ME242 – MECHANICAL ENGINEERING SYSTEMS

LECTURE 13:

- Laplace transform 7.2

LAPLACE TRANSFORM - DEFINITION

Function *f(t)* of time

Piecewise continuous and exponential order $|f(t)| < Ke^{bt}$

$$F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt \qquad \mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\alpha-j\infty}^{\alpha+j\infty} F(s)e^{st}ds$$

$0-$ limit is used to capture transients and discontinuities at $t=0$

s is a complex variable ($\sigma + j\omega$)

There is a need to worry about regions of convergence of the integral

Units of s are sec^{-1}=Hz

A frequency

If $f(t)$ is volts (amps) then $F(s)$ is volt-seconds (amp-seconds)

LAPLACE TRANSFORM – TABLE

Signal	Waveform	Transform
impulse	$\delta(t)$	1
step	$u(t)$	$\frac{1}{s}$
ramp	$tu(t)$	$\frac{1}{s^2}$
exponential	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
damped ramp	$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$
sine	$\sin(\beta t)u(t)$	$\frac{\beta}{s^2+\beta^2}$
cosine	$\cos(\beta t)u(t)$	$\frac{s}{s^2+\beta^2}$
damped sine	$e^{-\alpha t}\sin(\beta t)u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
damped cosine	$e^{-\alpha t}\cos(\beta t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$

LAPLACE TRANSFORM PROPERTIES

Linearity: (absolutely critical property)

$$\mathcal{L}\{Af_1(t)+Bf_2(t)\} = A\mathcal{L}\{f_1(t)\}+B\mathcal{L}\{f_2(t)\} = AF_1(s)+BF_2(s)$$

Integration property: $$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

Differentiation property: $$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s)-f(0-)$$

$$\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2F(s)-sf(0-)-f'(0-)$$

$$\mathcal{L}\left\{\frac{d^m f(t)}{dt^m}\right\} = s^mF(s)-s^{m-1}f(0-)-s^{m-2}f'(0-)-\cdots-f^{(m)}(0-)$$

LAPLACE TRANSFORM PROPERTIES

Translation properties:

s-domain translation: $\mathcal{L}\{e^{-\alpha t} f(t)\} = F(s+\alpha)$

t-domain translation: $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ for $a > 0$

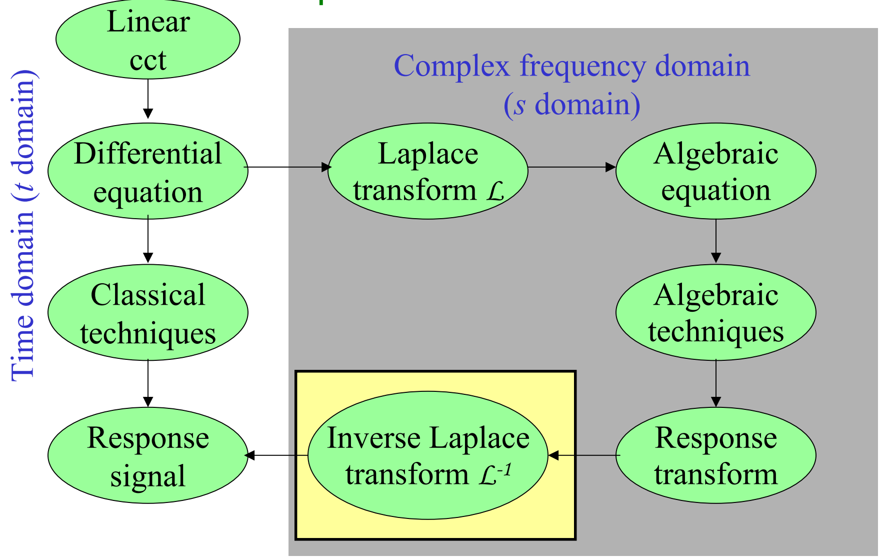
Initial Value Property:

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s)$$

Final Value Property:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Laplace transforms



The diagram commutes

Same answer whichever way you go

INVERTING LAPLACE TRANSFORM IN PRACTICE

We have a table of inverse LTs

Write F(s) as a partial fraction expansion

$$
\begin{aligned}
F(s) &= \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \\
&= K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \\
&= \frac{\alpha_1}{(s-p_1)} + \frac{\alpha_2}{(s-p_2)} + \frac{\alpha_{31}}{(s-p_3)} + \frac{\alpha_{32}}{(s-p_3)^2} + \frac{\alpha_{33}}{(s-p_3)^3} + \ldots + \frac{\alpha_q}{(s-p_q)}
\end{aligned}
$$

Now appeal to linearity to invert via the table

Surprise!

Nastiness: computing the partial fraction expansion is best done by calculating the residues

RATIONAL FUNCTIONS

We shall mostly be dealing with Laplace Transforms which are rational functions – ratios of polynomials in s

$$
\begin{aligned}
F(s) &= \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \\
&= K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}
\end{aligned}
$$

p_i are the poles and z_i are the zeros of the function

K is the scale factor or (sometimes) gain

A proper rational function has $n \geq m$

A strictly proper rational function has $n > m$

An improper rational function has $n < m$

RESIDUES AT SIMPLE POLES

Functions of a complex variable with isolated, finite order poles have *residues* at the poles

$$F(s)=K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}=\frac{k_1}{(s-p_1)}+\frac{k_2}{(s-p_2)}+\cdots+\frac{k_n}{(s-p_n)}$$

$$(s-p_i)F(s)=\frac{k_1(s-p_i)}{(s-p_1)}+\frac{k_2(s-p_i)}{(s-p_2)}+\cdots+k_i+\cdots+\frac{k_n(s-p_i)}{(s-p_n)}$$

