# 2-PRIMARY $v_1$-PERIODIC HOMOTOPY GROUPS OF $SU(n)$ REVISITED

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Abstract. In 1991, Bendersky and Davis used the $BP$-based unstable Novikov spectral sequence to study the 2-primary $v_1$-periodic homotopy groups of $SU(n)$. Here we use a $K$-theoretic approach to add more detail to those results. In particular, whereas only the order of the groups $v_1^{-1}\pi_{2k-1}(SU(n))$ was determined in the 1991 paper, here we determine the number of summands in these groups and much information about the orders of those summands. In addition, we give explicit conditions for certain differentials and extensions in a spectral sequence, which affect the homotopy groups. Finally, we give complete results for $v_1^{-1}\pi_*(SU(n))$ for $n \le 13$.

## 1. Statement of results

The 2-primary $v_1$-periodic homotopy groups $v_1^{-1}\pi_*(X)$ of a space $X$ are a localization of the portion of the actual homotopy groups of $X$ detected by 2-local $K$-theory. They form a good first approximation to $\pi_*(X)$; if $X$ is a sphere or compact Lie group, every group $v_1^{-1}\pi_i(X)$ is a direct summand of some group $\pi_{i+2^k}(X)$.([16])

In a 1991 paper ([2]), Bendersky and the first author used the $BP$-based unstable Novikov spectral sequence (UNSS) to study the 2-primary $v_1$-periodic homotopy groups $v_1^{-1}\pi_*(SU(n))$ of the special unitary groups. During the subsequent 14 years, $K$-theoretic approaches to $v_1$-periodic homotopy groups have been developed by Bendersky, Bousfield, Davis, and Thompson ([4, 6, 9, 8]. In this paper, we apply these methods to obtain some refinements of the results of [2].

The principal accomplishments of this paper are:

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- In [2], only the order of the groups $v_1^{-1}\pi_{2k-1}(SU(n))$ was determined. Here we determine the exact number of summands of these groups and establish how many of those summands have order 2 and order 4. We also present tabulations of the summands which present tantalizing patterns. (Theorems 1.2 and 1.4 and Section 2)
- The spectral sequence for $v_1^{-1}\pi_*(SU(n))$ which we compute here is isomorphic to that of [2]. When $n$ is even, there are differentials in the spectral sequence for which the determination in [2] was rather intractable. Here we give some more explicit information about these differentials. (Theorem 6.7 and Conjecture 6.6)
- Because [2] dealt primarily with orders of groups and not their summands, it did not give careful attention to extensions in the spectral sequence. We do that here. (Proposition 6.2, Theorem 7.1, and Conjecture 7.7)
- We give complete explicit results for $v_1^{-1}\pi_*(SU(n))$ for $n \le 13$, both to illustrate our methods and to confirm their efficacy. (Section 8)
- We extend results of [5] and [7] about relationships between $v_1^{-1}\pi_*(SU(n))$ and $v_1^{-1}\pi_*(Sp([n/2]))$ and $v_1^{-1}\pi_*(SU(n)/Sp([n/2]))$. (Section 4)
- The proof in [2] that, when $n$ is odd, the spectral sequence is 0 for $s > 2$ involves a complicated comparison of the UNSS with a more homotopy-theoretic approach, involving, for example, Toda brackets and Whitehead products. This was deemed there to be "the most delicate part" of the paper, and occupied well over half of the paper. Here we show how it is a routine computation using our current methods. (Theorem 3.4)
- We demonstrate the applicability of the Small Complex for calculating certain Ext groups introduced in [6, §11]. (Section 3 and most of Section 7)

- We present two combinatorial conjectures which will have important implications for $v_1^{-1}\pi_*(SU(n))$. (Section 9)

We begin describing our results with those for $SU(n)$ when $n$ is odd, since this is by far the simpler situation. All our results involve the numbers $e(k,n)$ in the following definition. Throughout the paper, $\nu(-)$ denotes the exponent of 2 in an integer.

