# ME242 – MECHANICAL ENGINEERING SYSTEMS

## LECTURE 5:

- Inertance Energy Storage 3.2



## Dynamic Systems

So far, we have introduced:

| | | |
|---|---|---|
| SOURCES | Emanate Energy | $\Big\}$ Steady or Equilibrium Systems |
| RESISTANCES | Dissipate Energy | |

We need new players to be able to represent dynamic systems

| | | |
|---|---|---|
| COMPLIANCES | Store Energy | $\Big\}$ Unsteady or Dynamic Systems |
| INERTANCES | Store Energy | |

- Dynamic physical systems contain mechanisms that store energy temporarily, for later release.

- The dynamics can be thought of as a sloshing of energy between different energy storage mechanisms, and/or a gradual dissipation of energy in resistances.

## Generalized Variables

Generalized Velocity or Flow: $f$

Generalized Displacement: $q$

$$f = \dot{q}, \text{ or } q = \int f dt$$

Generalized Force or Effort: $e$

Generalized Momentum: $p$

$$e = \dot{p}, \text{ or } p = \int e dt$$

$$\boxed{\mathrm{P} = ef = e\dot{q} = \dot{p}f}$$



## ENERGY STORAGE: COMPLIANCE & INERTANCE

Power: $P = ef$

Energy: $E = \int P dt = \int ef\, dt$

### Energy Storage Mechanisms

**Compliance**

Store energy by virtue of a generalized displacement

$$E = \int ef\, dt = \int e\dot{q} dt = \int e \frac{dq}{dt} dt = \int e dq$$

**Inertance**

Store energy by virtue of a generalized momentum

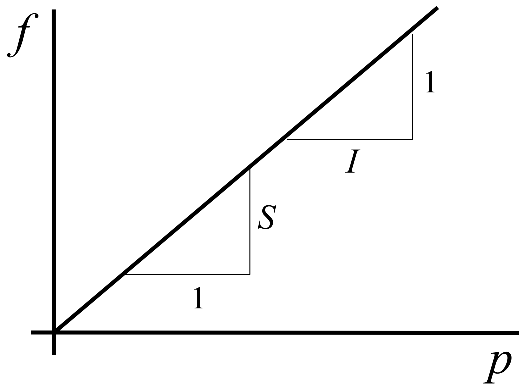
$$E = \int ef\, dt = \int \dot{p} f\, dt = \int \frac{dp}{dt} f\, dt = \int f\, dp$$

## ENERGY STORAGE: LINEAR INERTANCE

$$\frac{e = \dot{p}}{\dot{q}} \; I$$

$$\boxed{f = \frac{1}{I} p}$$

Constitutive Relationship




$I$ Inertance

$S = \dfrac{1}{I}$ Susceptance



## ENERGY STORAGE: LINEAR INERTANCE

$$\frac{e = \dot{p}}{\dot{q}} \; I$$

$$\boxed{f = \frac{1}{I} p}$$

Energy Storage:

Work from point 1 to point 2:

$$W_{1\to2} = \int_1^2 \dot{p} f\, dt = \int_1^2 \frac{dp}{dt} f\, dt = \int_{p_1}^{p_2} f\, dp = \int_{p_1}^{p_2} \frac{1}{I} p dp = \frac{1}{2I} p^2 \Big|_{p_1}^{p_2} = \frac{1}{2I}\left(p_2^2 - p_1^2\right)$$

Work from point 2 to point 1:

$$W_{2\to1} = \frac{1}{2I}\left(p_1^2 - p_2^2\right) = -W_{1\to2}$$

The energy is conserved!

Kinetic Energy:

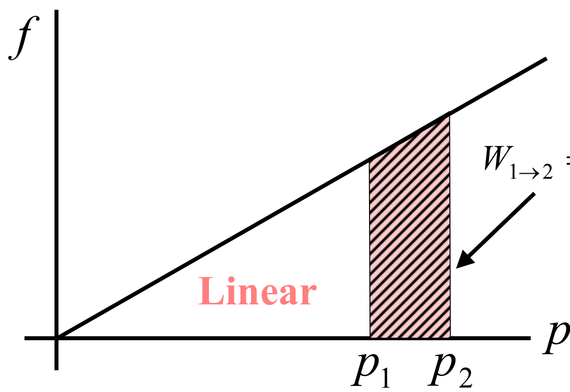
$$T = \frac{1}{2I} p^2 = \frac{1}{2} I \dot{q}^2 \quad \Rightarrow \quad W_{1\to2} = T_2 - T_1$$

# ENERGY STORAGE: LINEAR INERTANCE

Energy Storage:

