A LOWER BOUND FOR HIGHER TOPOLOGICAL COMPLEXITY OF REAL PROJECTIVE SPACE

DONALD M. DAVIS

Abstract. We obtain an explicit formula for the best lower bound for the higher topological complexity, TCₖ(RPⁿ), of real projective space implied by mod 2 cohomology.

1. Main theorem

The notion of higher topological complexity, TCₖ(X), of a topological space X was introduced in [2]. It can be thought of as one less than the minimal number of rules required to tell how to move consecutively between any k specified points of X. In [1], the study of TCₖ(Pⁿ) was initiated, where Pⁿ denotes real projective space. Using \( \mathbb{Z}_2 \) coefficients for all cohomology groups, define zclₖ(X) to be the maximal number of elements in \( \ker(\Delta^* : H^*(X)^{\otimes k} \to H^*(X)) \) with nonzero product. It is standard that

\[
TCₖ(X) \geq zclₖ(X).
\]

In [1], it was shown that

\[
zclₖ(Pⁿ) = \max\{a_1 + \cdots + a_{k-1} : (x_1 + x_k)^{a_1} \cdots (x_{k-1} + x_k)^{a_{k-1}} \neq 0\}
\]

in \( \mathbb{Z}_2[x_1, \ldots, x_k]/(x_1^{n+1}, \ldots, x_k^{n+1}) \). In Theorem 1.2 we give an explicit formula for zclₖ(Pⁿ), and hence a lower bound for TCₖ(Pⁿ).

Our main theorem, 1.2, requires some specialized notation.

Definition 1.1. If \( n = \sum \varepsilon_j 2^j \) with \( \varepsilon_j \in \{0, 1\} \) (so the numbers \( \varepsilon_j \) form the binary expansion of \( n \)), let

\[
Z_l(n) = \sum_{j=0}^{l} (1 - \varepsilon_j)2^j.
\]
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and let

\[ S(n) = \{ i : \varepsilon_i = \varepsilon_{i-1} = 1 \text{ and } \varepsilon_{i+1} = 0 \}. \]

Thus \( Z_i(n) \) is the sum of the 2-powers \( \leq 2^i \) which correspond to the 0’s in the binary expansion of \( n \). Note that \( Z_i(n) = 2^{i+1} - 1 - (n \mod 2^{i+1}) \). The \( i \)'s in \( S(n) \) are those that begin a sequence of two or more consecutive 1’s in the binary expansion of \( n \). Also, \( \nu(n) = \max \{ t : 2^t \text{ divides } n \} \).

**Theorem 1.2.** For \( n \geq 0 \) and \( k \geq 3 \),

\[ zcl_k(P^n) = kn - \max\{ 2^{(n+1)} - 1, 2^{i+1} - 1 - k \cdot Z_i(n) : i \in S(n) \}. \quad (1.3) \]

It was shown in [1] that, if \( 2^c \leq n < 2^{c+1} \), then \( zcl_2(P^n) = 2^{c+1} - 1 \), which follows immediately from our Theorem 1.6.

In Table 1, we tabulate \( zcl_k(P^n) \) for \( 1 \leq n \leq 17 \) and \( 2 \leq k \leq 8 \).

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The smallest value of \( n \) for which two values of \( i \) are significant in (1.3) is \( n = 102 = 2^6 + 2^5 + 2^2 + 2^1 \). With \( i = 2 \), we have \( 7 - k \) in the max, while with \( i = 6 \), we have \( 127 - 25k \). Hence

\[ zcl_k(P^{102}) = 102k - \begin{cases} 127 - 25k & 2 \leq k \leq 5 \\ 7 - k & 5 \leq k \leq 7 \\ 0 & 7 \leq k. \end{cases} \]

For all \( k \) and \( n \), \( TC_k(P^n) \leq kn \) for dimensional reasons ([1, Prop 2.2]). Thus we obtain a sharp result \( TC_k(P^n) = kn \) whenever \( zcl_k(P^n) = kn \). Corollary 3.4 tells exactly when this is true. Here is a simply-stated partial result.
Proposition 1.4. If $n$ is even, then $\text{TC}_k(P^n) = kn$ for $k \geq 2^{l+1} - 1$, where $\ell$ is the length of the longest string of consecutive 1’s in the binary expansion of $n$.