**Definition 1.1.** [1] *Let* $a(k,j) = \sum_{i\ odd} \binom{j}{i} i^k$. *Then*

$$e(k,n) = \min\{\nu(a(k,j)) : j \geq n\}.$$

**Theorem 1.2.** *Let* $n = 2m+1$ *be odd.*

(1) $v_1^{-1}\pi_{2k}(SU(n)) \approx \mathbf{Z}/2^{e(k,n)}$, *while* $v_1^{-1}\pi_{2k-1}(SU(n))$ *has order* $2^{e(k,n)}$ *and has exactly* $[\log_2(4m/3)]$ *summands.*

(2) *The summands of the groups* $v_1^{-1}\pi_{2k-1}(SU(n))$ *of order* 2 *or* 4 *are*

$$\begin{cases} L_m & k\ odd \\ L'_m & k\ even, \end{cases}$$

*where* $L_m$ *and* $L'_m$ *are as in the following definition.*

Throughout the paper, we use $\mathbf{Z}_n$ and $\mathbf{Z}/n$ interchangeably.

**Definition 1.3.** *For all positive integers* $m$, *abelian groups* $L_m$ *and* $L'_m$ *are defined inductively by*

$$L_1 = \mathbf{Z}_2,\ L_2 = 0,\ L_5 = \mathbf{Z}_4,\ L_6 = \mathbf{Z}_2 \oplus \mathbf{Z}_4,\ L_9 = \mathbf{Z}_4,$$

$$L'_1 = \mathbf{Z}_4,\ L'_2 = 0,\ L'_4 = \mathbf{Z}_2,\ L'_6 = \mathbf{Z}_2,\ L'_{10} = \mathbf{Z}_4,$$

*and for all other values of* $m$,

$$L_m = L_{[m/2]} \text{ and } L'_m = L'_{[m/2]}.$$

[1] Our definition of $a(k,j)$ differs from that in [2], but agrees with [9, 9.8]. This is the definition that makes $e(k,n)$ periodic in $k$ for all integers $k$. The two definitions give the same value of $e(k,n)$ if $k \geq n$.

*The groups* $L_m$ *with* $m \geq 5$ *and* $L'_m$ *with* $m \geq 6$ *can be written explicitly as, for* $e \geq 0$,

$$L_m = \begin{cases} \mathbf{Z}_4 & 5 \cdot 2^e \leq m < 6 \cdot 2^e \\ \mathbf{Z}_2 \oplus \mathbf{Z}_4 & 6 \cdot 2^e \leq m < 7 \cdot 2^e \\ \mathbf{Z}_2 & 7 \cdot 2^e \leq m < 8 \cdot 2^e \\ 0 & 8 \cdot 2^e \leq m < 9 \cdot 2^e \\ \mathbf{Z}_4 & 9 \cdot 2^e \leq m < 10 \cdot 2^e, \end{cases}$$

$$L'_m = \begin{cases} \mathbf{Z}_2 & 6 \cdot 2^e \leq m < 7 \cdot 2^e \\ \mathbf{Z}_4 & 7 \cdot 2^e \leq m < 8 \cdot 2^e \\ \mathbf{Z}_2 & 8 \cdot 2^e \leq m < 10 \cdot 2^e \\ \mathbf{Z}_4 & 10 \cdot 2^e \leq m < 11 \cdot 2^e \\ 0 & 11 \cdot 2^e \leq m < 12 \cdot 2^e. \end{cases}$$

The portion of Theorem 1.2 regarding the groups $v_1^{-1}\pi_{2k}(SU(n))$ and the order of $v_1^{-1}\pi_{2k-1}(SU(n))$ was proved in [2, 1.1a], but we will give a different proof. That the number of summands in $v_1^{-1}\pi_{2k-1}(SU(2m+1))$ is $\geq [\log_2(4m/3)]$ was proved in [7, 1.17]. A result similar to Theorem 1.2(1) is true for all $SU(n)$ localized at an odd prime $p$; the result for the number of summands in those cases was proved by the second author in [17].

In Section 2, we list calculations of the summand sizes of $v_1^{-1}\pi_{2k-1}(SU(2m+1))$ for several small values of $k$ and all $m < 96$. The pattern of these groups is quite tantalizing. Exact formulation of these patterns and proof that they persist remains for the future.

As we will show in Theorem 3.4, the reason that the description of $v_1^{-1}\pi_*(SU(n))$ is so simple when $n$ is odd is that the spectral sequence used to compute it (the UNSS in [2] and the BTSS (see Section 3) here) is nonzero only in filtrations 1 and 2, and hence necessarily collapses. When $n$ is even, the situation is no longer this simple.

The results for $v_1^{-1}\pi_*(SU(2m))$ in [2] were presented by listing groups in various cases. We feel that the following depiction of the spectral sequence which yields these groups is more enlightening.

