On the cohomology classes of planar polygon spaces

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Abstract. We obtain an explicit formula for the Poincaré duality isomorphism $H^{n-3}(\overline{M}(\ell);\mathbb{Z}_2)\to\mathbb{Z}_2$ for the space of isometry classes of n-gons with specified side lengths, if ℓ is monogenic in the sense of Hausmann-Rodriguez. This has potential application to topological complexity.

1. Main theorem

If $\ell=(\ell_1,\ldots,\ell_n)$ is an n-tuple of positive real numbers, let $\overline{M}(\ell)$ denote the space of isometry classes of oriented n-gons in the plane with the prescribed side lengths. In [**3**], a complete description of $H^*(\overline{M}(\ell);\mathbb{Z}_2)$ was given in terms of generators and a complicated set of relations. In [**1**], explicit calculations were made in $H^*(\overline{M}(\ell);\mathbb{Z}_2)$ for length vectors ℓ satisfying certain conditions, enabling us to prove that, for these ℓ, the topological complexity of $\overline{M}(\ell)$ satisfied

$$2n-6\le \mathrm{TC}(\overline{M}(\ell))\le 2n-5. \tag{1.1}$$

This is a result in topological robotics, as it specifies the number of motion planning rules required for a certain n-armed robot.([**2**]) However, our result only applied to a very restricted set of length vectors ℓ.

The groups $H^k(\overline{M}(\ell);\mathbb{Z}_2)$ are spanned by monomials $R^{k-r}V_{j_1}\cdots V_{j_r}$ for distinct positive subscripts $j\le n-1$. Here R and V_j are elements of $H^1(\overline{M}(\ell);\mathbb{Z}_2)$. Since

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$\overline{M}(\ell)$ is an $(n-3)$-manifold, there is a Poincaré duality isomorphism

$$\phi : H^{n-3}(\overline{M}(\ell);\mathbb{Z}_2) \to \mathbb{Z}_2. \tag{1.2}$$

For the cases considered in [**1**], we obtained an explicit formula for $\phi(R^{n-3-r}V_{j_1}\cdots V_{j_r})$. The contribution of this paper is to extend that formula to a broader class of length vectors. Note that it tells for each monomial whether it is 0 or the nonzero class, hence the title.

In order to describe these length vectors, we review the notion of genetic code introduced in [**4**]. Since permuting the length vectors does not affect the homeomorphism type of $\overline{M}(\ell)$, we may assume that $\ell_1 \le \cdots \le \ell_n$. A subset $S \subset \{1,\ldots,n\}$ is called *short* if $\sum_{i\in S}\ell_i < \sum_{i\notin S}\ell_i$. We assume that ℓ is *generic*, which says that there are no subsets S for which $\sum_{i\in S}\ell_i = \sum_{i\notin S}\ell_i$. We define a partial order on sets of integers by $\{s_1,\ldots,s_k\} \le T$ if there exist distinct $t_1,\ldots,t_k$ in T with $s_i \le t_i$. The *genetic code* of ℓ is the set of maximal elements in the set of short subsets for ℓ which contain n. An element in the genetic code is called a *gene*.

One of the main theorems of [**1**] was that, with three exceptions, (1.1) holds if ℓ has a single gene of size 4. In order to prove this, we needed and obtained the explicit formula for $\phi(R^{n-3-r}V_{j_1}\cdots V_{j_r})$ for such length vectors.([**1**, Thm 4.1]) In this paper, we extend this formula to all monogenic codes. We hope that this formula will enable us to study the cohomological implications for topological complexity of these spaces. The huge variety of genetic codes makes it seem unlikely that a formula such as ours might be extended to all genetic codes.

In [**1**], we introduced the term *gee* to refer to a gene with the n omitted. This is sensible since, for all genes G, $n \in G$. Also, most of the formulas do not involve n. We say that a *subgee* is any set of positive integers which is less than or equal to a gee. Thus the subgees are all sets S for which $S \cup \{n\}$ is short. We write the elements of a gee in decreasing order. Only those $\{j_1,\ldots,j_r\}$ which are subgees can have $R^{k-r}V_{j_1}\cdots V_{j_r} \ne 0$.([**3**, Cor 9.2])

Now we can state our theorem. It involves the following new definition.

Definition 1.3. *Let $\mathcal{S}_k$ denote the set of k-tuples of nonnegative integers such that, for all i, the sum of the first i components of the k-tuple is $\le i$.*

For $B = (b_1,\ldots,b_k)$, let $|B| = \sum b_i$.

