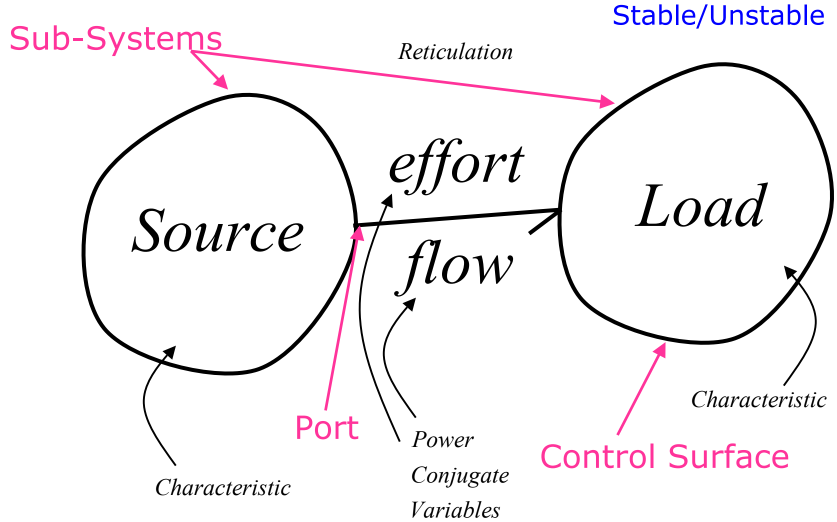
ME242 – MECHANICAL ENGINEERING SYSTEMS

LECTURE 16

- Review – Test 1

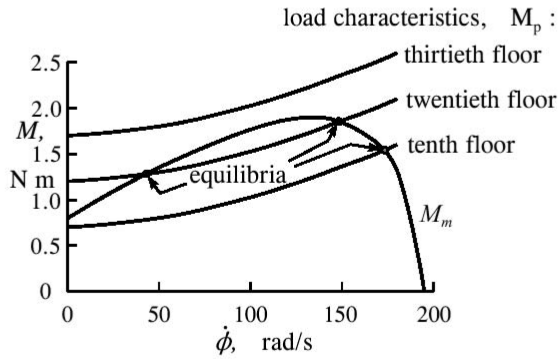
SOURCE-LOAD SYNTHESIS

Systems → Steady State/Equilibrium → Single/Multiple
Stable/Unstable

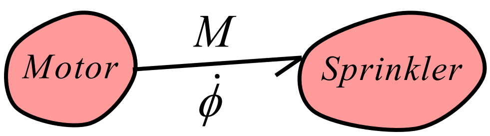


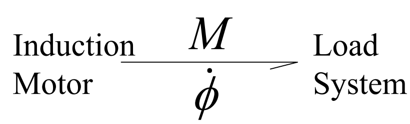
SOURCE-LOAD SYNTHESIS

Induction Motor ↔ Water Sprinkler



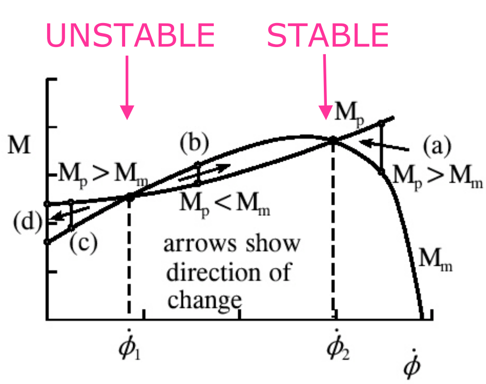
Equilibrium ⇔ Both conjugate variables have common values





SOURCE-LOAD SYNTHESIS: STABILITY

Induction Motor ↔ Water Sprinkler



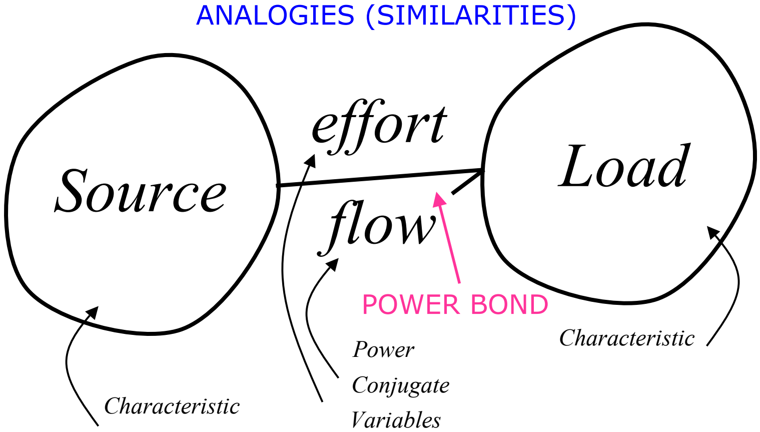
GENERALIZED FORCES AND VELOCITIES

Power is the product of two conjugate variables

Power = effort (generalized force) × flow (generalized velocity)

⇓

ANALOGIES (SIMILARITIES)



GENERALIZED FORCES AND VELOCITIES

effort or generalized force

----labeled as "e" or "$\dot{p}$"

flow or generalized velocity

---labeled as either "f" or "$\dot{q}$"

Power = effort × flow

$$\boxed{P = e\dot{q}}$$

GENERALIZED FORCES AND VELOCITIES

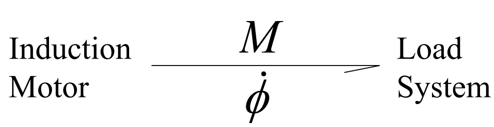
Convention:

- Effort variable written above a horizontal bond or to the left of a vertical bond
- Flow variable written below a horizontal bond or to the right of a vertical bond

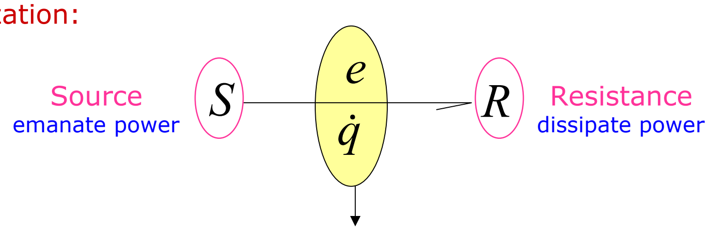
- The half arrow on the bonds indicate the direction that power when P>0
- The half arrow should be placed on the flow side of the bond

GENERALIZED SOURCES, SINKS, RESISTANCES

Case study of last class:



Generalization:



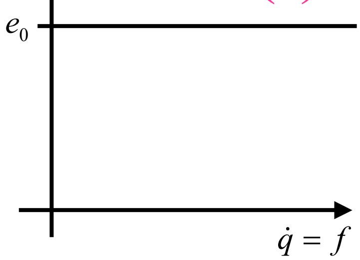
Any energy type:
-Mechanical (rotational, lateral, longitudinal)
-Fluid
-Electrical

INDEPENDENT EFFORT SOURCES AND SINKS

EFFORT:

An independent-effort source (usually called simply an effort source), and a independent-effort sink (usually called simply an effort sink), are defined to have efforts that are independent of their flows.

$$e = e(t)$$



effort source : $S_e \rightharpoonup$

effort sink : $\rightharpoonup S_e$

They are the SAME!!!

