ME242 – MECHANICAL ENGINEERING SYSTEMS

LECTURE 11:

- Solutions of Linear Differential Equations 3.5

FIRST ORDER DIFFERENTIAL EQUATIONS

$$\tau \frac{dx}{dt} + x = f_1(t) \Leftrightarrow \frac{dx}{dt} + \frac{1}{\tau} x = f_2(t) = \frac{1}{\tau} f_1(t)$$

These equations are derived from the RC and IR models

Solution: $x(t) = x_H(t) + x_P(t)$

Particular solution

Homogeneous solution

Homogeneous Solution: $\frac{dx_H}{dt} + \frac{1}{\tau} x_H = 0$

Particular Solution: $\frac{dx_P}{dt} + \frac{1}{\tau} x_P = f_2(t)$

FIRST ORDER DIFFERENTIAL EQUATIONS

$$\tau\frac{dx}{dt}+x=0 \Leftrightarrow \frac{dx}{dt}+\frac{1}{\tau}x=0$$ Unforced

Initial Condition: $x(0)=x_o$

Homogeneous Solution: $\frac{dx_H}{dt}+\frac{1}{\tau}x_H=0$

$$x_H=Ae^{st} \Rightarrow sAe^{st}+\frac{1}{\tau}Ae^{st}=0 \Rightarrow s+\frac{1}{\tau}=0 \Rightarrow s=-\frac{1}{\tau}$$

$$x_H(t)=Ae^{-\frac{t}{\tau}} \Rightarrow x_H(0)=A=x_o$$

$$x(t)=x_H(t)=x_o e^{-\frac{t}{\tau}}$$

FIRST ORDER DIFFERENTIAL EQUATIONS

$$\tau\frac{dx}{dt}+x=f_1(t) \Leftrightarrow \frac{dx}{dt}+\frac{1}{\tau}x=f_2(t)$$ Forced

Forcing Term: $f_2(t)=x_f=cte$ Step

Initial Condition: $x(0)=x_o$

Homogeneous Solution: $\frac{dx_H}{dt}+\frac{1}{\tau}x_H=0 \Rightarrow x_H(t)=Ae^{-\frac{t}{\tau}}$

Particular Solution: $\frac{dx_P}{dt}+\frac{1}{\tau}x_P=x_f \Rightarrow x_P(t)=\tau x_f$

$$x(t)=x_H(t)+x_P(t)=Ae^{-\frac{t}{\tau}}+\tau x_f$$

$$x(0)=A+\tau x_f=x_o \Rightarrow A=x_o-\tau x_f$$

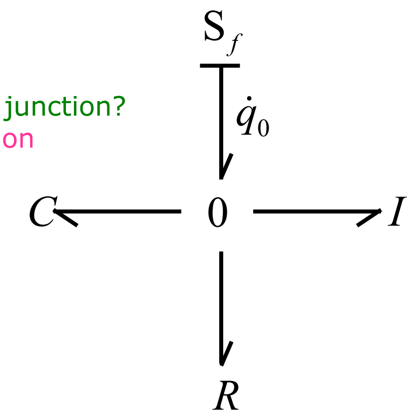
$$x(t)=\left(x_o-\tau x_f\right)e^{-\frac{t}{\tau}}+\tau x_f$$

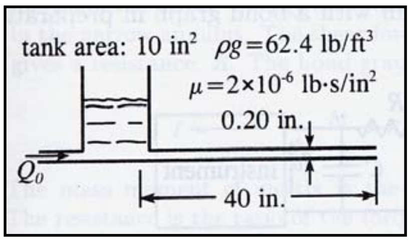
DYNAMIC MODELS

Example: Fluid System

Shared Variable?
Pressure (effort)