Theorem 1.4. *Suppose ℓ has a single gee, $G = \{g_1, \dots, g_k\}$ with $a_i = g_i - g_{i+1} > 0$. ($g_{k+1} = 0$.) If J is a set of distinct positive integers $\leq g_1$, let $\theta(J) = (\theta_1, \dots, \theta_k)$, where θ_i is the number of elements $j \in J$ satisfying $g_{i+1} < j \leq g_i$. Then, for ϕ as in (1.2),*

$$\phi(R^{n-3-r} V_{j_1} \cdots V_{j_r}) = \sum_B \prod_{i=1}^{k} \tbinom{a_i + b_i - 2}{b_i}, \tag{1.5}$$

where B ranges over all $(b_1, \dots, b_k)$ for which $|B| = k - r$ and $B + \theta(\{j_1, \dots, j_r\}) \in \mathcal{S}_k$.

Note that $J = \{j_1, \dots, j_r\}$ is a subgee if and only if $\theta(J) \in \mathcal{S}_k$. For example, if two j's are greater than g_2, then $\theta(J)$ has first component greater than 1, so $\theta(J)$ is not in $\mathcal{S}_k$, and J cannot be $\leq G$ since it has two elements greater than the second largest element of G. Thus (1.5) is only relevant when J is a subgee, but it yields 0 in other cases, anyway.

An important special case of the theorem appears in the following corollary.

Corollary 1.6. *If J is a subgee with $r = k$, then $\phi(R^{n-3-r} V_{j_1} \cdots V_{j_r}) = 1$.*

The following elegant independent proof of this corollary was provided by the referee.

Proof of Corollary 1.6. First note that $V_{j_1} \cdots V_{j_k}$ is nonzero, since this is also true in $H^*(\overline{M}(\ell); \mathbb{Z}_2)/\langle R \rangle$, which is an exterior face ring. By Poincaré duality, there must be an $X = \sum_i X_i \in H^{n-3-k}(\overline{M}(\ell); \mathbb{Z}_2)$ with $X \cdot V_{j_1} \cdots V_{j_k} = 1$. If X_i contains as a factor any V_t with $t \notin \{j_1, \dots, j_k\}$, then $X_i \cdot V_{j_1} \cdots V_{j_k} = 0$ since $\{t, j_1, \dots, j_k\}$ is not a subgee. Any factors V_t with $t \in \{j_1, \dots, j_k\}$ can be replaced by R, by the relation $V_t^2 = R V_t$. Thus each X_i with $X_i \cdot V_{j_1} \cdots V_{j_k} \neq 0$ can be replaced by R^{n-3-k}, and the number of such X_i must be odd. ■

In working with our formula, it is useful to denote by Y_T any term $R^{n-3-r} V_{j_1} \cdots V_{j_r}$ for which $\theta(\{j_1, \dots, j_r\}) = T$. The reader can verify that the case $k = 3$ of Theorem 1.4 agrees with [**1**, Thm 4.1], when the latter is expressed as in Example 1.7. For example, in $\phi(Y_{0,1,0})$, $B = (0,0,2)$, $(1,0,1)$, and $(0,1,1)$ satisfy $B + (0,1,0) \in \mathcal{S}_3$, but $B = (1,1,0)$ does not, since the sum of the first two entries of $(1,2,0)$ is greater than 2. (Our method of subscripting Y here differs from that in [**1**].) This $\phi(Y_{0,1,0})$ refers to $\phi(R^{n-4} V_j)$ for $g_3 < j \leq g_2$.

Example 1.7. *If ℓ has a single gee $\{a_1+a_2+a_3, a_2+a_3, a_3\}$, then, writing a_i' for a_i-1,*

$$\begin{aligned}
\phi(Y_T) &= 1 \textit{ if } |T|=3 \textit{ and } T\in\mathcal{S}_3\\
\phi(Y_{0,2,0})=\phi(Y_{1,1,0}) &= a_3'\\
\phi(Y_{1,0,1}) &= a_2'+a_3'\\
\phi(Y_{0,0,2})=\phi(Y_{0,1,1}) &= a_1'+a_2'+a_3'\\
\phi(Y_{1,0,0}) &= \tbinom{a_3}{2}+a_2'a_3'\\
\phi(Y_{0,1,0}) &= \tbinom{a_3}{2}+a_1'a_3'+a_2'a_3'\\
\phi(Y_{0,0,1}) &= \tbinom{a_2}{2}+\tbinom{a_3}{2}+a_1'a_2'+a_1'a_3'+a_2'a_3'\\
\phi(Y_{0,0,0}) &= \tbinom{a_3+1}{3}+\tbinom{a_2}{2}a_3'+\tbinom{a_3}{2}(a_1'+a_2')+a_1'a_2'a_3'.
\end{aligned}$$

