ME242 – MECHANICAL ENGINEERING SYSTEMS

LECTURE 10:

- Solutions of Linear Differential Equations 3.5

FIRST ORDER DIFFERENTIAL EQUATIONS

$$\tau \frac{dx}{dt} + x = f_1(t) \Leftrightarrow \frac{dx}{dt} + \frac{1}{\tau} x = f_2(t) = \frac{1}{\tau} f_1(t)$$

These equations are derived from the RC and IR models

Solution: $x(t) = x_H(t) + x_P(t)$

Particular solution

Homogeneous solution

Homogeneous Solution: $\frac{dx_H}{dt} + \frac{1}{\tau} x_H = 0$

Particular Solution: $\frac{dx_P}{dt} + \frac{1}{\tau} x_P = f_2(t)$

FIRST ORDER DIFFERENTIAL EQUATIONS

$$\tau \frac{dx}{dt} + x = 0 \Leftrightarrow \frac{dx}{dt} + \frac{1}{\tau} x = 0$$ Unforced

Initial Condition: $x(0) = x_o$

Homogeneous Solution: $\dfrac{dx_H}{dt} + \dfrac{1}{\tau} x_H = 0$

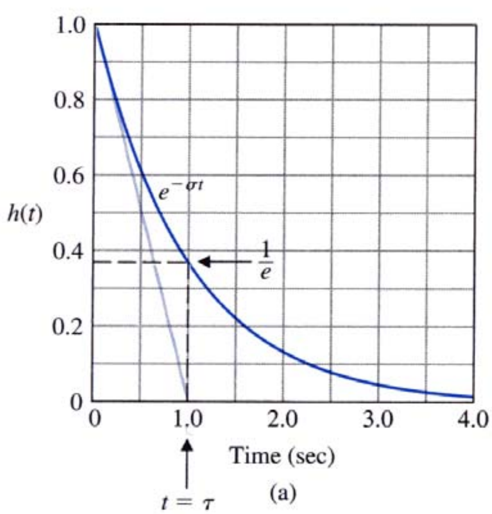
$$x_H = Ae^{st} \Rightarrow sAe^{st} + \frac{1}{\tau} Ae^{st} = 0 \Rightarrow s + \frac{1}{\tau} = 0 \Rightarrow s = -\frac{1}{\tau}$$

$$x_H(t) = Ae^{-\frac{t}{\tau}} \Rightarrow x_H(0) = A = x_o$$

$$x(t) = x_H(t) = x_o e^{-\frac{t}{\tau}}$$

FIRST ORDER DIFFERENTIAL EQUATIONS

$$x(t) = x_H(t) = x_o e^{-\frac{t}{\tau}} \Leftrightarrow h(t) = \frac{x(t)}{x_o} = e^{-\frac{t}{\tau}}$$



(a)

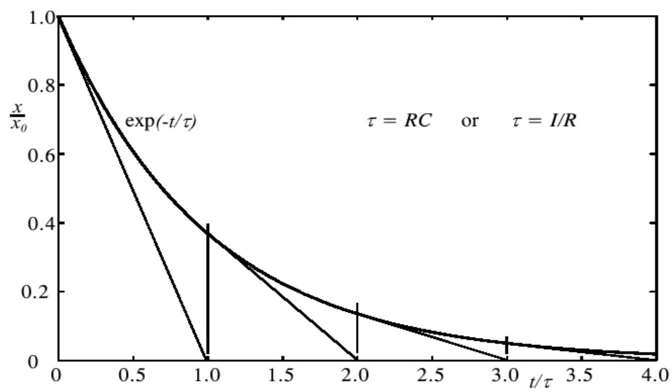
$\sigma > 0$ Stable

$\sigma < 0$ Unstable

$\tau = \dfrac{1}{\sigma}$ Time Constant

FIRST ORDER DIFFERENTIAL EQUATIONS

$$x(t) = x_H(t) = x_o e^{-\frac{t}{\tau}} \Leftrightarrow h(t) = \frac{x(t)}{x_o} = e^{-\frac{t}{\tau}}$$



$$\dot{h}(t) = -\frac{1}{\tau} e^{-\frac{t}{\tau}}$$

$$\Downarrow$$

$$\dot{h}(0) = -\frac{1}{\tau}$$

FIRST ORDER DIFFERENTIAL EQUATIONS

$$\tau \frac{dx}{dt} + x = f_1(t) \Leftrightarrow \frac{dx}{dt} + \frac{1}{\tau} x = f_2(t)$$ Forced