Proof. We use Theorem 1.2. We need to show that if $i \in S(n)$ begins a string of $j$ 1’s with $j \leq \ell$, then $2^{i+1} - 1 \leq (2^{\ell+1} - 1)Z_i(n)$. If $j < \ell$, then $Z_i(n) \geq 2^{j-1} + 1$, and the desired inequality reduces to $2^{i+1} + 2^{j-1} \leq 2^{\ell+1} + 2^{j-1} + 2^{\ell+1}$, which is satisfied since $2^{\ell+1+i-j}$ is strictly greater than both $2^{i+1}$ and $2^{j-1}$.

If $j = \ell$, then

$$Z_i(n) \geq 1 + \sum_{\alpha} 2^{i+1-\alpha(\ell+1)},$$

where $\alpha$ ranges over all positive integers such that $i + 1 - \alpha(\ell + 1) > 0$. This reflects the fact that the binary expansion of $n$ has a 0 starting in the $2^{-\ell}$ position and at least every $\ell + 1$ positions back from there, and also a 0 at the end since $n$ is even. The desired inequality follows easily from this. \[\blacksquare\]

Theorem 1.2 shows that $zcl_k(P^n) < kn$ when $n$ is odd. In the next proposition, we give complete information about when $zcl_k(n) = kn$ if $k = 3$ or 4.

Proposition 1.5. If $k = 3$ or 4, then $zcl_k(P^n) = kn$ if and only if $n$ is even and the binary expansion of $n$ has no consecutive 1’s.

Proposition 1.5 follows easily from Theorem 1.2 and the fact that if $i \in S(n)$, then $Z_i(n) \leq 2^{i-1} - 1$.

The following recursive formula for $zcl_k(P^n)$, which is interesting in its own right, is central to the proof of Theorem 1.2. It will be proved in Section 2.

Theorem 1.6. Let $n = 2^e + d$ with $0 \leq d < 2^e$, and $k \geq 2$. If $z_k(n) = zcl_k(P^n)$, then

$$z_k(n) = \min(z_k(d) + k2^e, (k - 1)(2^{e+1} - 1)), \text{ with } z_k(0) = 0.$$  

Equivalently, if $g_k(n) = kn - zcl_k(P^n)$, then

$$g_k(n) = \max(g_k(d), kn - (k - 1)(2^{e+1} - 1)), \text{ with } g_k(0) = 0. \quad (1.7)$$

We now use Theorem 1.6 to prove Theorem 1.2.

Proof of Theorem 1.2. We will prove that $g_k(n)$ of Theorem 1.6 satisfies

$$g_k(n) = \max\{2^{\nu(n+1)} - 1, 2^{i+1} - 1 - kZ_i(n) : i \in S(n)\} \quad (1.8)$$
if $k \geq 3$, which is clearly equivalent to Theorem 1.2. The proof is by induction, using
the recursive formula (1.7) for $g_k(n)$. Let $n = 2^e + d$ with $0 \leq d < 2^e$.

**Case 1:** $d = 0$. Then $n = 2^e$ and by (1.7) we have $g_k(n) = \max(0, k2^e - (k - 1)(2^{e+1} - 1))$. If $e = 0$, this equals 1, while if $e > 0$, it equals 0, since $k \geq 3$. These agree with the claimed answer $2^{\nu(n+1)} - 1$, since $S(2^e) = \emptyset$.

**Case 2:** $0 < d < 2^{e-1}$. Here $\nu(n + 1) = \nu(d + 1)$, $S(n) = S(d)$, and $Z_i(n) = Z_i(d)$ for any $i \in S(d)$. Substituting (1.8) with $n$ replaced by $d$ into (1.7), we obtain

$$g_k(n) = \max\{2^{\nu(n+1)} - 1, 2^{i+1} - 1 - kZ_i(n) : i \in S(n), kn - (k-1)(2^{e+1} - 1)\}.$$

We will be done once we show that $kn - (k - 1)(2^{e+1} - 1)$ is $\leq$ one of the other entries, and so may be omitted. If $i$ is the largest element of $S(n)$, we will show that $kn - (k - 1)(2^{e+1} - 1) \leq 2^{i+1} - 1 - kZ_i(n)$, i.e.,

$$kn_i \leq (k - 1)(2^{e+1} - 2^{i+1}), \quad (1.9)$$

where $n_i = n - (2^{i+1} - 1 - Z_i(n))$ is the sum of the $2$-powers in $n$ which are greater than $2^i$. The largest of these is $2^i$, and no two consecutive values of $i$ appear in this sum, hence $n_i \leq \sum 2^j$, taken over $j = e \ (2)$ and $i + 2 \leq j \leq e$. If $k = 3$, (1.9) is true because the above description of $n_i$ implies that $3n_i \leq 2(2^{e+1} - 2^{i+1})$, while for larger $k$, it is true since $\frac{k}{k-1} < \frac{3}{2}$. If $S(n)$ is empty, then $kn - (k - 1)(2^{e+1} - 1) \leq 2^{\nu(n+1)} - 1$ by a similar argument, since $n \leq 2^e + 2^{e-2} + 2^{e-4} + \cdots$, so $3n \leq 2(2^{e+1} - 1)$, and values of $k > 3$ follow as before.