**Theorem 1.4.**

- *The* $E_\infty$*-term and part of the* $E_3$*-term of the spectral sequence converging to* $v_1^{-1}\pi_*(SU(2m))$ *is as in Diagrams 1.5 and 1.6. Here* $C_k = \mathbf{Z}/2^{e(k,2m)}$ *and* $|G_k| = 2^{e(k,2m)}$.

- *Each group $G_k$ has exactly $[\log_2(4m/3)]$ summands. If $k$ is odd, $G_k \approx v_1^{-1}\pi_{2k-1}(SU(2m+1))$.*
- *Dotted differentials may or may not occur depending upon computable numerical conditions described in 6.5, 6.6, and 6.7.*
- *If $m = 2^e$, the smallest summand of $G_{4a+1}$ is $\mathbf{Z}/8$, and it supports a nontrivial extension and a nonzero differential. If $m = 3 \cdot 2^e$, $e > 0$, the smallest summands of $G_{4a+1}$ are $\mathbf{Z}/2 \oplus \mathbf{Z}/4$, and the $\mathbf{Z}/4$ summand supports a nontrivial extension. The differential from $G_{4a+1}$ does not emanate from a $\mathbf{Z}/2$ or $\mathbf{Z}/4$ summand.*
- *$G_{4a\pm 2}$ has a $\mathbf{Z}/2$ summand iff $m = 3$ or $4 \cdot 2^e < m < 5 \cdot 2^e$ or $6 \cdot 2^e \leq m < 7 \cdot 2^e$ for some $e \geq 0$. The differential is nonzero on the $\mathbf{Z}/2$ summand iff $m = 3 \cdot 2^e$ for $e \geq 0$.*

**Diagram 1.5. Spectral sequence converging to $v_1^{-1}\pi_*(SU(2m))$, $m > 3$ odd**

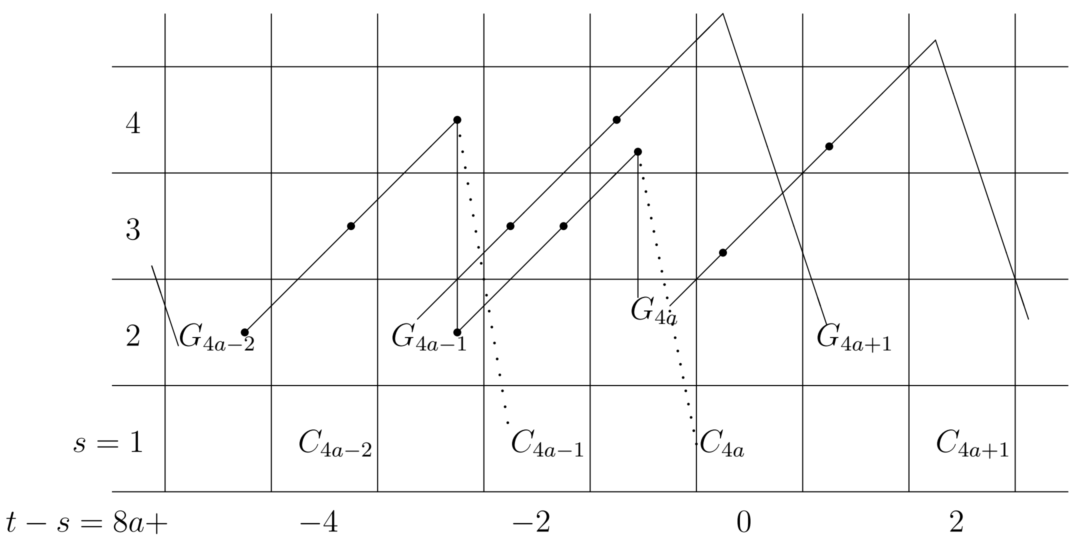
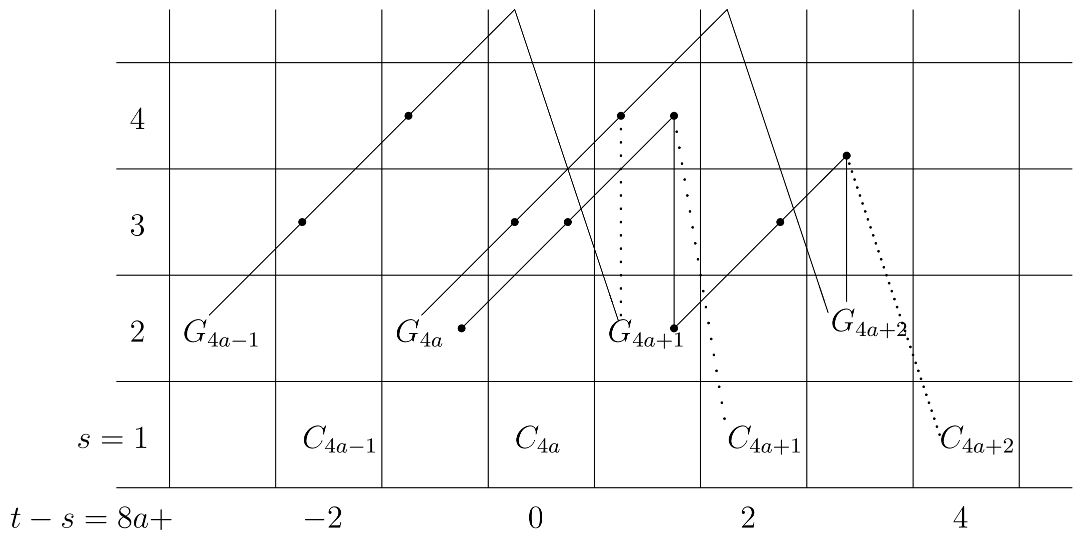