Type of junction?
0-Junction





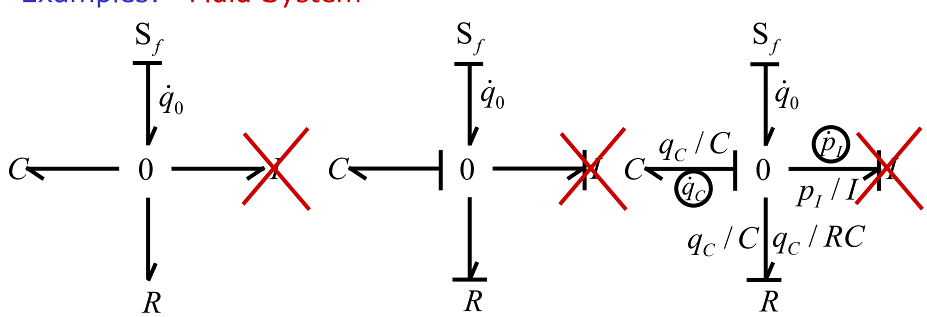
$$C = \frac{A}{\rho g},\ R = \frac{8\mu L}{\pi r^4},\ I = \frac{\rho L}{\pi r^2}$$

Type of behavior?
Coupled

$$\dot{q}_o = Q$$

DYNAMIC MODELS

Examples: Fluid System



a- Model

b- Causality added

c- Annotation causal bonds

$$\cancel{\dot{p}_I = \frac{q_C}{C}}$$

$$\dot{q}_o = \dot{q}_C + \frac{q_C}{RC} + \cancel{\frac{p_I}{I}}$$

$$\boxed{\begin{gathered}\dot{q}_o = Q,\quad q_C = V \\ \Downarrow \\ Q = \frac{dV}{dt} + \frac{1}{RC}V\end{gathered}}$$

DYNAMIC MODELS

Examples: Fluid System

$$\boxed{\frac{dV}{dt}+\frac{1}{\tau}V=Q}$$

IC: $V(0)=V_o \qquad \tau=RC$

Homogeneous Solution: $\frac{dV_H}{dt}+\frac{1}{\tau}V_H=0 \Rightarrow V_H(t)=Ae^{-\frac{t}{\tau}}$

Particular Solution: $\frac{dV_P}{dt}+\frac{1}{\tau}V_P=Q \Rightarrow V_P(t)=\tau Q$

General Solution: $V(t)=V_H(t)+V_P(t)=Ae^{-\frac{t}{\tau}}+\tau Q$

$$V(0)=A+\tau Q=V_o \Rightarrow A=V_o-\tau Q$$

$$\boxed{V(t)=(V_o-\tau Q)e^{-\frac{t}{\tau}}+\tau Q}$$

SECOND ORDER DIFFERENTIAL EQUATIONS

$$a_2\frac{d^2x}{dt^2}+a_1\frac{dx}{dt}+a_0x=f_1(t) \Leftrightarrow \frac{d^2x}{dt^2}+\frac{a_1}{a_2}\frac{dx}{dt}+\frac{a_0}{a_2}x=f_2(t)=\frac{1}{a_2}f_1(t)$$

$$\boxed{\begin{array}{l}\omega_n=\sqrt{\frac{a_0}{a_2}}\\ \varsigma=\frac{a_1}{2\sqrt{a_0a_2}}\end{array}} \qquad \frac{d^2x}{dt^2}+2\varsigma\omega_n\frac{dx}{dt}+\omega_n^2x=f_2(t)$$

These equations are derived from the RIC models

Solution: $x(t)=x_H(t)+x_P(t)$

Particular solution

Homogeneous solution

Homogeneous Solution: $\frac{d^2x_H}{dt^2}+2\varsigma\omega_n\frac{dx_H}{dt}+\omega_n^2x_H=0$

Particular Solution: $\frac{d^2x_P}{dt^2}+2\varsigma\omega_n\frac{dx_P}{dt}+\omega_n^2x_P=f_2(t)$

SECOND ORDER DIFFERENTIAL EQUATIONS

$$\frac{d^2x}{dt^2} + 2\varsigma\omega_n \frac{dx}{dt} + \omega_n^2 x = 0 \qquad \text{Unforced}$$