**Case 3:** $d \geq 2^{e-1}$. If $e - 1 \in S(d)$, then it is replaced by $e$ in $S(n)$, while other elements of $S(d)$ form the rest of $S(n)$. If $e - 1 \notin S(d)$, then $S(n) = S(d) \cup \{e\}$. If $i \in S(n) - \{e\}$, then $Z_i(n) = Z_i(d)$, so its contribution to the set of elements whose max equals $g_k(n)$ is $2^{i+1} - 1 - kZ_i(n)$, as desired. For $i = e$, the claimed term is $2^{e+1} - 1 - kZ_e(n) = kn - (k - 1)(2^{e+1} - 1)$, which is present by the induction from (1.7). If $e - 1 \in S(d)$, then the $i = e - 1$ term in the max for $g_k(n)$ is $2^e - 1 - kZ_e(n)$ and contributes to $g_k(n)$ less than the term described in the preceding sentence, and hence cannot contribute to the max. The $2^{\nu(n+1)} - 1$ term is obtained from the induction since $\nu(n + 1) = \nu(d + 1)$.

The author wishes to thank Jesus González for many useful suggestions.
In this section, we prove Theorem 1.6 and the following variant.

**Theorem 2.1.** Let $n = 2^e + d$ with $0 \leq d < 2^e$, and $k \geq 2$. If $h_k(n) = \text{zcl}_k(P^n) - (k - 1)n$, then

$$h_k(n) = \min(h_k(d) + 2^e, (k - 1)(2^{e+1} - 1 - n)), \text{ with } h_k(0) = 0. \quad (2.2)$$

**Proof of Theorems 1.6 and 2.1.** It is elementary to check that the formulas for $z_k$, $g_k$, and $h_k$ are equivalent to one another. We prove (2.2). We first look for nonzero monomials in $(x_1 + x_k)^a_1 \cdots (x_{k-1} + x_k)^a_{k-1}$ of the form $x_1^n \cdots x_{k-1}^n x_k^\ell$ with $\ell \leq n$.

Letting $a_i = n + b_i$, the analogue of $h_k(n)$ for such monomials is given by

$$\tilde{h}_k(n) = \max\{\sum_{i=1}^{k-1} b_i : \binom{n+b_i}{n} \cdots \binom{n+b_{k-1}}{n} \text{ is odd and } \sum_{i=1}^{k-1} b_i \leq n\}, \quad (2.3)$$

since $\sum b_i$ is the exponent of $x_k$. We will begin by proving

$$\tilde{h}_k(n) = \min(\tilde{h}_k(d) + 2^e, (k - 1)(2^{e+1} - 1 - n)). \quad (2.4)$$

For a nonzero integer $m$, let $Z(m)$ (resp. $P(m)$) denote the set of 2-powers corresponding to the 0’s (resp. 1’s) in the binary expansion of $m$, with $Z(0) = P(0) = \emptyset$. By Lucas’s Theorem, $\binom{n+b_i}{n}$ is odd iff $P(b_i) \subset Z(n)$. Note that the integers $Z_i(n)$ considered earlier are sums of elements of subsets of $Z(n)$.

For a multiset $S$, let $|S|$ denote the sum of its elements, and let

$$\phi(S, n) = \max\{|T| \leq n : T \subset S\}.$$  

Note that $|Z(n)| = 2^{\log_2(n)+1} - 1 - n$, where $\log(n) = \lceil \log_2(n) \rceil$, $(\log(0) = -1)$. Let $Z(n)^j$ denote the multiset consisting of $j$ copies of $Z(n)$, and let

$$m_j(n) = \phi(Z(n)^j, n).$$

Then, from (2.3), we obtain the key equation $\tilde{h}_k(n) = m_{k-1}(n)$. Thus (2.4) follows from Lemma 2.5 below.