**Diagram 1.6. Spectral sequence converging to $v_1^{-1}\pi_*(SU(2m))$, $m$ even**




These charts employ the usual Adams spectral sequence conventions. Dots are $\mathbf{Z}/2$, lines of slope 1 indicate the action of the Hopf map $\eta$, and lines of slope $-3$ are differentials. Vertical lines are exotic multiplications by 2. The dotted extension in $8a+1$ in Diagram 1.6 is nontrivial if $m=2^e$ or $3\cdot 2^e$ (7.1) and conjectured (in 7.7) to be trivial for other values of $m$. If an extension hits an element which is in the image of a differential, then of course the extension is trivial. We have not drawn in a dotted extension in $8a+1$ in Diagram 1.5 because we conjecture in 7.7 that such extension does not exist unless $m=3$. Similarly, in 6.6 we conjecture that for odd $m>3$, there is no differential from $C_{4a+1}$, and so have not drawn them in. The explicit chart for $SU(6)$ is given in 8.4.

In our diagrams, we have not pictured most of the elements which are involved in $d_3$-differentials, since they do not survive to homotopy classes, and their inclusion leads to a more cluttered diagram. For example, in the box in $(x,y)=(8a+1,2)$ in Diagram 1.5, the $E_2$-term contains an additional (unpictured) $\mathbf{Z}/2$ summand, and an unpictured eta-tower (line of slope 1 with a dot in each box) rises from it. This eta tower supports a differential which hits the (unpictured) remainder of the second eta tower passing through $(8a,5)$. The complete $E_2$-term in $s>2$ consists of two eta towers passing through each box $(x,y)$ with $x+y$ odd.

Some of Theorem 1.4 was proved in [2], but not presented this way; we shall discuss its proof in Section 6. The reason for the attention to $\mathbf{Z}/2$ summands in Theorem 1.4 is that differentials on them affect the number of summands in $v_1^{-1}\pi_*(SU(2m))$. We will not write it out explicitly, but the information in 1.4 determines the number of summands in these groups, modulo the only-conjectural non-extensions from $G_{4a+1}$ and the only-conjectural non-differential on $C_{4a+1}$ when $m$ is odd.

## 2. Sizes of summands of $v_1^{-1}\pi_{2k-1}(SU(2m+1))$ and $E_2^{2,2k+1}(SU(2m))$

In this section we list calculations of the summand sizes of $v_1^{-1}\pi_{2k-1}(SU(2m+1))$ for $k = 0$, 6, 1, and 13, and all $m < 96$. The results are presented in Tables 1, 2, 3, and 4. The pattern of these groups is quite tantalizing. Exact formulation of these patterns and proof that they persist remains for the future. These are computed using Proposition 4.7 and the algorithm described in the proof of 7.1.