Work from point 1 to point 2:

$$W_{1\rightarrow 2} = \int_1^2 \dot{p} f \, dt = \int_1^2 \frac{dp}{dt} f \, dt = \int_{p_1}^{p_2} f \, dp = \int_{p_1}^{p_2} \frac{1}{I} p dp = \frac{1}{2I} p^2 \Big|_{p_1}^{p_2} = \frac{1}{2I}\left(p_2^2 - p_1^2\right)$$




$$W_{1\rightarrow 2} = T_2 - T_1$$



# LINEAR INERTANCE

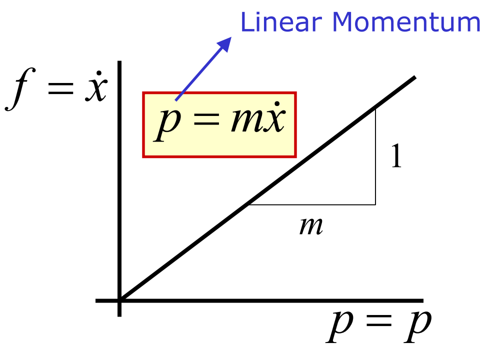
Examples:

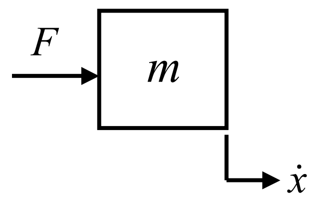
- Translating or rotating masses
- Electrical inductor
- Fluid inertance: flowing liquid in a pipe

## ACCELERATING MASS

$$\overset{F}{\underset{\dot{x}}{\rightharpoonup}}\ I$$

Mass




The mass impedes the rate of change of velocity



## ACCELERATING MASS

Momentum Method:

$$F = m\ddot{x} \Rightarrow p = \int F dt = \int m\ddot{x} dt = \int m d\dot{x} = m\dot{x}$$

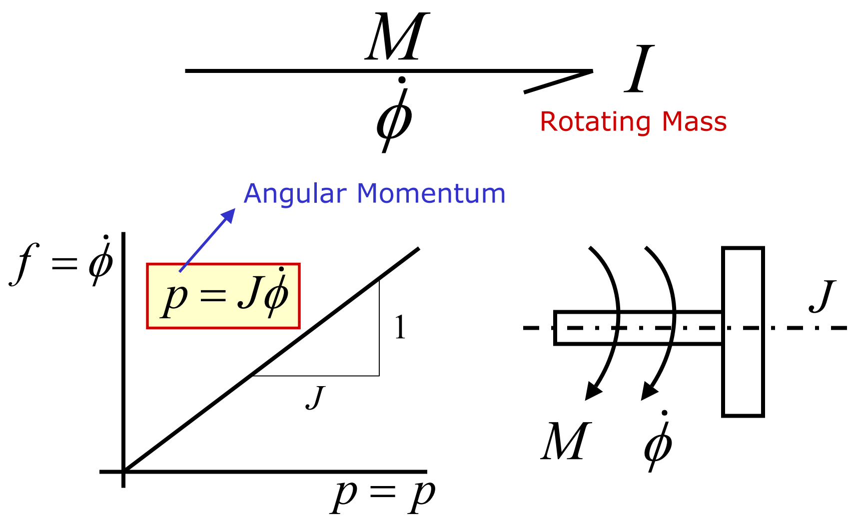
$$\boxed{p = If \Rightarrow I = m}$$

Energy Method:

$$F = m\ddot{x} \Rightarrow T = \overbrace{\int F dx}^{\text{Work}} = \int m\ddot{x}\dot{x} dt = \int m\dot{x} d\dot{x} = \frac{1}{2} m\dot{x}^2$$

$$\boxed{T = \frac{1}{2} I\dot{q}^2 \Rightarrow I = m}$$

## ROTATING MASS




The moment of inertia impedes the rate of change of angular velocity



## ROTATING MASS

Momentum Method:

$$p = \int r v dm = \int r(r\dot{\phi})dm = \left[\int r^2 dm\right]\dot{\phi}$$

$$\boxed{p = If \Rightarrow I = \int r^2 dm}$$

Energy Method:

$$T = \int \frac{1}{2} v^2 dm = \frac{1}{2}\int (r\dot{\phi})^2 dm = \frac{1}{2}\left[\int r^2 dm\right]\dot{\phi}^2$$

$$\boxed{T = \frac{1}{2} I\dot{q}^2 \Rightarrow I = \int r^2 dm}$$

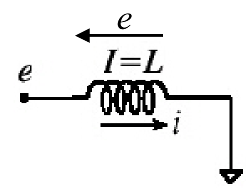
$$I = \int r^2 dm = \frac{1}{2} mR^2$$

Disk of radius $R$

## ELECTRICAL INDUCTOR

$$\frac{e\,(\text{voltage})}{i\,(\text{current})} \rightharpoonup I$$

Inductor




$$\boxed{i = \frac{1}{L}\int e\,dt} \;\Rightarrow\; e = L\frac{di}{dt} \Rightarrow\; I = L$$