**Lemma 2.5.** If $n = 2^e + d$ with $0 \leq d < 2^e$, and $j \geq 1$, then

$$m_j(n) = \min(m_j(d) + 2^e, j(2^{e+1} - 1 - n)).$$
Proof. The result is clear if \( j = 1 \) since \( 2^{e+1} - 1 - n < 2^e \), so we assume \( j \geq 2 \). Let \( S \subset Z(d)^j \) satisfy \( |S| = m_j(d) \).

First assume \( d < 2^{e-1} \). Then \( 2^{e-1} \in Z(n) \). Let \( T = S \cup \{2^{e-1}, 2^{e-1}\} \). No other subset of \( Z(n)^j \) can have larger sum than \( T \) which is \( \leq n \) due to maximality of \( |S| \) and the fact that the 2-powers in \( Z(n)^j - Z(d)^j \) are larger than those in \( Z(d)^j \). Thus \( m_j(n) = m_j(d) + 2^e \) in this case, and this is \( \leq j(2^{e+1} - 1 - n) = |Z(n)^j| \).

If, on the other hand, \( d \geq 2^{e-1} \), then \( Z(d)^j = Z(n)^j \). If \( |Z(n)^j - S| < 2^e \), then let \( T = Z(n)^j \) with \( |T| = j(2^{e+1} - 1 - n) \), as large as it could possibly be, and less than \( m_j(d) + 2^e \). Otherwise, since any multiset of 2-powers whose sum is \( \geq 2^e \) has a subset whose sum equals \( 2^e \), we can let \( T = S \cup V \), where \( V \) is a subset of \( Z(n)^j - S \) with \( |V| = 2^e \). As before, no subset of \( Z(n)^j \) can have size greater than that.

Now we wish to consider more general monomials. We claim that for any multiset \( S \) and positive integers \( m \) and \( n \),

\[
\phi(Z(m - 1) \cup S, n) \leq \phi(Z(m) \cup S, n) + 1. \tag{2.6}
\]

This follows from the fact that subtracting 1 from \( m \) can affect \( Z(m) \) by adding 1, or changing \( 1, 2, \ldots, 2^{e-1} \) to \( 2^t \). These changes cannot add more than 1 to the largest subset of size \( \leq n \). We show now that this implies that \( h_k(n) = m_{k-1}(n) = \bar{h}_k(n) \), and hence (2.2) follows from (2.4).

Suppose that \( x_1^{n-\varepsilon_1} \cdots x_{k-1}^{n-1} x_k^\varepsilon \) with \( \varepsilon \geq 0 \) and \( \ell \leq n \) is a nonzero monomial in the expansion of \( (x_1 + x_k)^{n+b_1} \cdots (x_{k-1} + x_k)^{n+b_{k-1}} \). We wish to show that \( \sum b_i \leq m_{k-1}(n) \).

It follows from (2.6) that

\[
\phi\left( \bigcup_{i=1}^{k-1} Z(n - \varepsilon_i), n \right) \leq \phi(Z(n)^{k-1}, n) + \sum \varepsilon_i = m_{k-1}(n) + \sum \varepsilon_i.
\]

The odd binomial coefficients \( \binom{n+b_i}{n-\varepsilon_i} \) imply that \( P(b_i + \varepsilon_i) \subset Z(n - \varepsilon_i) \). Thus

\[
\phi\left( \bigcup_{i=1}^{k-1} P(b_i + \varepsilon_i), n \right) \leq m_{k-1}(n) + \sum \varepsilon_i. \tag{2.7}
\]

Since \( |P(b_i + \varepsilon_i)| = b_i + \varepsilon_i \) and \( \sum (b_i + \varepsilon_i) \leq n \), the left hand side of (2.7) equals \( \sum (b_i + \varepsilon_i) \), hence \( \sum b_i \leq m_{k-1}(n) \), as desired. \( \blacksquare \)
3. Examples and comparisons

In this section, we examine some special cases of our results (in Propositions 3.1 and 3.5) and make comparisons with some work in [1].

The numbers $z_3(n) = zcl_3(P^n)$ are 1 less than a sequence which was listed by the author as A290649 at [3] in August 2017. They can be characterized as in Proposition 3.1, the proof of which is a straightforward application of the recursive formula

$$z_3(2^e + d) = \min(z_3(d) + 3 \cdot 2^e, 2(2^{e+1} - 1)) \text{ for } 0 \leq d < 2^e,$$

from Theorem 1.6.

**Proposition 3.1.** For $n \geq 0$, $zcl_3(n)$ is the largest even integer $z$ satisfying $z \leq 3n$ and $(\frac{z+1}{n}) = 1 (2)$.

We have not found similar characterizations for $z_k(n)$ when $k > 3$.