We also list summand sizes of $E_2^{2,2k+1}(SU(2m))$ when $k = 0$ and 6 and $m < 48$. The results are presented in Tables 5 and 6. The pattern for these is similar, but not identical, to the groups for $SU(2m+1)$. They are computed by a different algorithm, which appears in the second half of Section 7. We show in Corollary 4.5 that if $k$ is odd, $E_2^{2,2k+1}(SU(2m)) \approx v_1^{-1}\pi_{2k-1}(SU(2m+1)) \oplus \mathbf{Z}/2$, and so a separate listing of $E_2^{2,2k+1}(SU(2m))$ is not needed when $k$ is odd.

## 3. Proof of Theorem 1.2(1)

In this section, we prove Theorem 1.2(1). This section also contains many of the preliminaries that will be used in future sections, as it introduces the way in which the Small Complex of [6, §11] is used to make explicit calculations.

In [8], a $K$-based spectral sequence (the BTSS) converging to $v_1^{-1}\pi_*(X)$ was introduced, and in [4], it was proved that, for a category of spaces which includes spheres and compact Lie groups, its $E_2^{s,t}$-term was $\operatorname{Ext}_{\mathcal{A}}^{s,t}(QK^1(X)/\operatorname{im}(\psi^2))$, where $\mathcal{A}$ denotes the category of $\mathbf{Z}_2$-graded stable 2-adic Adams modules ([9]),[2] $\operatorname{Ext}_{\mathcal{A}}^{s,t}(M)$ means $\operatorname{Ext}_{\mathcal{A}}^{s}(M, K^*(S^t))$, $K$-groups are $\mathbf{Z}_2$-graded 2-adic $K$-theory, and $Q$ denotes the indecomposable quotient. Since $QK^0(SU(n)) = 0$ and $K^1(S^t) = 0$ if $t$ is even, it

[2]Stable 2-adic Adams modules involve an action of $\psi^t$ for odd integers $t$, while unstable 2-adic Adams modules involve $\psi^t$ for all integers $t$.

Table 1. Exponents of 2 of summands of $v_1^{-1}\pi_{2k-1}(SU(2m+1))$, $k=0$

| $m$ | | $m$ | | $m$ | |
|---|---|---|---|---|---|
| | | 32 | 1,3,7,15,38 | 64 | 1,3,7,15,31,71 |
| 1 | 2 | 33 | 1,3,7,18,37 | 65 | 1,3,7,15,34,70 |
| 2 | 4 | 34 | 1,3,8,20,36 | 66 | 1,3,7,16,36,69 |
| 3 | 2,4 | 35 | 1,3,10,20,36 | 67 | 1,3,7,18,37,68 |
| 4 | 1,7 | 36 | 1,4,9,19,39 | 68 | 1,3,8,17,36,71 |
| 5 | 4,6 | 37 | 1,4,11,20,38 | 69 | 1,3,8,20,36,70 |
| 6 | 1,3,8 | 38 | 1,5,11,19,40 | 70 | 1,3,9,20,35,72 |
| 7 | 2,4,8 | 39 | 1,6,11,20,40 | 71 | 1,3,10,20,36,72 |
| 8 | 1,3,12 | 40 | 2,5,10,19,44 | 72 | 1,4,9,19,35,76 |
| 9 | 1,6,11 | 41 | 2,5,10,22,43 | 73 | 1,4,9,19,38,75 |
| 10 | 2,6,12 | 42 | 2,6,10,22,44 | 74 | 1,4,10,19,38,76 |
| 11 | 4,6,12 | 43 | 2,6,12,22,44 | 75 | 1,4,11,20,38,76 |
| 12 | 1,3,5,15 | 44 | 3,6,11,21,47 | 76 | 1,5,11,19,37,79 |
| 13 | 1,3,8,14 | 45 | 3,6,11,24,46 | 77 | 1,5,11,19,40,78 |
| 14 | 2,3,7,16 | 46 | 4,6,11,23,48 | 78 | 1,6,11,19,39,80 |
| 15 | 2,4,8,16 | 47 | 4,6,12,24,48 | 79 | 1,6,11,20,40,80 |
| 16 | 1,3,7,21 | 48 | 1,3,5,11,23,53 | 80 | 2,5,10,19,39,85 |
| 17 | 1,3,10,20 | 49 | 1,3,5,11,26,52 | 81 | 2,5,10,19,42,84 |
| 18 | 1,4,11,20 | 50 | 1,3,5,12,27,52 | 82 | 2,5,10,20,43,84 |
| 19 | 1,6,11,20 | 51 | 1,3,5,14,27,52 | 83 | 2,5,10,22,43,84 |
| 20 | 2,5,10,23 | 52 | 1,3,6,13,26,55 | 84 | 2,6,10,21,42,87 |
| 21 | 2,6,12,22 | 53 | 1,3,6,14,28,54 | 85 | 2,6,11,22,44,86 |
| 22 | 3,6,11,24 | 54 | 1,3,7,14,27,56 | 86 | 2,6,11,22,43,88 |
| 23 | 4,6,12,24 | 55 | 1,3,8,14,28,56 | 87 | 2,6,12,22,44,88 |
| 24 | 1,3,5,11,28 | 56 | 2,3,7,13,27,60 | 88 | 3,6,11,21,43,92 |
| 25 | 1,3,5,14,27 | 57 | 2,3,7,13,30,59 | 89 | 3,6,11,21,46,91 |
| 26 | 1,3,6,14,28 | 58 | 2,3,7,14,30,60 | 90 | 3,6,11,22,46,92 |
| 27 | 1,3,8,14,28 | 59 | 2,3,7,16,30,60 | 91 | 3,6,11,24,46,92 |
| 28 | 2,3,7,13,31 | 60 | 2,4,7,15,29,63 | 92 | 4,6,11,23,45,95 |
| 29 | 2,3,7,16,30 | 61 | 2,4,7,15,32,62 | 93 | 4,6,11,23,48,94 |
| 30 | 2,4,7,15,32 | 62 | 2,4,8,15,31,64 | 94 | 4,6,12,23,47,96 |
| 31 | 2,4,8,16,32 | 63 | 2,4,8,16,32,64 | 95 | 4,6,12,24,48,96 |