Initial Conditions: $x(0) = x_o, \quad \dot{x}(0) = \dot{x}_o$

Homogeneous Solution: $\displaystyle \frac{d^2x_H}{dt^2} + 2\varsigma\omega_n \frac{dx_H}{dt} + \omega_n^2 x_H = 0$

$$x_H = Ae^{st} \Rightarrow As^2e^{st} + 2\varsigma\omega_n Ase^{st} + \omega_n^2 Ae^{st} = 0$$

$$s^2 + 2\varsigma\omega_n s + \omega_n^2 = 0$$

SECOND ORDER DIFFERENTIAL EQUATIONS

Characteristic Equation: $s^2 + 2\varsigma\omega_n s + \omega_n^2 = 0$

$$(s + \varsigma\omega_n)^2 + \omega_n^2(1 - \varsigma^2) = 0$$

ω_n : Undamped natural frequency

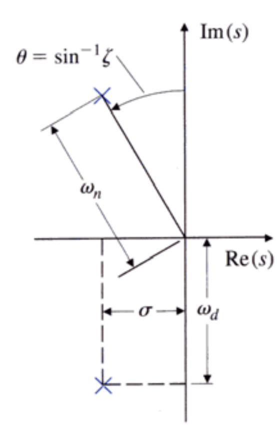
ζ : Damping ratio

$$(s + \sigma + j\omega_d)(s + \sigma - j\omega_d) = 0$$

$$(s + \sigma)^2 + \omega_d^2 = 0$$

$$\sigma = \zeta\omega_n, \quad \omega_d = \omega_n\sqrt{1 - \zeta^2}$$

ω_d : Damped natural frequency



SECOND ORDER DIFFERENTIAL EQUATIONS

$$(s+\sigma)^2+\omega_d^2=0 \Rightarrow s_{1,2}=-\sigma \pm j\omega_d=-\varsigma\omega_n \pm j\sqrt{\varsigma^2-1}\omega_n$$

$$x_H(t)=C_1e^{s_1t}+C_2e^{s_2t} \Rightarrow \quad x_H(0)=C_1+C_2=x_o$$

$$\dot{x}_H(0)=C_1s_1+C_2s_2=\dot{x}_o$$

$$\boxed{C_1=\frac{x_os_2-\dot{x}_o}{s_2-s_1}, C_2=\frac{\dot{x}_o-x_os_1}{s_2-s_1}}$$

Complex Poles:

$$x_H(t)=e^{-\sigma t}\left[A_1\cos(\omega_dt)+A_2\sin(\omega_dt)\right] \Rightarrow x_H(0)=A_1=x_o$$

$$\dot{x}_H(0)=\sigma A_1+\omega_dA_2=\dot{x}_o$$

$$\boxed{A_1=x_o, A_2=\frac{\dot{x}_o-\sigma x_o}{\omega_d}}$$

SECOND ORDER DIFFERENTIAL EQUATIONS

$$\frac{d^2x}{dt^2}+2\varsigma\omega_n\frac{dx}{dt}+\omega_n^2x=f_2(t) \qquad \text{Forced}$$

Forcing Term: $f_2(t)=x_f=cte$

Initial Conditions: $x(0)=x_o, \quad \dot{x}(0)=\dot{x}_o$

Homogeneous Solution:

$$\frac{d^2x_H}{dt^2}+2\varsigma\omega_n\frac{dx_H}{dt}+\omega_n^2x_H=0$$

$$x_H(t)=C_1e^{s_1t}+C_2e^{s_2t}$$

Particular Solution:

$$\frac{d^2x_P}{dt^2}+2\varsigma\omega_n\frac{dx_P}{dt}+\omega_n^2x_P=x_f$$

$$x_P(t)=\frac{x_f}{\omega_n^2}$$

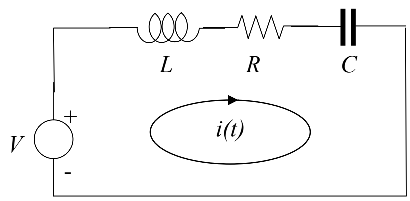
$$x(t)=x_H(t)+x_P(t)=C_1e^{s_1t}+C_2e^{s_2t}+\frac{x_f}{\omega_n^2}$$

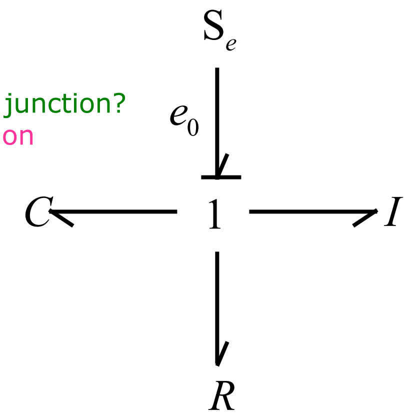
DYNAMIC MODELS

Example: RLC Series

Shared Variable?
Current (flow)