In [1, Thm 5.7], it is shown that our $g_k(n)$ in Theorem 1.6 is a decreasing function of $k$, and achieves a stable value of $2^{v(n+1)} - 1$ for sufficiently large $k$. They defined $s(n)$ to be the minimal value of $k$ such that $g_k(n) = 2^{v(n+1)} - 1$. We obtain a formula for the precise value of $s(n)$ in our next result.

Let $S'(n)$ denote the set of integers $i$ such that the $2^i$ position begins a string of two or more consecutive 1’s in the binary expansion of $n$ which stops prior to the $2^0$ position. For example, $S'(187) = \{5\}$ since its binary expansion is 10111011.

**Proposition 3.2.** Let $s(\cdot)$ and $S'(\cdot)$ be the functions just described. Then

$$s(n) = \begin{cases} 
2 & \text{if } n + 1 \text{ is a 2-power} \\
3 & \text{if } n + 1 \text{ is not a 2-power and } S'(n) = \emptyset \\
\max\left\{\frac{2^{i+1} - 2^{v(n+1)}}{Z_i(n)} : i \in S'(n)\right\} & \text{otherwise.}
\end{cases}$$

**Proof.** It is shown in [1, Expl 5.8] that $g_k(2^v - 1) = 2^v - 1$ for all $k \geq 2$, hence $s(2^v - 1) = 2$. This also follows readily from (1.7).

If the binary expansion of $n$ has a string of $i + 1$ 1’s at the end and no other consecutive 1’s (so that $S(n) = \{i\}$ in (1.3)), then $Z_i(n) = 0$. Thus by (1.8) $g_k(n) = 2^{i+1} - 1 = 2^{v(n+1)} - 1$ for $k \geq 3$. If $n \neq 2^{i+1} - 1$, then $s(n) = 3$, since $g_3(n) > 2^{i+1} - 1$. 
Now assume $S'(n)$ is nonempty. By (1.8), $s(n)$ is the smallest $k$ such that
\[ 2^{i+1} - 1 - kZ_i(n) \leq 2^{\ell(n+1)} - 1 \quad (3.3) \]
for all $i \in S(n)$, which easily reduces to the claimed value. Note that if the string of 1’s beginning at position $2^i$ goes all the way to the end, then (3.3) is satisfied; this case is omitted from $S'(n)$ in the theorem, because it would yield $0/0$.

The following corollary is immediate.

**Corollary 3.4.** If $n$ is even and
\[ k \geq \max\{3, \left\lfloor \frac{2^{i+1} - 1}{Z_i(n)} \right\rfloor : i \in S(n)\} \]
then $TC_k(P^n) = kn$. These are the only values of $n$ and $k$ for which $zcl_k(P^n) = kn$.

In [1, Def 5.10], a complicated formula was presented for numbers $r(n)$, and in [1, Thm 5.11], it was proved that $s(n) \leq r(n)$. It was conjectured there that $s(n) = r(n)$. However, comparison of the formula for $s(n)$ established in Proposition 3.2 with their formula for $r(n)$ showed that there are many values of $n$ for which $s(n) < r(n)$. The first is $n = 50$, where we prove $s(50) = 5$, whereas their $r(50)$ equals 7. Apparently their computer program did not notice that
\[(x_1 + x_3)^{63}(x_2 + x_3)^{63}(x_3 + x_4)^{62}(x_4 + x_5)^{62}\]
contains the nonzero monomial $x_1^{50}x_2^{50}x_3^{50}x_4^{50}x_5^{50}$, showing that our $z_5(50) = 250$ and $g_5(50) = 0$, so $s(50) \leq 5$.

In Table 2, we present a table of some values of $s(-)$, omitting $s(2^v - 1) = 2$ and $s(2^v) = 3$ for $v > 0$.

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<td>31</td>
</tr>
</tbody>
</table>

In [1], there seems to be particular interest in $TC_k(P^{3k-2})$. We easily read off from Theorem 1.2 the following result.
Proposition 3.5. For $k \geq 2$ and $e \geq 1$, we have

$$zcl_k(P^{3\cdot 2^e}) = \begin{cases} (k-1)(2^{e+2} - 1) & \text{if } (e = 1, k \leq 6) \text{ or } (e \geq 2, k \leq 4) \\ k \cdot 3 \cdot 2^e & \text{otherwise.} \end{cases}$$

This shows that the estimate $s(3 \cdot 2^e) \leq 5$ for $e \geq 2$ in [1] is sharp.

REFERENCES


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