TABLE 2. Exponents of 2 of summands of $v_1^{-1}\pi_{2k-1}(SU(2m+1))$, $k=6$

| $m$ | | $m$ | | $m$ | |
|---|---|---|---|---|---|
| | | 32 | 1,6,6,16,34 | 64 | 1,6,6,16,32,66 |
| 1 | 2 | 33 | 1,6,6,18,34 | 65 | 1,6,6,16,34,66 |
| 2 | 3 | 34 | 1,6,7,16,36 | 66 | 1,6,6,17,32,68 |
| 3 | 2,6 | 35 | 1,6,8,18,39 | 67 | 1,6,6,18,34,71 |
| 4 | 1,8 | 36 | 1,5,8,18,41 | 68 | 1,6,7,16,34,73 |
| 5 | 3,6 | 37 | 1,5,8,20,39 | 69 | 1,6,7,16,36,71 |
| 6 | 1,3,6 | 38 | 1,6,8,19,40 | 70 | 1,6,8,16,34,73 |
| 7 | 2,6,7 | 39 | 1,6,10,22,40 | 71 | 1,6,8,18,37,73 |
| 8 | 1,6,8 | 40 | 2,4,10,22,41 | 72 | 1,5,8,18,37,74 |
| 9 | 1,6,10 | 41 | 2,4,10,23,42 | 73 | 1,5,8,18,39,74 |
| 10 | 2,4,12 | 42 | 2,4,11,21,44 | 74 | 1,5,8,19,38,75 |
| 11 | 3,6,15 | 43 | 2,4,12,23,47 | 75 | 1,5,8,20,39,79 |
| 12 | 1,3,4,16 | 44 | 3,4,12,21,48 | 76 | 1,6,8,19,38,80 |
| 13 | 1,3,6,15 | 45 | 3,4,12,23,47 | 77 | 1,6,8,19,40,79 |
| 14 | 2,3,6,15 | 46 | 3,5,12,23,47 | 78 | 1,6,9,19,40,79 |
| 15 | 2,6,7,16 | 47 | 3,6,15,23,48 | 79 | 1,6,10,22,40,80 |
| 16 | 1,6,6,18 | 48 | 1,3,4,14,23,50 | 80 | 2,4,10,22,39,82 |
| 17 | 1,6,8,18 | 49 | 1,3,4,16,23,50 | 81 | 2,4,10,22,41,82 |
| 18 | 1,5,8,20 | 50 | 1,3,4,15,23,52 | 82 | 2,4,10,21,41,84 |
| 19 | 1,6,10,23 | 51 | 1,3,4,15,26,55 | 83 | 2,4,10,23,42,87 |
| 20 | 2,4,10,25 | 52 | 1,3,5,13,26,57 | 84 | 2,4,11,21,42,89 |
| 21 | 2,4,12,23 | 53 | 1,3,5,13,28,55 | 85 | 2,4,11,21,44,87 |
| 22 | 3,4,12,23 | 54 | 1,3,6,13,28,55 | 86 | 2,4,12,21,44,87 |
| 23 | 3,6,15,23 | 55 | 1,3,6,15,31,55 | 87 | 2,4,12,23,47,87 |
| 24 | 1,3,4,16,23 | 56 | 2,3,6,13,31,56 | 88 | 3,4,12,21,48,87 |
| 25 | 1,3,4,15,26 | 57 | 2,3,6,13,31,58 | 89 | 3,4,12,21,47,90 |
| 26 | 1,3,5,13,28 | 58 | 2,3,6,14,29,60 | 90 | 3,4,12,22,45,92 |
| 27 | 1,3,6,15,31 | 59 | 2,3,6,15,31,63 | 91 | 3,4,12,23,47,95 |
| 28 | 2,3,6,13,32 | 60 | 2,4,6,15,29,64 | 92 | 3,5,12,23,45,96 |
| 29 | 2,3,6,15,31 | 61 | 2,4,6,15,31,63 | 93 | 3,5,12,23,47,95 |
| 30 | 2,4,6,15,31 | 62 | 2,5,6,15,31,63 | 94 | 3,6,12,23,47,95 |
| 31 | 2,6,7,16,32 | 63 | 2,6,7,16,32,64 | 95 | 3,6,15,23,48,96 |