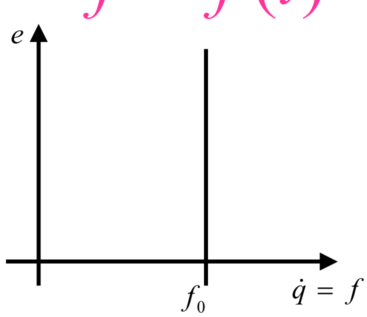
Effort SINK = Effort SOURCE with $P<0$

INDEPENDENT FLOWS SOURCES AND SINKS

FLOW:

An independent-flow source (usually called simply a flow source), and a independent-flow sink (usually called simply a flow sink), are defined to have efforts that are independent of their flows.

$$f = f(t)$$



flow source : $S_f \rightharpoonup$

flow sink : $\rightharpoonup S_f$

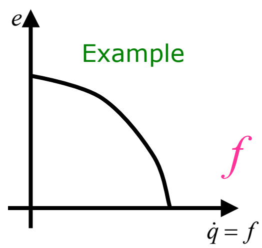
They are the SAME!!!

Flow SINK = Flow SOURCE with $P<0$

GENERAL SOURCES AND SINKS (RESISTANCE)

A GENERAL SOURCE can represent any prescribed (static) relationship between its effort and its flow

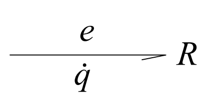
A GENERAL SINK (RESISTANCE) can represent any prescribed (static) relationship between its effort and its flow

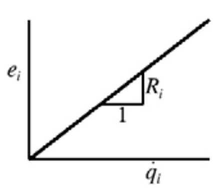


$$f = f(e), e = e(f)$$

LINEAR RESISTANCES

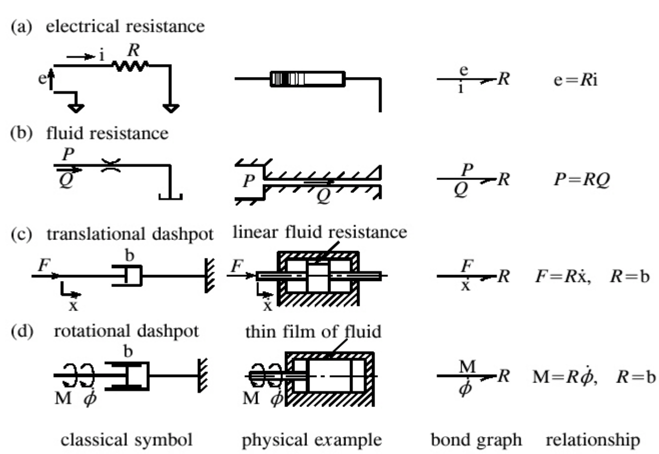
Examples:





(a) linear

$$e = R\dot{q}$$



Algebra is never included within a bond graph! The algebraic relationship is written separately

DYNAMIC SYSTEMS

So far, we have introduced:

SOURCES	Emanate Energy	Steady or Equilibrium Systems
RESISTANCES	Dissipate Energy	

We need new players to be able to represent dynamic systems

COMPLIANCES	Store Energy	Unsteady or Dynamic Systems
INERTANCES	Store Energy	

- Dynamic physical systems contain mechanisms that store energy temporarily, for later release.

- The dynamics can be thought of as a sloshing of energy between different energy storage mechanisms, and/or a gradual dissipation of energy in resistances.

GENERALIZED VARIABLES

Generalized Velocity or Flow: f

Generalized Displacement: q

$$f = \dot{q}, \text{ or } q = \int f dt$$

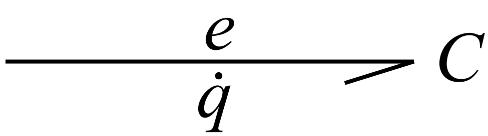
Generalized Force or Effort: e

Generalized Momentum: p

$$e = \dot{p}, \text{ or } p = \int e dt$$

$$\boxed{P = ef = e\dot{q} = \dot{p}f}$$

ENERGY STORAGE: LINEAR COMPLIANCE



$$e = \frac{1}{C}q$$

Energy Storage:

Work from point 1 to point 2:

$$W_{1\to 2} = \int_1^2 e\dot{q}dt = \int_1^2 e\frac{dq}{dt}dt = \int_{q_1}^{q_2} edq = \int_{q_1}^{q_2} \frac{1}{C}qdq = \frac{1}{2C}q^2\Big|_{q_1}^{q_2} = \frac{1}{2C}\left(q_2^2 - q_1^2\right)$$

Work from point 2 to point 1:

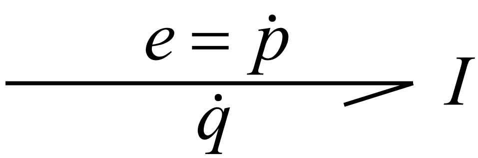
$$W_{2\to 1} = \frac{1}{2C}\left(q_1^2 - q_2^2\right) = -W_{1\to 2}$$

The energy is conserved!

Potential Energy:

$$V = \frac{1}{2C}q^2 = \frac{1}{2}Ce^2 \quad \Rightarrow \quad W_{1\to 2} = V_2 - V_1$$

ENERGY STORAGE: LINEAR INERTANCE



$$f = \frac{1}{I}p$$

Energy Storage:

Work from point 1 to point 2:

$$W_{1\to 2} = \int_1^2 \dot{p}f\,dt = \int_1^2 \frac{dp}{dt}f\,dt = \int_{p_1}^{p_2} f\,dp = \int_{p_1}^{p_2} \frac{1}{I}pdp = \frac{1}{2I}p^2\Big|_{p_1}^{p_2} = \frac{1}{2I}\left(p_2^2 - p_1^2\right)$$

Work from point 2 to point 1:

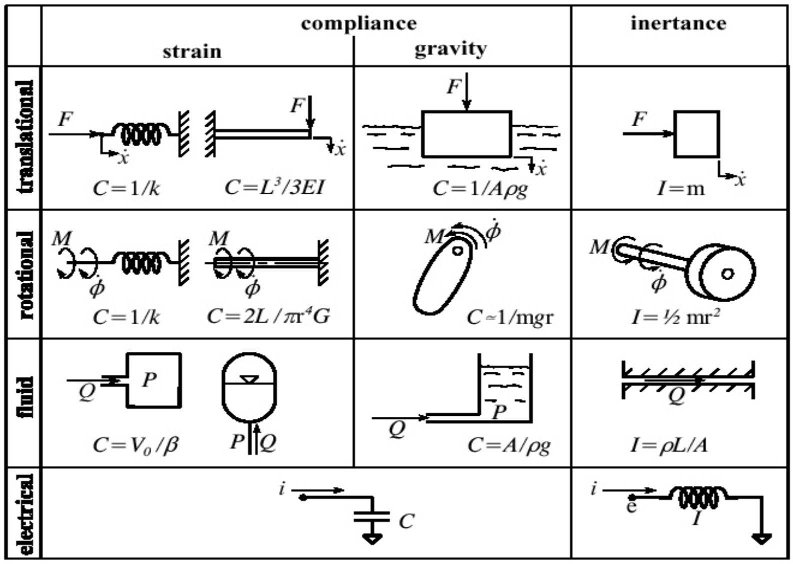
$$W_{2\to 1} = \frac{1}{2I}\left(p_1^2 - p_2^2\right) = -W_{1\to 2}$$

The energy is conserved!