Forcing Term: $f_2(t) = x_f = cte$ Step

Initial Condition: $x(0) = x_o$

Homogeneous Solution: $\frac{dx_H}{dt} + \frac{1}{\tau} x_H = 0 \Rightarrow x_H(t) = Ae^{-\frac{t}{\tau}}$

Particular Solution: $\frac{dx_P}{dt} + \frac{1}{\tau} x_P = x_f \Rightarrow x_P(t) = \tau x_f$

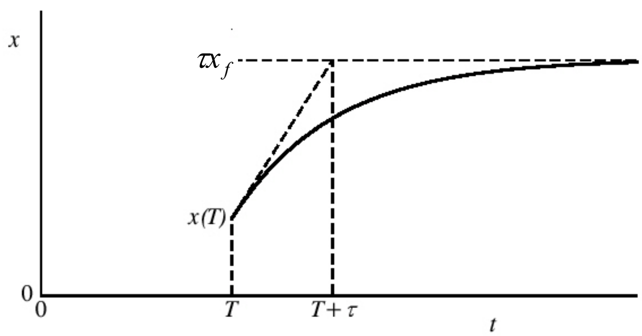
$$x(t) = x_H(t) + x_P(t) = Ae^{-\frac{t}{\tau}} + \tau x_f$$

$$x(0) = A + \tau x_f = x_o \Rightarrow A = x_o - \tau x_f$$

$$x(t) = \left(x_o - \tau x_f\right) e^{-\frac{t}{\tau}} + \tau x_f$$

FIRST ORDER DIFFERENTIAL EQUATIONS

$$x(t) = x_H(t) + x_P(t) = \left(x_o - \tau x_f\right)e^{-\frac{t}{\tau}} + \tau x_f \qquad (T = 0)$$



SECOND ORDER DIFFERENTIAL EQUATIONS

$$a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0 x = f_1(t) \Leftrightarrow \frac{d^2x}{dt^2} + \frac{a_1}{a_2}\frac{dx}{dt} + \frac{a_0}{a_2}x = f_2(t) = \frac{1}{a_2}f_1(t)$$

$$\boxed{\begin{aligned}\omega_n &= \sqrt{\frac{a_0}{a_2}} \\ \varsigma &= \frac{a_1}{2\sqrt{a_0 a_2}}\end{aligned}} \qquad \frac{d^2x}{dt^2} + 2\varsigma\omega_n\frac{dx}{dt} + \omega_n^2 x = f_2(t)$$

These equations are derived from the RIC models

Solution: $x(t) = x_H(t) + x_P(t)$

Particular solution

Homogeneous solution

Homogeneous Solution: $\dfrac{d^2x_H}{dt^2} + 2\varsigma\omega_n\dfrac{dx_H}{dt} + \omega_n^2 x_H = 0$

Particular Solution: $\dfrac{d^2x_P}{dt^2} + 2\varsigma\omega_n\dfrac{dx_P}{dt} + \omega_n^2 x_P = f_2(t)$

SECOND ORDER DIFFERENTIAL EQUATIONS

$$\frac{d^2x}{dt^2} + 2\varsigma\omega_n \frac{dx}{dt} + \omega_n^2 x = 0 \qquad \text{Unforced}$$