TABLE 3. Exponents of 2 of summands of $v_1^{-1}\pi_{2k-1}(SU(2m+1))$, $k=1$

| $m$ | | $m$ | | $m$ | |
|---|---|---|---|---|---|
| | | 32 | 3,4,8,16,32 | 64 | 3,4,8,16,32,64 |
| 1 | 1 | 33 | 3,4,6,14,38 | 65 | 3,4,7,14,30,71 |
| 2 | 3 | 34 | 3,3,6,16,39 | 66 | 3,4,6,14,32,72 |
| 3 | 1,4 | 35 | 3,3,6,20,37 | 67 | 3,4,6,14,36,70 |
| 4 | 3,4 | 36 | 2,3,8,20,38 | 68 | 3,3,6,16,36,71 |
| 5 | 2,7 | 37 | 2,3,8,21,39 | 69 | 3,3,6,16,38,71 |
| 6 | 1,2,8 | 38 | 2,3,10,20,40 | 70 | 3,3,6,18,37,72 |
| 7 | 1,4,8 | 39 | 2,3,12,20,40 | 71 | 3,3,6,20,37,72 |
| 8 | 3,4,8 | 40 | 2,4,12,21,40 | 72 | 2,3,8,20,38,72 |
| 9 | 2,3,12 | 41 | 2,4,12,19,44 | 73 | 2,3,8,20,36,76 |
| 10 | 2,4,13 | 42 | 2,4,12,20,45 | 74 | 2,3,8,21,36,77 |
| 11 | 2,7,12 | 43 | 2,4,12,23,44 | 75 | 2,3,8,21,39,76 |
| 12 | 1,2,8,12 | 44 | 2,6,11,24,44 | 76 | 2,3,10,20,40,76 |
| 13 | 1,2,7,15 | 45 | 2,6,11,23,47 | 77 | 2,3,10,20,39,79 |
| 14 | 1,4,6,16 | 46 | 2,7,12,22,48 | 78 | 2,3,12,20,38,80 |
| 15 | 1,4,8,16 | 47 | 2,7,12,24,48 | 79 | 2,3,12,20,40,80 |
| 16 | 3,4,8,16 | 48 | 1,2,8,12,24,48 | 80 | 2,4,12,21,40,80 |
| 17 | 3,3,6,21 | 49 | 1,2,8,11,22,53 | 81 | 2,4,12,20,38,85 |
| 18 | 2,3,8,22 | 50 | 1,2,8,10,24,54 | 82 | 2,4,12,19,40,86 |
| 19 | 2,3,12,20 | 51 | 1,2,8,10,28,52 | 83 | 2,4,12,19,44,84 |
| 20 | 2,4,12,21 | 52 | 1,2,7,12,28,53 | 84 | 2,4,12,20,44,85 |
| 21 | 2,4,12,23 | 53 | 1,2,7,12,28,55 | 85 | 2,4,12,20,44,87 |
| 22 | 2,6,11,24 | 54 | 1,2,7,14,27,56 | 86 | 2,4,12,22,43,88 |
| 23 | 2,7,12,24 | 55 | 1,2,7,15,28,56 | 87 | 2,4,12,23,44,88 |
| 24 | 1,2,8,12,24 | 56 | 1,4,6,16,28,56 | 88 | 2,6,11,24,44,88 |
| 25 | 1,2,8,10,28 | 57 | 1,4,6,16,26,60 | 89 | 2,6,11,24,42,92 |
| 26 | 1,2,7,12,29 | 58 | 1,4,6,15,28,61 | 90 | 2,6,11,23,44,93 |
| 27 | 1,2,7,15,28 | 59 | 1,4,6,15,31,60 | 91 | 2,6,11,23,47,92 |
| 28 | 1,4,6,16,28 | 60 | 1,4,8,14,32,60 | 92 | 2,7,12,22,48,92 |
| 29 | 1,4,6,15,31 | 61 | 1,4,8,14,31,63 | 93 | 2,7,12,22,47,95 |
| 30 | 1,4,8,14,32 | 62 | 1,4,8,16,30,64 | 94 | 2,7,12,24,46,96 |
| 31 | 1,4,8,16,32 | 63 | 1,4,8,16,32,64 | 95 | 2,7,12,24,48,96 |