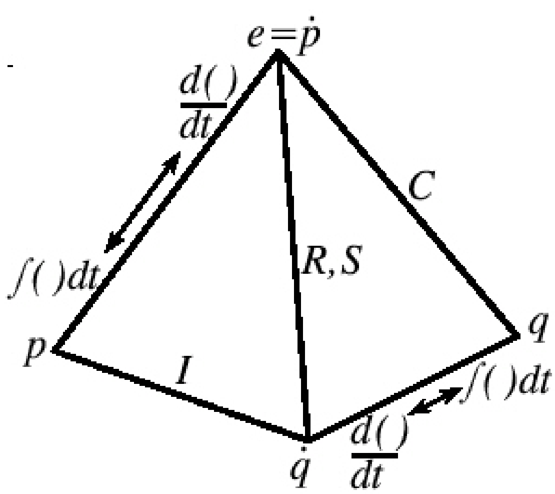
Kinetic Energy:

$$T = \frac{1}{2I}p^2 = \frac{1}{2}I\dot{q}^2 \quad \Rightarrow \quad W_{1\to 2} = T_2 - T_1$$

ENERGY STORAGE: COMPLIANCE & INERTANCE



ENERGY STORAGE: COMPLIANCE & INERTANCE



JUNCTIONS

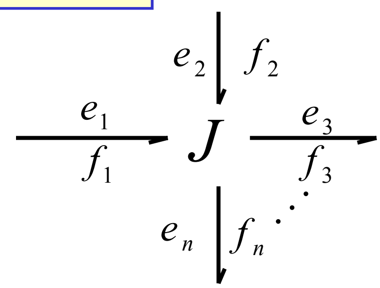
Elements introduced so far → ONE PORT

$$S, S_e, S_f, R, C, I$$

At termination (beginning or end)

JUNCTIONS:
- Branching
- Constraints

POWER CONSTRAINT: The junction is IDEAL, neither storing, creating, nor dissipating energy



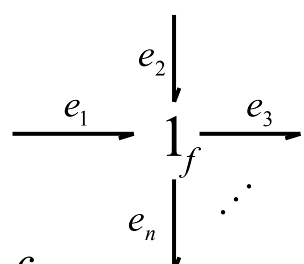
$$\sum P = 0 \Rightarrow \sum P_{in} = \sum P_{out}$$

$$e_1 f_1 + e_2 f_2 = e_3 f_3 + \cdots + e_n f_n$$

JUNCTIONS: 1-JUNCTION

COMMON FLOW CONSTRAINT:

$$f_1 = f_2 = f_3 = \cdots = f_n = f$$



POWER CONSTRAINT:

$$e_1 f_1 + e_2 f_2 = e_3 f_3 + \cdots + e_n f_n$$

Then, we have

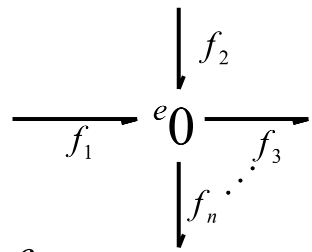
$$e_1 + e_2 = e_3 + \cdots + e_n \quad \Rightarrow \quad \sum e_{in} = \sum e_{out} \;\; \textbf{or} \;\; \sum e = 0$$

The **common flow – sum of effort** junction

JUNCTIONS: 0-JUNCTION

COMMON EFFORT CONSTRAINT:

$$e_1 = e_2 = e_3 = \cdots = e_n = e$$



POWER CONSTRAINT:

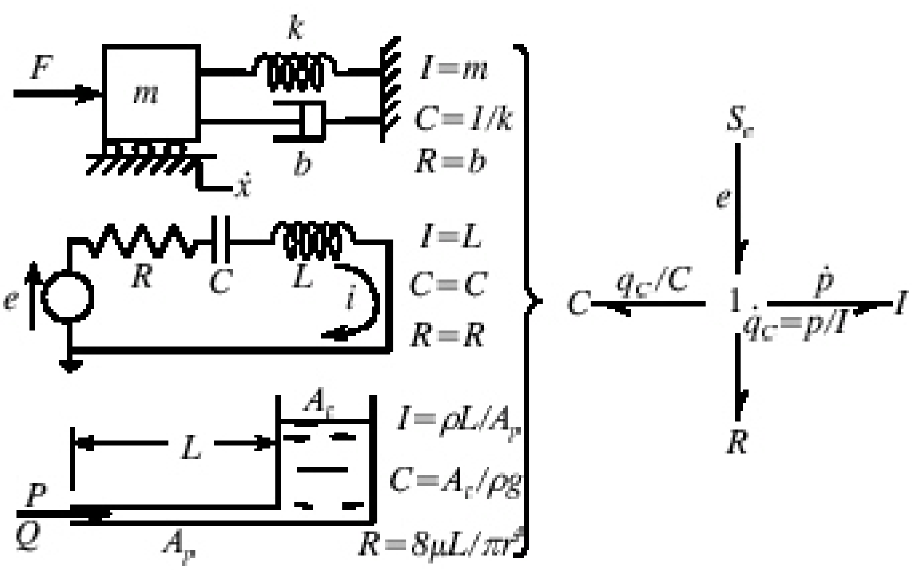
$$e_1 f_1 + e_2 f_2 = e_3 f_3 + \cdots + e_n f_n$$

Then, we have

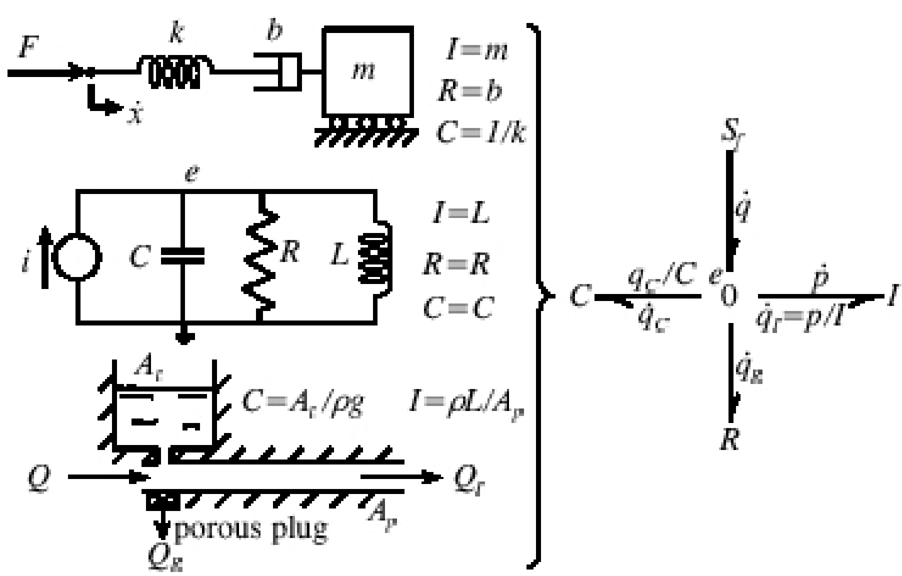
$$f_1 + f_2 = f_3 + \cdots + f_n \quad \Rightarrow \quad \sum f_{in} = \sum f_{out} \quad \textbf{or} \quad \sum f = 0$$