$L$ stands for Inductance

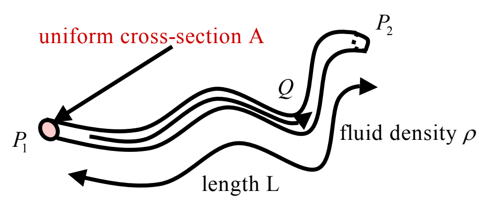
The inductance impedes the rate of change of current



## INCOMPRESSIBLE FLUID IN A CHANNEL

$$\frac{P_1\text{-}P_2\ \ (\text{pressure})}{Q\ \ (\text{flow})} \rightharpoonup I$$

Fluid




Momentum Method:

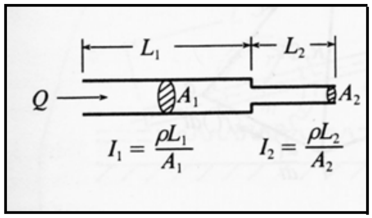
$$I = \frac{p}{f} = \frac{p}{Q} = \frac{dp/dt}{dQ/dt} = \frac{P_1 - P_2}{A\,dv/dt} = \frac{(F_1 - F_2)/A}{A\,dv/dt} = \frac{\left(\rho A L\,dv/dt\right)/A}{A\,dv/dt} = \frac{\rho L}{A}$$

Energy Method:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}(\rho A L)\left(\frac{Q}{A}\right)^2 = \frac{1}{2}\frac{\rho L}{A}Q^2 \Rightarrow I = \frac{\rho L}{A}$$

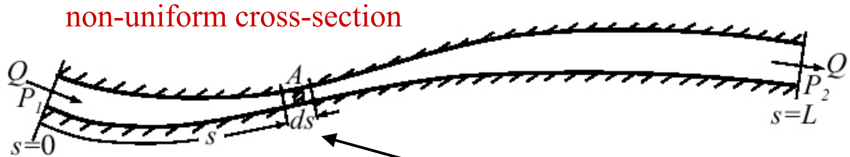
The inertance impedes the rate of change of the volumetric flow

# INCOMPRESSIBLE FLUID IN A CHANNEL




$$I = I_1 + I_2$$

non-uniform cross-section




Uniform cross-section: $$I = \frac{\rho L}{A} \Rightarrow I = \frac{\rho ds}{A}$$

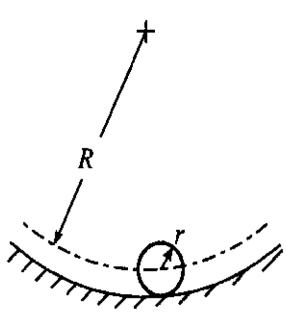
Non-uniform cross-section: $$I = \rho \int_0^L \frac{1}{A(s)} ds$$



# LINEAR INERTANCE

Example: Problem 3.13, page 109

a- Find the kinetic energy of the disk, first in terms of any velocity but finally in terms of $d\phi/dt$, where $\phi$ is the angle between the vertical and a line drawn from the center of curvature to the center of the cylinder.
b- Use the result of a- to find the inertance of the disk relative to $d\phi/dt$.




Solution:

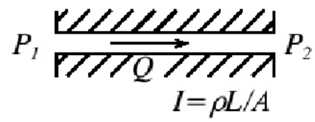
$$I = \frac{3}{2} m R^2$$

# LINEAR INERTANCE

## Example: Problem 3.10, page 109

a- A tube of length $L$ and internal area $A$ is filled with and its entrance surrounded by an inviscid fluid of density $\rho$. The pressure at the entrance is a constant $P = \rho g h$ and is zero at the other end. Represent the situation with a bond-graph, using the volume flow rate $Q$ as the flow variable but neglecting entrance and exit losses. Find the rate of change of the flow.

$$I = \frac{p}{\dot{q}} = \frac{p}{Q} = \frac{dp/dt}{dQ/dt} = \frac{e}{dQ/dt} = \frac{P_1 - P_2}{dQ/dt} = \frac{\rho g h}{dQ/dt}$$




Then $$\frac{dQ}{dt} = \frac{\rho g h}{I} = \frac{\rho g h}{\rho L / A} = \frac{A g h}{L}$$

Solution:

$$\frac{dQ}{dt} = \frac{Agh}{L}$$