \quad \Rightarrow \quad Q^* = \left.\frac{\partial Q}{\partial P_d}\right|_{P_d=\bar{P}_d} P_d^* + \left.\frac{\partial Q}{\partial x}\right|_{x=\bar{x}} x^*$$

$$Q^* = \frac{P_d^*}{R*} + kx^*,$$

$$x^* \equiv x - \bar{x},$$

$$\frac{1}{R^*} = \left.\frac{\partial Q}{\partial P_d}\right|_{x=\bar{x}},$$

$$k = \left.\frac{\partial Q}{\partial x}\right|_{\substack{P_d=\bar{P}_d \\ x=\bar{x}}} = \left.\frac{\partial Q}{\partial A}\right|_{P_d=\bar{P}_d} \left.\frac{\partial A}{\partial x}\right|_{x=\bar{x}}$$

LINEARIZATION – FUNCTION OF TWO VARIABLES

$$Q = Q(P_d, x)$$

$$P_d = \frac{\rho}{2}\left(\frac{Q}{c_D A}\right)^2$$ supplies both $\left.\frac{\partial Q}{\partial P_d}\right|_{P_d=\bar{P}_d}$ and $\left.\frac{\partial Q}{\partial A}\right|_{A=\bar{A}}$

$$A = A_0\left(\frac{x}{x_0}\right)^{1.5}$$ supplies $\left.\frac{\partial A}{\partial x}\right|_{x=\bar{x}}$

Start non-linear differential equation

(Single First Order Diff-E-Q)

$$\frac{dx}{dt} = f\left(x(t), u(t)\right)$$

Total, Nominal and Pertubation Variables

$$x(t) = \bar{x} + x^*(t)$$

$$u(t) = \bar{u} + u^*(t)$$

Find nominal solution(s)

$$f\left(\bar{x}, \bar{u}\right) = 0$$

Linearize function

$$\bar{a} = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}} \qquad \bar{b} = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\bar{x} \\ u=\bar{u}}}$$

Linearize differential equation

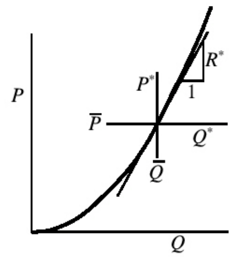
$$\frac{dx^*}{dt} \approx \bar{a}x^* + \bar{b}u^*$$

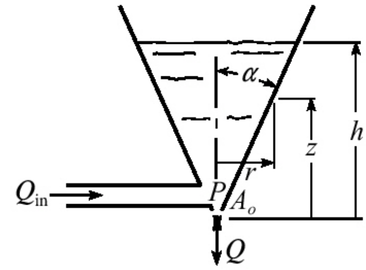
LINEARIZATION – FIRST ORDER ODEs

Example: Conic Tank

Resistance

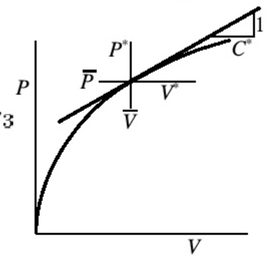
$$P = \frac{1}{2}\rho\left(\frac{Q}{A_0}\right)^2$$

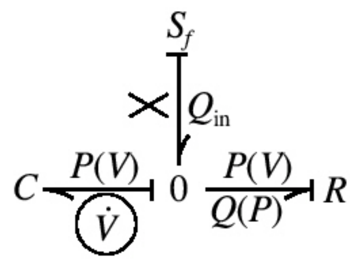




Compliance

$$P = \rho g\left(\frac{3V}{\pi\tan^2\alpha}\right)^{1/3}$$





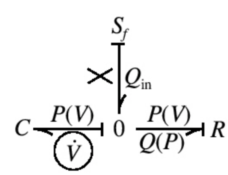
LINEARIZATION – FIRST ORDER ODEs

Find differential equation

$$\frac{dV}{dt} = Q_{in} - Q(P) = Q_{in} - A_0\sqrt{\frac{2}{\rho}P} = Q_{in} - A_0\sqrt{2g\left(\frac{3V}{\pi\tan^2\alpha}\right)^{1/3}}$$