The **common effort – sum of flow** junction

1-JUNCTION: SIMPLE IRC MODELS



0-JUNCTION: SIMPLE IRC MODELS



DYNAMICS

Dynamic behavior of well-posed model with energy storage elements

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DIFFERENTIAL EQUATION

Analytical Solution ↙ ↘ Numerical Solution

Approach: Each independent energy storage element

↓

One first-order differential equation

↓

STATE VARIABLE REPRESENTATION

CAUSALITY OF EFFORT SOURCES

Effort Source:

$$\mathbf{S}_e \xrightarrow[f]{e} \qquad e = e(t), e \neq e(f)$$

Effort is imposed by the source

The effort e is CAUSED by action of S_e

Flow is imposed by…? Whatever system is attached to the bond

The flow f is CAUSED by system reaction

CAUSALITY OF EFFORT SOURCES

This Bilateral CAUSALITY can be indicated as:

$$\mathbf{S}_e \xrightarrow[\leftarrow f]{e \rightarrow} \quad \text{or} \quad \mathbf{S}_e \xrightarrow[f]{e}|$$

This is not a power flow concept, it is a CAUSALITY concept

$$\mathbf{S}_e \xrightarrow[f]{e}| \ \mathbf{Load}$$

$\mathbf{S}_e$ causes e **Load** causes f

$$f = f_{Load}(e)$$

CAUSALITY OF FLOW SOURCES

Flow Source:

$$\mathbf{S}_f \overset{e}{\underset{f}{\rightharpoonup}} \quad f = f(t), f \neq f(e)$$

Flow is imposed by the source

The flow f is CAUSED by action of S_f

Effort is imposed by…? Whatever system is attached to the bond

The effort e is CAUSED by system reaction

CAUSALITY OF FLOW SOURCES

This Bilateral CAUSALITY can be indicated as:

$$\mathbf{S}_f \overset{\leftarrow e}{\underset{f \rightarrow}{\rightharpoonup}} \quad \text{or} \quad \mathbf{S}_f \vdash\overset{e}{\underset{f}{\rightharpoonup}}$$

This is not a power flow concept, it is a CAUSALITY concept

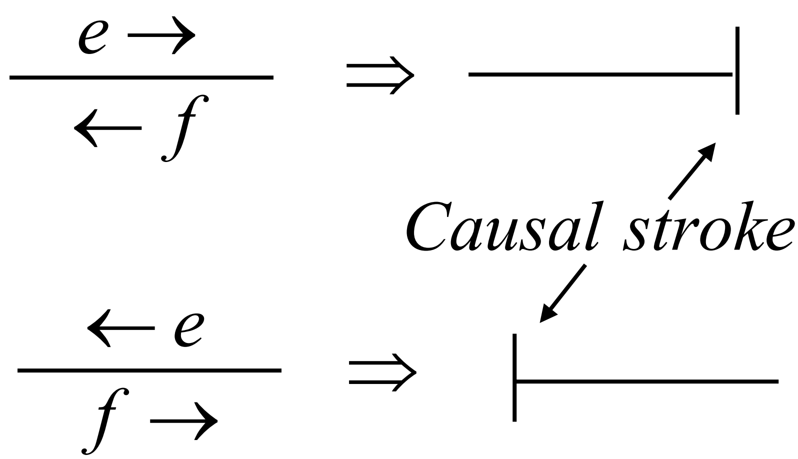
$$\mathbf{S}_f \vdash\overset{e}{\underset{f}{\text{——}}} \mathbf{Load}$$

$\mathbf{S}_f$ causes f **Load** causes e

$$e = e_{Load}\left(f\right)$$

CAUSALITY

The CAUSALITY is Bilateral



CAUSALITY TYPES FOR C AND I

Differential Causality:

compliance : $\overset{e_i}{\underset{\dot{q}_i}{\rightharpoonup}}\!\dashv C_i \qquad \dot{q}_i = C_i \dfrac{de_i}{dt},$

inertance : $\vdash\!\overset{e_i}{\underset{\dot{q}_i}{\rightharpoonup}} I_i \qquad e_i = I_i \dfrac{d\dot{q}}{dt}$

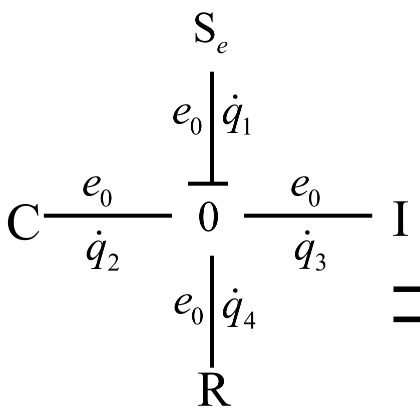
Integral Causality:

compliance : $\vdash\!\overset{e_i}{\underset{\dot{q}_i}{\rightharpoonup}} C_i \qquad e_i = e_i(q_i) = \dfrac{q_i}{C_i} = \dfrac{1}{C_i}\displaystyle\int \dot{q}_i \, dt,$

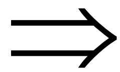
inertance : $\overset{e_i}{\underset{\dot{q}_i}{\rightharpoonup}}\!\dashv I_i \qquad \dot{q}_i = \dot{q}(p_i) = \dfrac{p_i}{I_i} = \dfrac{1}{I_i}\displaystyle\int e_i \, dt$

S_e AND THE 0 JUNCTION

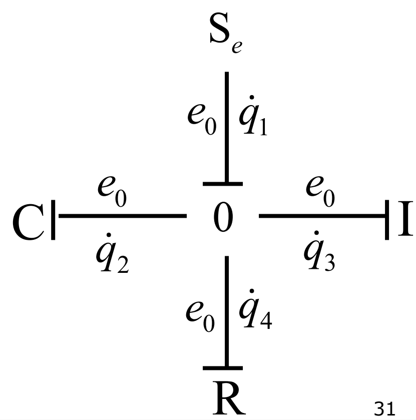
Example:



(a) model

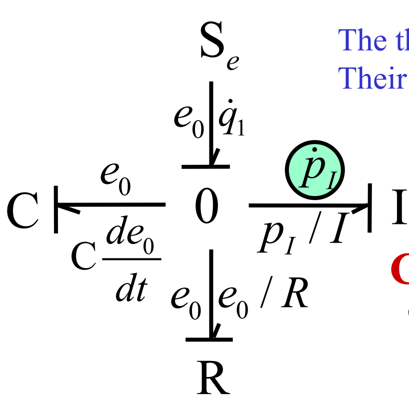


(b) causality added



S_e AND THE 0 JUNCTION

(c) annotation of causal bonds



The three elements have admittance causality
Their flow respond to an imposed effort

C has **differential** causality.