Residue at a simple pole: $k_i=\lim_{s\to p_i}(s-p_i)F(s)$

RESIDUES AT SIMPLE POLES

Exercise: Find the Laplace transform $V(s)$

$$\frac{dv(t)}{dt}+6v(t)=4u(t)$$
$$v(0-)=-3$$

$$V(s)=\frac{4}{s(s+6)}-\frac{3}{s+6}$$
$$V(s)=\frac{4-3s}{s(s+6)}$$
$$v(t)=\frac{2}{3}u(t)-\frac{11}{3}e^{-6t}u(t)$$

Exercise: Find the Laplace transform $V(s)$

$$\frac{d^2v(t)}{dt^2}+4\frac{dv(t)}{dt}+3v(t)=5e^{-2t}$$
$$v(0-)=-2, v'(0-)=2$$

$$V(s)=\frac{5}{(s+1)(s+2)(s+3)}-\frac{2}{s+1}$$
$$V(s)=\frac{5-2(s+2)(s+3)}{(s+1)(s+2)(s+3)}$$
$$v(t)=\frac{1}{2}e^{-t}u(t)-5e^{-2t}u(t)+\frac{5}{2}e^{-3t}u(t)$$

What about $v(t)$?

RESIDUES AT MULTIPLE POLES

Let us assume that *V(s)* has a pole of order *n* at *s=-a*

Residues at a multiple pole:

$$k_m = \frac{1}{(m-1)!} \lim_{s \to a} \frac{d^{m-1}}{ds^{m-1}} \left[(s-a)^n F(s) \right], \quad m = 1 \cdots n$$

RESIDUES AT MULTIPLE POLES

Example: $V(s) = \frac{2s^2 + 5s}{(s+1)^3} = \frac{k_1}{s+1} + \frac{k_2}{(s+1)^2} + \frac{k_3}{(s+1)^3}$

$$k_1 = \frac{1}{2!} \lim_{s \to -1} \frac{d^2}{ds^2} \left[\frac{(s+1)^3 (2s^2 + 5s)}{(s+1)^3} \right] = 2$$

$$k_2 = \lim_{s \to -1} \frac{d}{ds} \left[\frac{(s+1)^3 (2s^2 + 5s)}{(s+1)^3} \right] = 1$$

$$k_3 = \lim_{s \to -1} \frac{(s+1)^3 (2s^2 + 5s)}{(s+1)^3} = -3$$

$$L^{-1}\left(\frac{2s^2 + 5s}{(s+1)^3} \right) = L^{-1}\left(\frac{2}{s+1} + \frac{1}{(s+1)^2} - \frac{3}{(s+1)^3} \right) = e^{-t}\left(2 + t - 3t^2\right)u(t)$$

RESIDUES AT COMPLEX POLES

Compute residues at the poles $\lim_{s \to a} (s-a)F(s)$

Bundle complex conjugate pole pairs into second-order terms if you want. But you will need to be careful!

$$(s-\alpha-j\beta)(s-\alpha+j\beta) = \left[s^2 - 2\alpha s + \left(\alpha^2+\beta^2\right)\right]$$

Inverse Laplace Transform is a sum of complex exponentials

RESIDUES AT COMPLEX POLES

Example: $V(s) = \dfrac{20(s+3)}{(s+1)(s^2+2s+5)} = \dfrac{k_1}{s+1} + \dfrac{k_2}{s+1-j2} + \dfrac{k_2^*}{s+1+j2}$

$$k_1 = \lim_{s \to -1} (s+1)F(s) = \left.\frac{20(s+3)}{s^2+2s+5}\right|_{s=-1} = 10$$

$$k_2 = \lim_{s \to -1+2j} (s+1-2j)F(s) = \left.\frac{20(s+3)}{(s+1)(s+1+2j)}\right|_{s=-1+2j} = -5-5j = 5\sqrt{2}e^{j\frac{5}{4}\pi}$$

$$L^{-1}\left\{\frac{20(s+3)}{(s+1)(s^2+2s+5)}\right\} = L^{-1}\left\{\frac{10}{s+1} + \frac{5\sqrt{2}e^{j\frac{5}{4}\pi}}{s+1-j2} + \frac{5\sqrt{2}e^{-j\frac{5}{4}\pi}}{s+1+j2}\right\}$$

$$= \left[10e^{-t} + 5\sqrt{2}e^{(-1+j2)t+j\frac{5}{4}\pi} + 5\sqrt{2}e^{(-1-j2)t-j\frac{5}{4}\pi}\right]u(t)$$

$$= \left[10e^{-t} + 10\sqrt{2}e^{-t}\cos(2t+\frac{5\pi}{4})\right]u(t)$$

NOT STRICTLY PROPER LAPLACE TRANSFORMS

Find the inverse LT of $V(s)=\dfrac{s^3+6s^2+12s+8}{s^2+4s+3}$

Convert to polynomial plus strictly proper rational function

Use polynomial division

$$\begin{aligned} V(s) &= s+2+\frac{s+2}{s^2+4s+3} \\ &= s+2+\frac{0.5}{s+1}+\frac{0.5}{s+3} \end{aligned}$$

Invert as normal

$$v(t)=\left[\frac{d\delta(t)}{dt}+2\delta(t)+0.5e^{-t}+0.5e^{-3t}\right]u(t)$$