Nonlinear differential equation

$$\frac{dV}{dt} = Q_{in} - \frac{A_0\sqrt{2g}}{(\pi\tan^2\alpha)^{1/6}}(3V)^{1/6}$$



Find nominal solution(s)

$$0 = \overline{Q} - A_0\sqrt{2g}\left(\frac{3\overline{V}}{\pi\tan^2\alpha}\right)^{1/6}$$

$$\left(\text{Pick either } \overline{V} \text{ or } \overline{Q}\right)$$

Linearize function

$$\frac{dV}{dt} = Q_{in} - \frac{A_0\sqrt{2g}}{(\pi \tan^2\alpha)^{1/6}}(3V)^{1/6}$$

$$a = \frac{d}{dV}\left(-\frac{A_0\sqrt{2g}}{\left(\pi \tan^2\alpha\right)^{1/6}}\left(3V\right)^{1/6} \right) = -\frac{A_0\sqrt{2g}}{6}\left(\frac{3}{\pi \bar{V}^5 \tan^2\alpha}\right)^{1/6}$$

$$b = \frac{d}{dQ_{in}}\left(Q_{in}\right) = 1$$

Linearize differential equation

$$\frac{dV^*}{dt} \simeq Q_{in}^* - \frac{A_0\sqrt{2g}}{6}\left(\frac{3}{\pi \overline{V}^5 \tan^2\alpha}\right)^{1/6} V^*$$

Examine the solution

$$\frac{dV^*}{dt} \simeq Q_{in}^* - \frac{A_0\sqrt{2g}}{6}\left(\frac{3}{\pi \overline{V}^5 \tan^2\alpha}\right)^{1/6} V^*$$

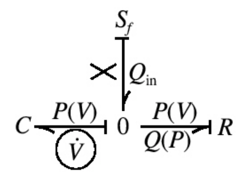
Of linear first order form

$$\tau \frac{dV^*}{dt} + V^* = \tau Q_{in}^*$$

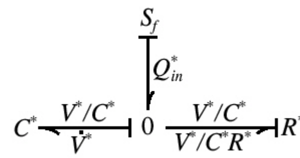
Has time constant

$$\tau = \frac{6}{A_0\sqrt{2g}}\left(\frac{\pi \overline{V}^5 \tan^2\alpha}{3}\right)^{1/6}$$

LINEARIZATION – FIRST ORDER ODEs



$\Rightarrow$



Compliance $$P = \rho g \left(\frac{3V}{\pi \tan^2 \alpha} \right)^{1/3}$$

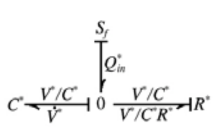
$$\frac{1}{C^*} = \left.\frac{dP}{dV}\right|_{V=\overline{V}} = \frac{\rho g}{3} \left(\frac{3}{\overline{V}^2 \pi \tan^2 \alpha} \right)^{1/3}$$

Resistance $$P = \frac{1}{2}\rho \left(\frac{Q}{A_0} \right)^2$$

$$R^* = \left.\frac{dP}{dQ}\right|_{Q=\overline{Q}} = \frac{\rho \overline{Q}}{(c_d A)^2} = \frac{\sqrt{2\rho \overline{P}}}{c_d A} = \frac{2\rho^{3/2}}{c_d A} \left(\frac{3\overline{V}}{\pi \tan^2 \alpha} \right)^{1/6}$$

LINEARIZATION – FIRST ORDER ODEs

Find differential equation



$\Rightarrow$

$$\frac{dV^*}{dt} = Q^*_{in} - \frac{1}{C^* R^*} V^*$$

Of linear first order form

$$\tau \frac{dV^*}{dt} + V^* = \tau Q^*_{in}$$

LINEARIZATION – FIRST ORDER ODEs

$$\tau \frac{dV^*}{dt} + V^* = \tau Q^*_{in}$$

Has time constant

$$\tau = R^* C^* = \frac{6}{A_0\sqrt{2g}} \left(\frac{\pi \overline{V}^5 \tan^2 \alpha}{3} \right)^{1/6}$$

Has forced solution

$$V^* = (1 - e^{-t/\tau})\tau Q^*_{in}$$