Causal Output = Derivative of Causal Input

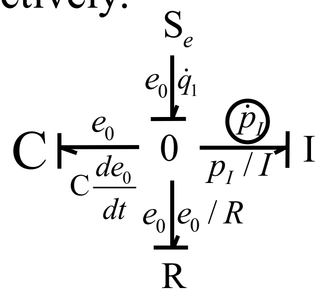
I has **integral** causality.

Causal Output = Integral of Causal Input

Element behavior is **uncoupled**.

S_e AND THE 0 JUNCTION

The state of the inertance and the capacitance is determined by q_C and p_I respectively.



$$\dot{p}_I = e_0$$

$$q_C = Ce_0 \Leftrightarrow \dot{q}_C = C\frac{de_0}{dt}$$

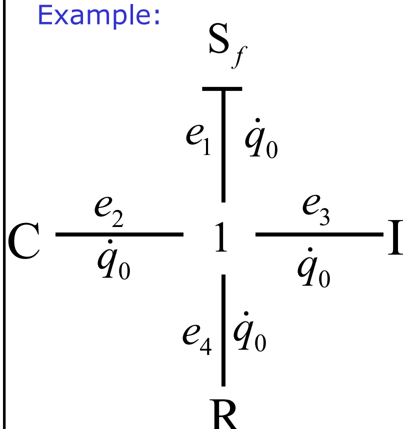
The order of the systems is two

The flow required of the effort source is

$$\dot{q}_1 = C\frac{de_0}{dt} + \frac{e_0}{R} + \frac{p_I}{I}$$

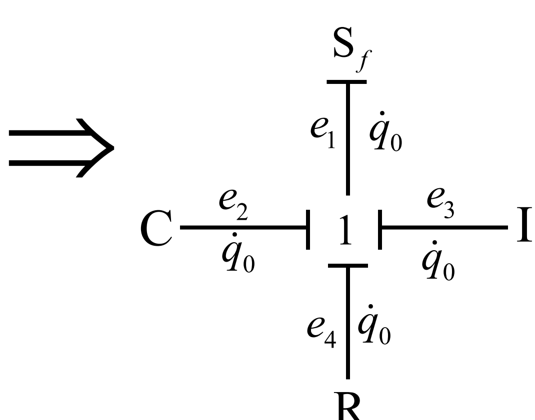
S_f AND THE 1 JUNCTION

Example:



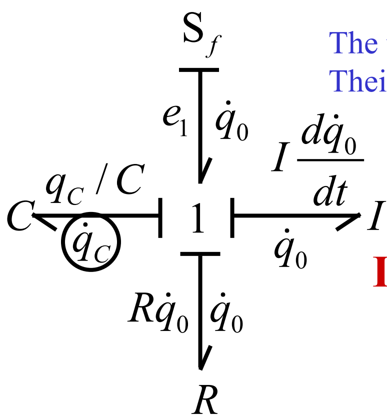
(a) model

(b) causality added



S_f AND THE 1 JUNCTION

(c) annotation of causal bonds



The three elements have impedance causality
Their efforts respond to an imposed flow

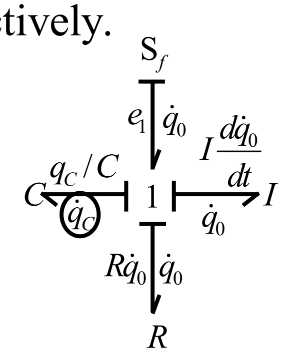
I has **differential** causality.
Causal Output = Derivative of Causal Input

C has **integral** causality.
Causal Output = Integral of Causal Input

Element behavior is **uncoupled**.

S_f AND THE 1 JUNCTION

The state of the inertance and the capacitance is determined by q_C and p_I respectively.



$$\dot{q}_C = \dot{q}_0$$

$$\dot{p}_I = e_I = I\frac{d\dot{q}_0}{dt}$$

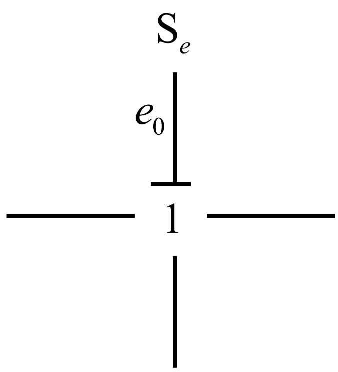
The order of the systems is two

The effort required of the flow source is

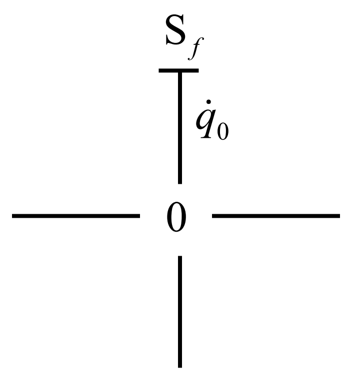
$$e_1 = I\frac{d\dot{q}_0}{dt} + R\dot{q}_0 + q_c / C$$

JUNCTIONS WITH COUPLED BEHAVIORS

S_e and the 1 junction:



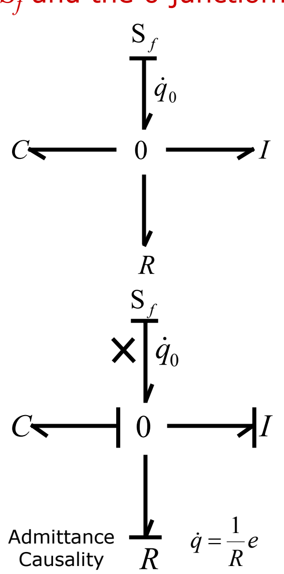
S_f and the 0 junction:



The source does NOT directly determine either the efforts or the flows on any bonds other than the source bond itself

JUNCTIONS WITH COUPLED BEHAVIORS

S_f and the 0 junction:



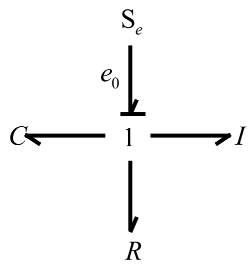
- In this case, the source does **NOT** directly determine the effort associated with the 0-Junction
- Who does? It must be one of the attached bonds!
- How? One bond will have its causal stroke adjacent to the junction. It will imposed flow to the attached element and will imposed effort to the junction as reaction.
- Due to the properties of the 0-Junction, the other bonds will have the causal strokes placed at the outer end. The effort is imposed by the junction to these bonds.
- We have, in this case, three possible patterns!
- Which one do we use? We use INTEGRAL CAUSALITY! C or I with stroke adjacent to junction?