Initial Conditions: $x(0) = x_o, \quad \dot{x}(0) = \dot{x}_o$

Homogeneous Solution: $$\frac{d^2x_H}{dt^2} + 2\varsigma\omega_n \frac{dx_H}{dt} + \omega_n^2 x_H = 0$$

$$x_H = Ae^{st} \Rightarrow As^2e^{st} + 2\varsigma\omega_n Ase^{st} + \omega_n^2 Ae^{st} = 0$$

$$s^2 + 2\varsigma\omega_n s + \omega_n^2 = 0$$

SECOND ORDER DIFFERENTIAL EQUATIONS

Characteristic Equation: $$s^2 + 2\varsigma\omega_n s + \omega_n^2 = 0$$

$$(s + \varsigma\omega_n)^2 + \omega_n^2(1 - \varsigma^2) = 0$$

ω_n : Undamped natural frequency

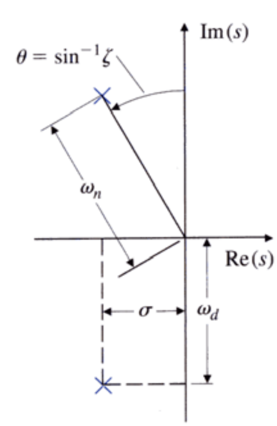
ζ : Damping ratio

$$(s + \sigma + j\omega_d)(s + \sigma - j\omega_d) = 0$$

$$(s + \sigma)^2 + \omega_d^2 = 0$$

$$\sigma = \zeta\omega_n, \quad \omega_d = \omega_n\sqrt{1 - \zeta^2}$$

ω_d : Damped natural frequency



SECOND ORDER DIFFERENTIAL EQUATIONS

Complex poles:

$$(s+\varsigma\omega_n)^2+\omega_n^2(1-\varsigma^2)=0$$

$$\Downarrow$$

$$s=-\varsigma\omega_n \pm j\sqrt{\varsigma^2-1}\omega_n=-\sigma \pm j\omega_d$$

CASES:

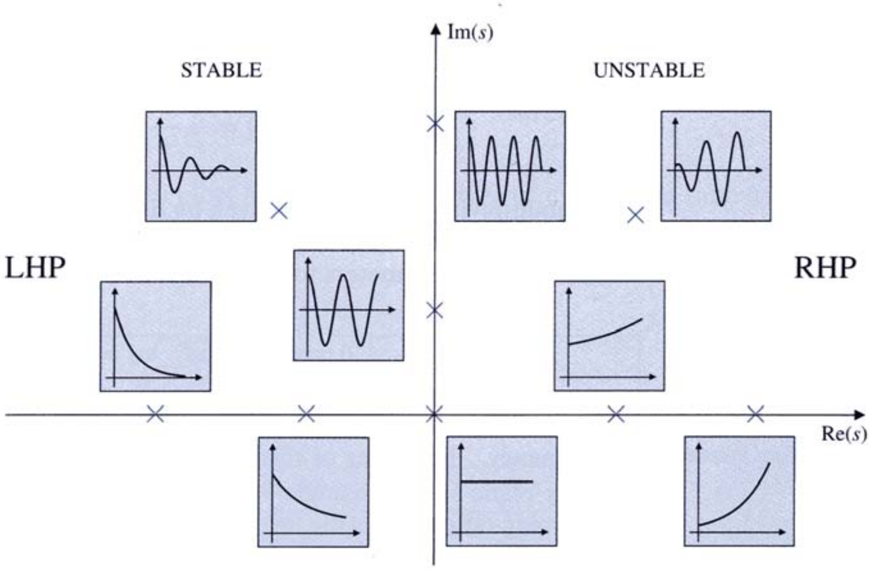
$\zeta=0: s^2+\omega_n^2$ — Undamped

$\zeta<1: (s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)$ — Underdamped

$\zeta=1: (s+\omega_n)^2$ — Critically damped

$\zeta>1: \left[s+\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n\right]\left[s+\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n\right]$ — Overdamped

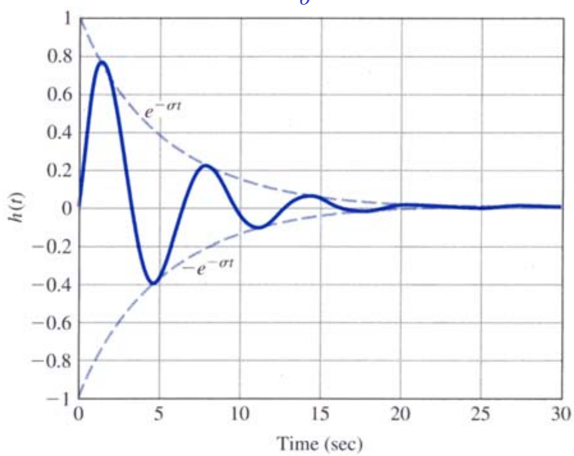
SECOND ORDER DIFFERENTIAL EQUATIONS



SECOND ORDER DIFFERENTIAL EQUATIONS

Complex poles:

$$h(t) = \frac{x(t)}{x_o} = e^{-\sigma t}\left[C\cos(\omega_d t) + D\sin(\omega_d t)\right]$$