Type of junction?
1-Junction





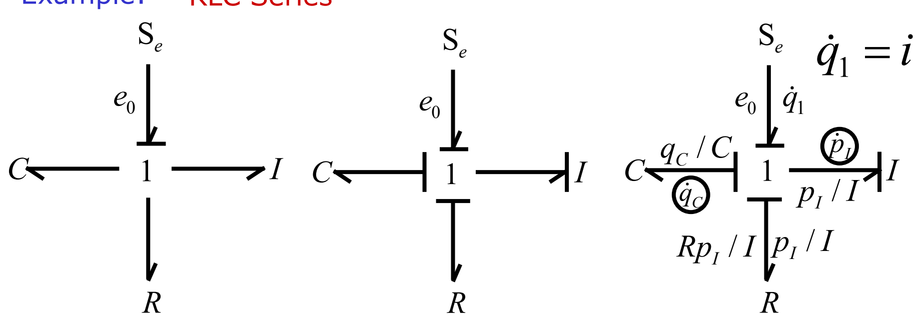
$$\mathrm{C} = C,\ \ \mathrm{R} = R,\ \ \mathrm{I} = L$$

$$e_o = V$$

Type of behavior?
Coupled

DYNAMIC MODELS

Example: RLC Series



a- Model b- Causality added c- Annotation causal bonds

$$\dot{q}_C = \frac{p_I}{I}$$

$$e_o = \frac{q_C}{C} + \dot{p}_I + R\frac{p_I}{I}$$

$$e_o = V, \quad V_C = \frac{q_C}{C}$$

$$\Downarrow$$

$$V = V_C + LC\frac{d^2V_C}{dt^2} + RC\frac{dV_C}{dt}$$

DYNAMIC MODELS

Example: RLC Series

$$\boxed{\frac{d^2V_C}{dt^2}+\frac{R}{L}\frac{dV_C}{dt}+\frac{1}{LC}V_C=\frac{V}{LC}}$$

IC: $V_C(0)=V_o$

$\dot{V}_C(0)=0$

Homogeneous Solution: $$\frac{d^2V_{CH}}{dt^2}+2\varsigma\omega_n\frac{dV_{CH}}{dt}+\omega_n^2V_{CH}=0$$

$$V_{CH}(t)=C_1e^{s_1t}+C_2e^{s_2t}$$

Complex Conjugate Poles

$$V_{CH}(t)=e^{\sigma t}\left[A_1\cos(\omega_d t)+A_2\sin(\omega_d t)\right]$$

Particular Solution: $$\frac{d^2V_{CP}}{dt^2}+2\varsigma\omega_n\frac{dV_{CP}}{dt}+\omega_n^2V_{CP}=\omega_n^2V$$

$$V_{CP}(t)=V$$

DYNAMIC MODELS

Example: RLC Series

$$\boxed{\frac{d^2V_C}{dt^2}+\frac{R}{L}\frac{dV_C}{dt}+\frac{1}{LC}V_C=\frac{V}{LC}}$$

IC: $V_C(0)=V_o$

$\dot{V}_C(0)=0$

General Solution:

$$V_C(t)=V_{CH}(t)+V_{CP}(t)=e^{\sigma t}\left[A_1\cos(\omega_d t)+A_2\sin(\omega_d t)\right]+V$$

$$\dot{V}_C(t)=\sigma e^{\sigma t}\left[A_1\cos(\omega_d t)+A_2\sin(\omega_d t)\right]+e^{\sigma t}\omega_d\left[-A_1\sin(\omega_d t)+A_2\cos(\omega_d t)\right]$$

$$V_C(0)=A_1+V=V_o\Rightarrow A_1=V_o-V$$

$$\dot{V}_C(0)=\sigma A_1+\omega_d A_2=0\Rightarrow A_2=-\frac{\sigma}{\omega_d}(V_o-V)$$

$$\boxed{V_C(t)=e^{\sigma t}\left[(V_o-V)\cos(\omega_d t)-\frac{\sigma}{\omega_d}(V_o-V)\sin(\omega_d t)\right]+V}$$