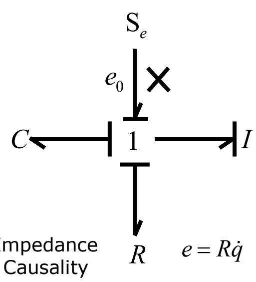
$$C: \quad q = Ce \Rightarrow \dot{q} = C\frac{de}{dt} \quad \text{or} \quad e = \frac{1}{C}\int \dot{q}dt$$

$$I: \quad p = I\dot{q} \Rightarrow e = \dot{p} = I\frac{d\dot{q}}{dt} \quad \text{or} \quad \dot{q} = \frac{1}{I}\int edt$$

JUNCTIONS WITH COUPLED BEHAVIORS

S_e and the 1 junction:





Impedance Causality

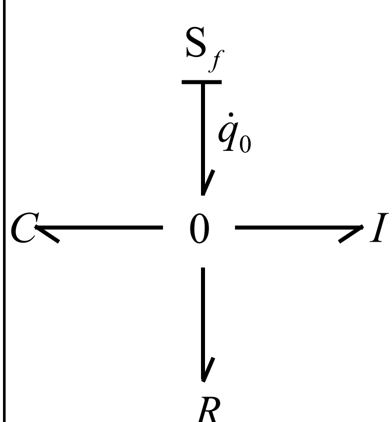
- In this case, the source does **NOT** directly determine the flow associated with the 1-Junction
- Who does? It must be one of the attached bonds!
- How? One bond will have its causal stroke at the outer end of the junction. It will imposed effort to the attached element and will imposed flow to the junction as reaction.
- Due to the properties of the 1-Junction, the other bonds will have the causal strokes adjacent to the junction. The flow is imposed by the junction to these bonds.
- We have, in this case, three possible patterns!
- Which one do we use? We use INTEGRAL CAUSALITY! C or I with stroke adjacent to junction?

$$C: \quad q = Ce \Rightarrow \dot{q} = C\frac{de}{dt} \quad \text{or} \quad e = \frac{1}{C}\int \dot{q}dt$$

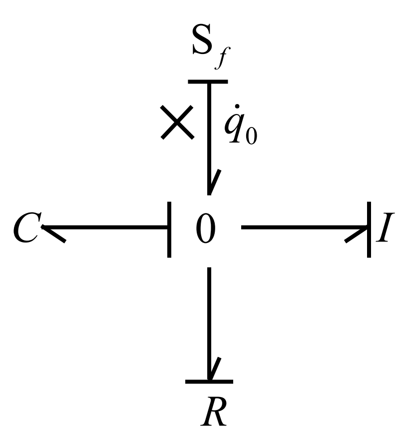
$$I: \quad p = I\dot{q} \Rightarrow e = \dot{p} = I\frac{d\dot{q}}{dt} \quad \text{or} \quad \dot{q} = \frac{1}{I}\int edt$$

S_f AND THE 0 JUNCTION

Second Order Problem: IRC Model

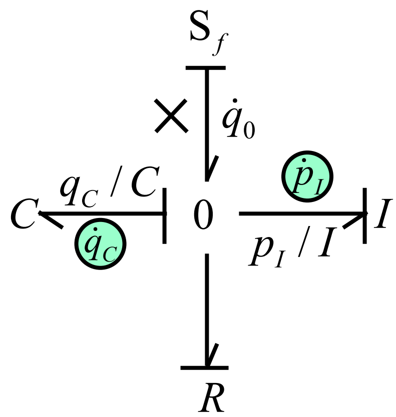


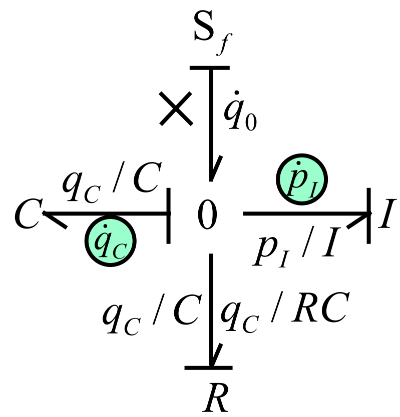
(a) model



(b) integral causality added

S_f AND THE 0 JUNCTION





(c) annotation of causal bonds (d) annotation completion

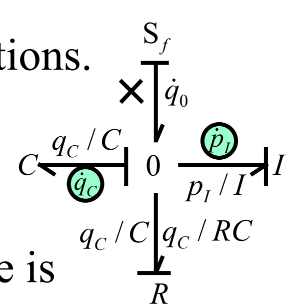
S_f AND THE 0 JUNCTION

The state of the inertance and the capacitance is determined by q_C and p_I respectively.

$$\dot{p}_I = q_C / C \qquad \dot{q}_C = \dot{q}_0 - p_I / I - q_C / RC$$

There are two state differential equations.

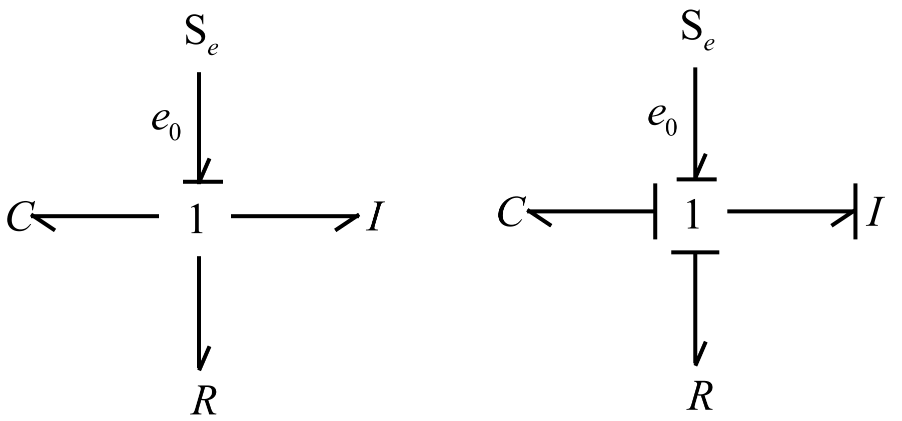
The order of the system is two.



The effort required of the flow source is

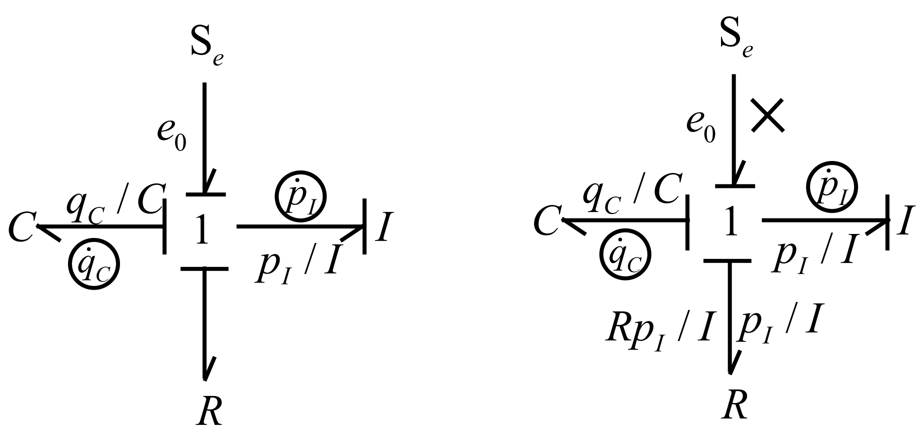
$$e_f = q_C / C$$

S_e AND THE 1 JUNCTION



(a) model (b) integral causality added

S_e AND THE 1 JUNCTION



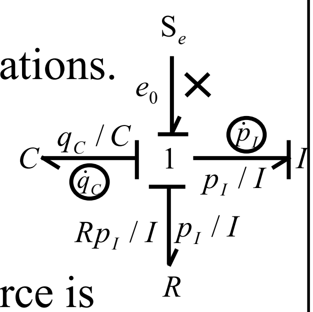
(c) annotation of causal bonds (d) annotation completion

S_e AND THE 1 JUNCTION

The state of the inertance and the capacitance is determined by q_C and p_I respectively.

$$\dot{p}_I = e_0 - q_C / C - Rp_I / I \qquad \dot{q}_C = p_I / I$$

There are two state differential equations. The order of the system is two.