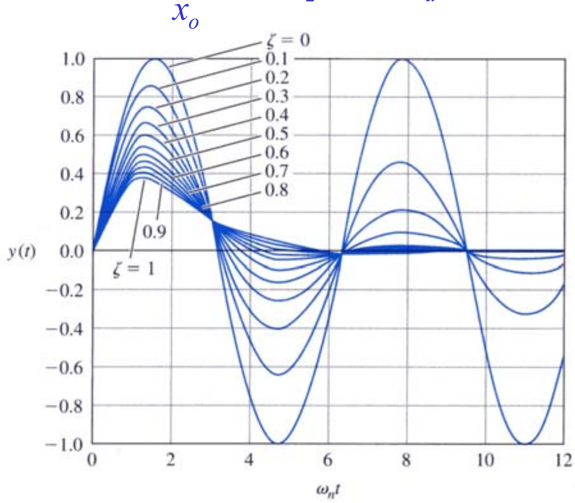
$\sigma > 0$ Stable

$\sigma < 0$ Unstable

SECOND ORDER DIFFERENTIAL EQUATIONS

Complex poles:

$$h(t) = \frac{x(t)}{x_o} = e^{-\sigma t}\left[C\cos(\omega_d t) + D\sin(\omega_d t)\right]$$



SECOND ORDER DIFFERENTIAL EQUATIONS

$$\frac{d^2x}{dt^2} + 2\varsigma\omega_n \frac{dx}{dt} + \omega_n^2 x = f_2(t) \qquad \text{Forced}$$

Forcing Term: $f_2(t) = x_f = cte$

Initial Conditions: $x(0) = x_o, \quad \dot{x}(0) = \dot{x}_o$

Homogeneous Solution:
$$\frac{d^2x_H}{dt^2} + 2\varsigma\omega_n \frac{dx_H}{dt} + \omega_n^2 x_H = 0$$
$$x_H(t) = e^{-\sigma t}\left[C\cos(\omega_d t) + D\sin(\omega_d t)\right]$$

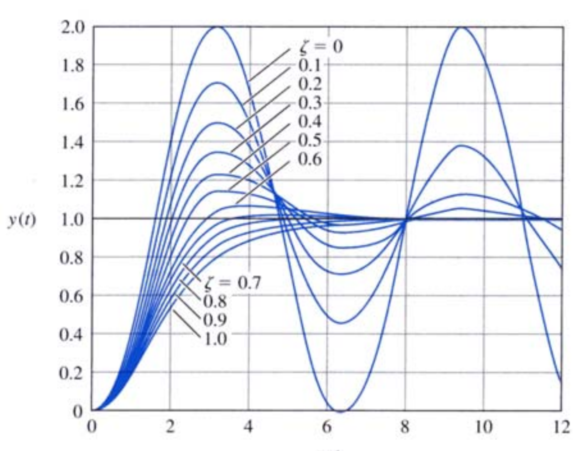
Particular Solution:
$$\frac{d^2x_P}{dt^2} + 2\varsigma\omega_n \frac{dx_P}{dt} + \omega_n^2 x_P = x_f$$
$$x_P(t) = \frac{x_f}{\omega_n^2}$$

$$x(t) = x_H(t) + x_P(t) = e^{-\sigma t}\left[C\cos(\omega_d t) + D\sin(\omega_d t)\right] + \frac{x_f}{\omega_n^2}$$

SECOND ORDER DIFFERENTIAL EQUATIONS

Complex poles:

$$x(t) = e^{-\sigma t}\left[C\cos(\omega_d t) + D\sin(\omega_d t)\right] + \frac{x_f}{\omega_n^2}$$



Step Response