2. Proof

In this section, we prove Theorem 1.4. As noted above, $H^{n-3}(\overline{M}(\ell);\mathbb{Z}_2)$ is spanned by monomials $R^{n-3-r}V_{j_1}\cdots V_{j_r}$ for which $J=\{j_1,\ldots,j_r\}\le G$, i.e., J is a subgee. Using [**3**, Cor 9.2] as interpreted in [**1**, Thm 2.1], a complete set of relations is given by relations $\mathcal{R}_I$ for each subgee I except the empty set. This relation $\mathcal{R}_I$ says

$$\sum_{J\not\supset I} R^{n-3-|J|}\prod_{j\in J}V_j=0, \tag{2.1}$$

where the sum is taken over all subgees J disjoint from I. To prove our theorem, it suffices to show that our proposed ϕ sends each relation $\mathcal{R}_I$ to 0, since it will then be the unique nonzero homomorphism (1.2).

Similarly to [**1**, (4.2)], the number of subgees J disjoint from I and satisfying $\theta(J)=C=(c_1,\ldots,c_k)$ is $\prod\limits_{i=1}^{k}\binom{a_i-m_i}{c_i}$, if $\theta(I)=(m_1,\ldots,m_k)$. Note that $m_i\le a_i$ since m_i is the size of a subset of a_i integers. Since our ϕ applied to a term in (2.1) is determined by $\theta(J)$, $\phi(\mathcal{R}_I)$ becomes

$$\sum_C\prod_{i=1}^k\tbinom{a_i-m_i}{c_i}\sum_B\prod_{i=1}^k\tbinom{a_i+b_i-2}{b_i}, \tag{2.2}$$

where $|B| = k - |C|$ and $B + C \in \mathcal{S}_k$. Letting $T = B + C$, this can be rewritten, with the outer sum taken over all $T \in \mathcal{S}_k$ satisfying $|T| = k$, as

$$\begin{aligned}
&\sum_T \sum_{B \le T} \prod_{i=1}^k \tbinom{a_i - m_i}{t_i - b_i} \tbinom{a_i + b_i - 2}{b_i} \\
&= \sum_T \prod_{i=1}^k \sum_{b_i} \tbinom{a_i - m_i}{t_i - b_i} \tbinom{a_i + b_i - 2}{b_i} \\
&\equiv \sum_T \prod_{i=1}^k \sum_{b_i} \tbinom{a_i - m_i}{t_i - b_i} \tbinom{1 - a_i}{b_i} \quad \bmod 2 \\
&= \sum_T \prod_{i=1}^k \tbinom{1 - m_i}{t_i}. \qquad (2.3)
\end{aligned}$$

Here we use that $\binom{a+b-2}{b} = \pm\binom{1-a}{b}$, and the Vandermonde identity.

The sum (2.3) can be considered as a sum Σ_1 over *all* k-tuples T of nonnegative integers summing to k minus the sum Σ_2 over those which are not in $\mathcal{S}_k$. The sum Σ_1 equals $\binom{k - \sum m_i}{k}$, which is 0 unless $\sum m_i = 0$, but that case has been excluded. ($I \ne \emptyset$.)

If $t_j, \dots, t_k$ are nonnegative integers for which $t_j + \cdots + t_k \le k - j$, but $t_{j'} + \cdots + t_k > k - j'$ for all $j' > j$, let $U(t_j, \dots, t_k)$ denote the set of k-tuples T indexing the sum Σ_2 which end with $(t_j, \dots, t_k)$. Since $t_j + \cdots + t_k \le k - j$ is equivalent to saying that the sum of the first $j - 1$ components is not $\le j - 1$, these sets $U(t_j, \dots, t_k)$ partition the set of T's which occur in Σ_2. We show that the sum Σ_2 restricted to any such set $U(t_j, \dots, t_k)$ is 0, which will complete the proof. We have

$$\begin{aligned}
&\sum_{T \in U(t_j, \dots, t_k)} \prod_{i=1}^k \tbinom{1 - m_i}{t_i} \\
&= \prod_{i=j}^k \tbinom{1 - m_i}{t_i} \cdot \sum_{|T'| = k - t_j - \cdots - t_k} \prod_{i=1}^{j-1} \tbinom{1 - m_i}{t_i} \\
&= \prod_{i=j}^k \tbinom{1 - m_i}{t_i} \cdot \tbinom{j - 1 - m_1 - \cdots - m_{j-1}}{k - t_j - \cdots - t_k}.
\end{aligned}$$

In the second line, $T' = (t_1, \ldots, t_{j-1})$. Since $(m_1, \ldots, m_k)$ arises from a subgee, $m_1 + \cdots + m_{j-1} \le j - 1$. But $k - t_j - \cdots - t_k \ge j$. Thus the final binomial coefficient consists of a nonnegative integer atop a larger integer, and hence is 0.

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