The flow required of the effort source is

$$f_e = p_I / I$$

FIRST ORDER DIFFERENTIAL EQUATIONS

$$\tau \frac{dx}{dt} + x = f_1(t) \Leftrightarrow \frac{dx}{dt} + \frac{1}{\tau} x = f_2(t)$$

Forcing Term: $f_2(t)$ Initial Condition: $x(0) = x_o$

$$x(t) = x_H(t) + x_P(t)$$

1- Homogeneous Solution: $\frac{dx_H}{dt} + \frac{1}{\tau} x_H = 0 \Rightarrow x_H(t) = Ae^{-\frac{t}{\tau}}$

2- Particular Solution: $x_P(t) : \frac{dx_P}{dt} + \frac{1}{\tau} x_P = f_2(t)$

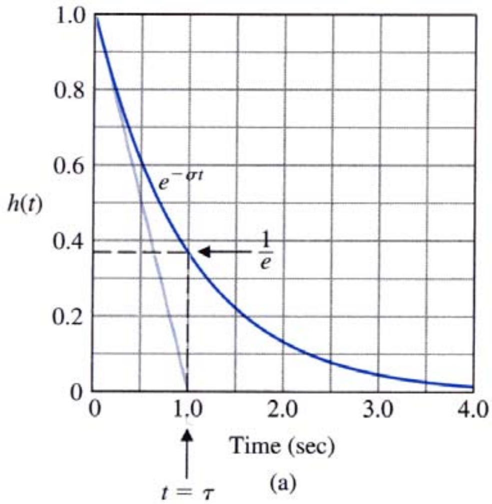
$$x(t) = x_H(t) + x_P(t) = Ae^{-\frac{t}{\tau}} + x_P(t)$$

3- Initial Condition: $x(0) = A + x_P(0) = x_o \Rightarrow A = x_o - x_P(0)$

$$x(t) = (x_o - x_P(0))e^{-\frac{t}{\tau}} + x_P(t)$$

FIRST ORDER DIFFERENTIAL EQUATIONS

$$x(t) = x_H(t) = x_o e^{-\frac{t}{\tau}} \Leftrightarrow h(t) = \frac{x(t)}{x_o} = e^{-\frac{t}{\tau}}$$



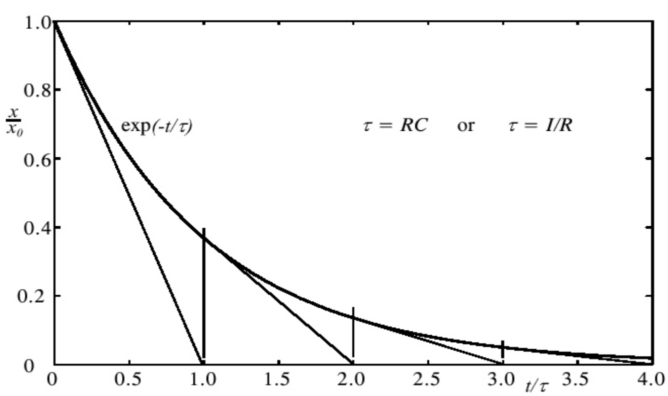
$\sigma > 0$ Stable

$\sigma < 0$ Unstable

$\tau = \frac{1}{\sigma}$ Time Constant

FIRST ORDER DIFFERENTIAL EQUATIONS

$$x(t) = x_H(t) = x_o e^{-\frac{t}{\tau}} \Leftrightarrow h(t) = \frac{x(t)}{x_o} = e^{-\frac{t}{\tau}}$$



$$\dot{h}(t) = -\frac{1}{\tau} e^{-\frac{t}{\tau}}$$

$$\Downarrow$$

$$\dot{h}(0) = -\frac{1}{\tau}$$

SECOND ORDER DIFFERENTIAL EQUATIONS

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = f_1(t) \Leftrightarrow \frac{d^2x}{dt^2} + \frac{a_1}{a_2}\frac{dx}{dt} + \frac{a_0}{a_2} x = f_2(t) = \frac{1}{a_2} f_1(t)$$

$$\omega_n = \sqrt{\frac{a_0}{a_2}} \qquad \varsigma = \frac{a_1}{2\sqrt{a_0 a_2}}$$

$$\frac{d^2x}{dt^2} + 2\varsigma\omega_n \frac{dx}{dt} + \omega_n^2 x = f_2(t)$$

These equations are derived from the RIC models

Solution: $x(t) = x_H(t) + x_P(t)$

Particular solution

Homogeneous solution

Homogeneous Solution: $$\frac{d^2x_H}{dt^2} + 2\varsigma\omega_n \frac{dx_H}{dt} + \omega_n^2 x_H = 0$$

Particular Solution: $$\frac{d^2x_P}{dt^2} + 2\varsigma\omega_n \frac{dx_P}{dt} + \omega_n^2 x_P = f_2(t)$$

SECOND ORDER DIFFERENTIAL EQUATIONS

Homogeneous Solution: $$\frac{d^2x_H}{dt^2} + 2\varsigma\omega_n \frac{dx_H}{dt} + \omega_n^2 x_H = 0$$

Characteristic Equation:

$$s^2 + 2\varsigma\omega_n s + \omega_n^2 = 0$$

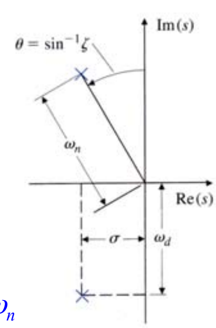
$$(s + \varsigma\omega_n)^2 + \omega_n^2(1 - \varsigma^2) = 0$$

$$(s + \sigma)^2 + \omega_d^2 = 0$$

$$(s + \sigma + j\omega_d)(s + \sigma - j\omega_d) = 0$$

$$\Downarrow$$

$$s_{1,2} = -\sigma \pm j\omega_d = -\varsigma\omega_n \pm j\sqrt{\varsigma^2 - 1}\omega_n$$



$$\sigma = \zeta\omega_n \qquad \omega_d = \omega_n\sqrt{1-\zeta^2} \Leftrightarrow \omega_n = \sqrt{\omega_d^2 + \sigma^2} \qquad \zeta = \frac{\sigma}{\omega_n}$$

ω_n : Undamped natural frequency

ζ : Damping ratio

$$x_H(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

SECOND ORDER DIFFERENTIAL EQUATIONS

Different real poles: $x_H(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$

