ME242 – MECHANICAL ENGINEERING SYSTEMS

LECTURE 12:

- Laplace transform 7.2

HOMEWORK 4: Due on 2/18/05

- Problem 3.42
- Problem 3.43
- Problem 3.47
- Problem 3.48
- Problem 3.50 (Using MATLAB)

LAPLACE TRANSFORM - DEFINITION

Function *f(t)* of time

Piecewise continuous and exponential order $|f(t)| < Ke^{bt}$

$$F(s) = \int_{0-}^{\infty} f(t) e^{-st} dt \qquad \mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\alpha-j\infty}^{\alpha+j\infty} F(s) e^{st} ds$$

0- limit is used to capture transients and discontinuities at $t=0$

s is a complex variable ($\sigma + j\omega$)

There is a need to worry about regions of convergence of the integral

Units of s are sec^{-1}=Hz

A frequency

If *f(t)* is volts (amps) then *F(s)* is volt-seconds (amp-seconds)

LAPLACE TRANSFORM - EXAMPLES

Step function – unit Heavyside Function

After Oliver Heavyside (1850-1925)

$$u(t)=\begin{cases}0, & \text{for } t<0\\ 1, & \text{for } t\geq 0\end{cases}$$

$$F(s)=\int_{0-}^{\infty}u(t)e^{-st}dt=\int_{0-}^{\infty}e^{-st}dt=-\frac{e^{-st}}{s}\Big|_0^{\infty}=-\frac{e^{-(\sigma+j\omega)t}}{\sigma+j\omega}\Big|_0^{\infty}=\frac{1}{s}\ \text{if}\ \ \sigma>0$$

Exponential function

After Oliver Exponential (1176 BC- 1066 BC)

$$F(s)=\int_{0}^{\infty}e^{-\alpha t}e^{-st}dt=\int_{0}^{\infty}e^{-(s+\alpha)t}dt=-\frac{e^{-(s+\alpha)t}}{s+\alpha}\Big|_0^{\infty}=\frac{1}{s+\alpha}\ \text{if}\ \sigma>\alpha$$

Delta (impulse) function $\delta(t)$

$$F(s)=\int_{0-}^{\infty}\delta(t)e^{-st}dt=1\ \text{ for all } s$$

LAPLACE TRANSFORM – PAIR TABLES

Signal	Waveform	Transform
impulse	$\delta(t)$	1
step	$u(t)$	$\frac{1}{s}$
ramp	$tu(t)$	$\frac{1}{s^2}$
exponential	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$
damped ramp	$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$
sine	$\sin(\beta t)u(t)$	$\frac{\beta}{s^2+\beta^2}$
cosine	$\cos(\beta t)u(t)$	$\frac{s}{s^2+\beta^2}$
damped sine	$e^{-\alpha t}\sin(\beta t)u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
damped cosine	$e^{-\alpha t}\cos(\beta t)u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$

LAPLACE TRANSFORM PROPERTIES

Linearity: (absolutely critical property)

Follows from the integral definition

$$\mathcal{L}\{Af_1(t)+Bf_2(t)\}=A\mathcal{L}\{f_1(t)\}+B\mathcal{L}\{f_2(t)\}=AF_1(s)+BF_2(s)$$

Example:

$$\mathcal{L}(A\cos(\beta t))=\mathcal{L}\left(\frac{A}{2}\left[e^{j\beta t}+e^{-j\beta t}\right]\right)=\frac{A}{2}\mathcal{L}\left(e^{j\beta t}\right)+\frac{A}{2}\mathcal{L}\left(e^{-j\beta t}\right)$$

$$=\frac{A}{2}\frac{1}{s-j\beta}+\frac{A}{2}\frac{1}{s+j\beta}$$

$$=\frac{As}{s^2+\beta^2}$$

LAPLACE TRANSFORM PROPERTIES

Integration property:

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}=\frac{F(s)}{s}$$

Differentiation property:

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\}=sF(s)-f(0-)$$

$$\mathcal{L}\left\{\frac{d^2f(t)}{dt^2}\right\}=s^2F(s)-sf(0-)-f'(0-)$$

$$\vdots$$

$$\mathcal{L}\left\{\frac{d^mf(t)}{dt^m}\right\}=s^mF(s)-s^{m-1}f(0-)-s^{m-2}f'(0-)-\cdots-f^{(m)}(0-)$$

LAPLACE TRANSFORM PROPERTIES

Translation properties:

s-domain translation: $\mathcal{L}\{e^{-\alpha t} f(t)\} = F(s+\alpha)$

t-domain translation: $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ for $a > 0$

Initial Value Property:

$$\lim_{t \to 0+} f(t) = \lim_{s \to \infty} sF(s)$$

Final Value Property:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

LAPLACE TRANSFORM

Exercise: Find the Laplace transform of the following waveform

$$f(t) = [2 + 2\sin(2t) - 2\cos(2t)]u(t) \qquad F(s) = \frac{4(s+2)}{s(s^2+4)}$$

Exercise: Find the Laplace transform of the following waveform

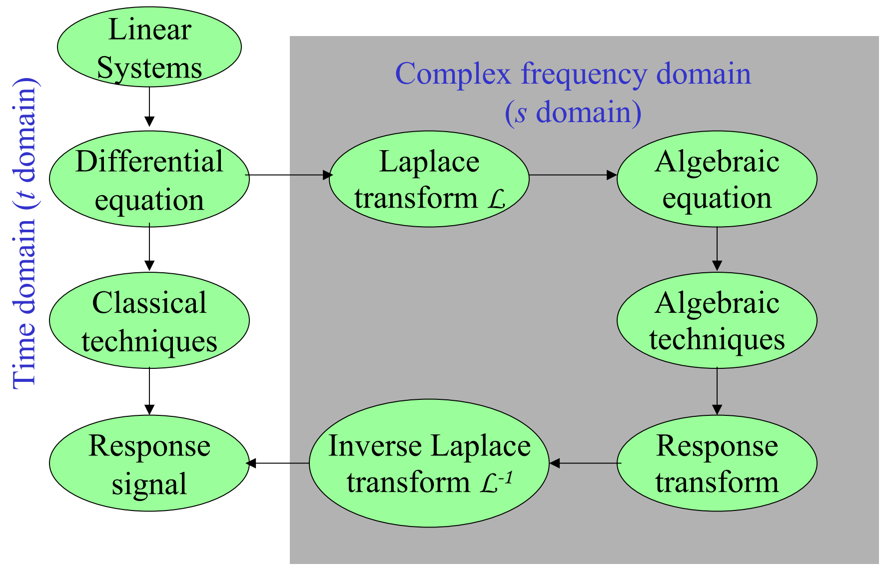
$$f(t) = e^{-4t}u(t) + 5\int_0^t \sin(4x)dx \qquad F(s) = \frac{s^3 + 36s + 80}{s(s+4)(s^2+16)}$$

$$f(t) = 5e^{-40t}u(t) + \frac{d\left[5te^{-40t}\right]}{dt}u(t) \qquad F(s) = \frac{10s + 200}{(s+40)^2}$$

Exercise: Find the Laplace transform of the following waveform

$$f(t) = Au(t) - 2Au(t-T) + Au(t-2T) \qquad F(s) = \frac{A\left(1 - e^{-Ts}\right)^2}{s}$$

LAPLACE TRANSFORM-SOLVING ODEs



The diagram commutes

Same answer whichever way you go

LAPLACE TRANSFORM

Exercise: Find the Laplace transform $V(s)$

$$\frac{dv(t)}{dt} + 6v(t) = 4u(t) \qquad V(s) = \frac{4}{s(s+6)} - \frac{3}{s+6}$$

$$v(0-) = -3$$

Exercise: Find the Laplace transform $V(s)$

$$\frac{d^2v(t)}{dt^2} + 4\frac{dv(t)}{dt} + 3v(t) = 5e^{-2t} \qquad V(s) = \frac{5}{(s+1)(s+2)(s+3)} - \frac{2}{s+1}$$

$$v(0-) = -2, v'(0-) = 2$$

What about $v(t)$?