$\zeta > 1$

Equal real poles: $x_H(t) = C_1 e^{s_1 t} + C_2 t e^{s_2 t}$

$\zeta = 1$

Complex conjugate poles: $x_H(t) = e^{-\sigma t}\left[C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)\right]$

$0 < \zeta < 1$

SECOND ORDER DIFFERENTIAL EQUATIONS

$$\frac{d^2x}{dt^2} + 2\varsigma\omega_n \frac{dx}{dt} + \omega_n^2 x = f_2(t)$$

Forcing Term: $f_2(t)$ Initial Conditions: $x(0) = x_o, \quad \dot{x}(0) = \dot{x}_o$

$$x(t) = x_H(t) + x_P(t)$$

1- Homogeneous Solution: $x_H : \frac{d^2x_H}{dt^2} + 2\varsigma\omega_n \frac{dx_H}{dt} + \omega_n^2 x_H = 0$

2- Particular Solution: $x_P : \frac{d^2x_P}{dt^2} + 2\varsigma\omega_n \frac{dx_P}{dt} + \omega_n^2 x_P = f_2(t)$

$$x(t) = x_H(t) + x_P(t)$$

3- Initial Conditions: $x(0) = x_o, \quad \dot{x}(0) = \dot{x}_o$

$$x(t) = x_H(t) + x_P(t)$$

LAPLACE TRANSFORM - DEFINITION

Function *f(t)* of time

Piecewise continuous and exponential order $|f(t)| < Ke^{bt}$

$$F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt \qquad \mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\alpha-j\infty}^{\alpha+j\infty} F(s)e^{st}ds$$

0- limit is used to capture transients and discontinuities at $t=0$

s is a complex variable ($\sigma+j\omega$)

There is a need to worry about regions of convergence of the integral

LAPLACE TRANSFORM – TABLE

Signal	Waveform	Transform
impulse	$\delta(t)$	1
step	$u(t)$	$\frac{1}{s}$
ramp	$tu(t)$	$\frac{1}{s^2}$
exponential	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
damped ramp	$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$
sine	$\sin(\beta t)u(t)$	$\frac{\beta}{s^2+\beta^2}$
cosine	$\cos(\beta t)u(t)$	$\frac{s}{s^2+\beta^2}$
damped sine	$e^{-\alpha t}\sin(\beta t)u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
damped cosine	$e^{-\alpha t}\cos(\beta t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$

LAPLACE TRANSFORM PROPERTIES

Linearity: (absolutely critical property)

$$\mathcal{L}\{Af_1(t)+Bf_2(t)\}=A\mathcal{L}\{f_1(t)\}+B\mathcal{L}\{f_2(t)\}=AF_1(s)+BF_2(s)$$

Integration property:
$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}=\frac{F(s)}{s}$$

Differentiation property:
$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\}=sF(s)-f(0-)$$

$$\mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\}=s^2F(s)-sf(0-)-f'(0-)$$

$$\mathcal{L}\left\{\frac{d^mf(t)}{dt^m}\right\}=s^mF(s)-s^{m-1}f(0-)-s^{m-2}f'(0-)-\cdots-f^{(m)}(0-)$$

LAPLACE TRANSFORM PROPERTIES

Translation properties:

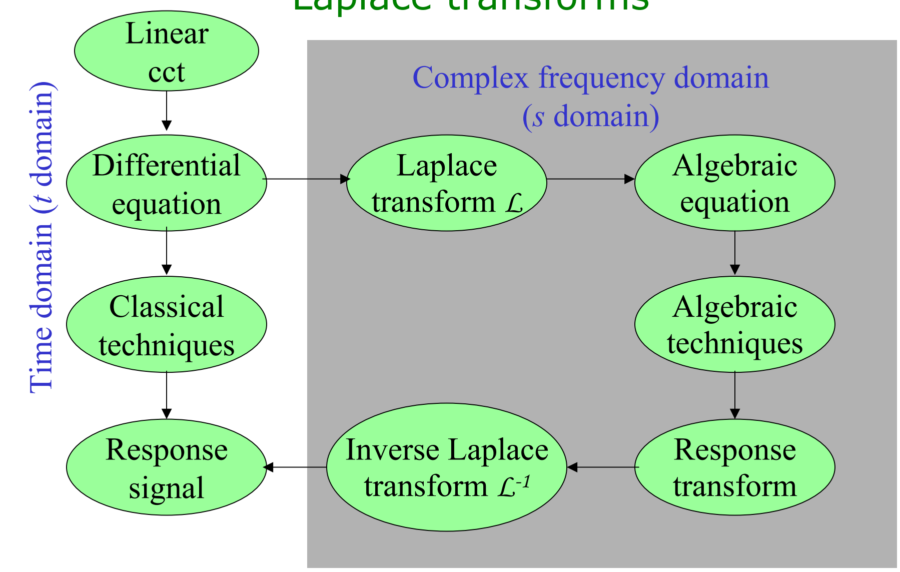
s-domain translation: $\mathcal{L}\{e^{-\alpha t}f(t)\}=F(s+\alpha)$

t-domain translation: $\mathcal{L}\{f(t-a)u(t-a)\}=e^{-as}F(s)$ for $a>0$

Initial Value Property:
$$\lim_{t\to 0+} f(t)=\lim_{s\to\infty} sF(s)$$

Final Value Property:
$$\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$$

Laplace transforms



The diagram commutes

Same answer whichever way you go

MODEL REPRESENTATION

Scalar Differential Equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^n} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u$$

$$\mathcal{L}\left\{\frac{d^m f(t)}{dt^m}\right\} = s^m F(s) - s^{m-1} f(0-) - s^{m-2} f'(0-) - \cdots - f^{(m)}(0-)$$

↓ L

Forcing function $u(t)$
Initial Conditions

Rational Function

$$Y(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

↓ $n > m$

Partial Fraction Expansion

$$Y(s) = \frac{\alpha_1}{(s-p_1)} + \frac{\alpha_2}{(s-p_2)} + \frac{\alpha_{31}}{(s-p_3)} + \frac{\alpha_{32}}{(s-p_3)^2} + \frac{\alpha_{33}}{(s-p_3)^3} + \ldots + \frac{\alpha_q}{(s-p_q)}$$

↓ L^{-1}

$y(t)$

RESIDUES

Residues at simple poles:

$$k_i = \lim_{s \to p_i} (s - p_i) F(s)$$

Residues at multiple poles:

$$k_m = \frac{1}{(m-1)!} \lim_{s \to a} \frac{d^{m-1}}{ds^{m-1}} \left[(s-a)^n V(s) \right], \quad m = 1 \cdots n$$

Residues at complex poles = Residues at simple poles

NOT STRICTLY PROPER LAPLACE TRANSFORMS

Rational Function

$$Y(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \qquad \boxed{n \le m}$$

↓ Polynomial Division

Rational Function

$$Y(s) = \alpha(s) + \frac{b_q s^q + b_{q-1} s^{q-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \qquad \boxed{n > q}$$

STATE VARIABLE → SCALAR FORM

State Variable Representation

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x} + Bu$$
$$y = C\boldsymbol{x} + Du$$

This is the outcome of the bondgraph modeling process

Scalar Differential Equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^n} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u$$