TABLE 4. Exponents of 2 of summands of $v_1^{-1}\pi_{2k-1}(SU(2m+1))$, $k=13$

| $m$ | | $m$ | | $m$ | |
|---|---|---|---|---|---|
| | | 32 | 3,5,10,13,32 | 64 | 3,5,10,13,32,64 |
| 1 | 1 | 33 | 3,4,10,12,35 | 65 | 3,5,10,12,31,67 |
| 2 | 3 | 34 | 3,3,11,13,36 | 66 | 3,4,10,12,33,68 |
| 3 | 1,7 | 35 | 3,3,11,15,39 | 67 | 3,4,10,12,35,71 |
| 4 | 3,5 | 36 | 2,3,13,16,37 | 68 | 3,3,11,13,36,69 |
| 5 | 2,6 | 37 | 2,3,13,16,37 | 69 | 3,3,11,13,36,69 |
| 6 | 1,2,7 | 38 | 2,3,15,15,38 | 70 | 3,3,11,15,36,69 |
| 7 | 1,5,15 | 39 | 2,3,14,18,42 | 71 | 3,3,11,15,37,75 |
| 8 | 3,5,13 | 40 | 2,4,15,18,42 | 72 | 2,3,13,16,37,75 |
| 9 | 2,3,16 | 41 | 2,4,12,18,46 | 73 | 2,3,13,16,34,79 |
| 10 | 2,4,15 | 42 | 2,4,11,20,47 | 74 | 2,3,13,16,35,80 |
| 11 | 2,6,13 | 43 | 2,4,11,22,46 | 75 | 2,3,13,16,37,78 |
| 12 | 1,2,7,13 | 44 | 2,6,10,23,46 | 76 | 2,3,15,15,38,78 |
| 13 | 1,2,6,14 | 45 | 2,6,10,22,47 | 77 | 2,3,15,15,37,79 |
| 14 | 1,4,5,15 | 46 | 2,6,12,21,48 | 78 | 2,3,14,17,37,80 |
| 15 | 1,5,10,13 | 47 | 2,6,13,25,47 | 79 | 2,3,14,18,42,78 |
| 16 | 3,5,10,13 | 48 | 1,2,7,13,25,47 | 80 | 2,4,15,18,42,78 |
| 17 | 3,3,11,15 | 49 | 1,2,7,11,25,50 | 81 | 2,4,13,18,42,81 |
| 18 | 2,3,13,16 | 50 | 1,2,7,10,27,51 | 82 | 2,4,12,18,44,82 |
| 19 | 2,3,15,18 | 51 | 1,2,7,10,29,53 | 83 | 2,4,12,18,46,83 |
| 20 | 2,4,15,18 | 52 | 1,2,6,12,30,52 | 84 | 2,4,11,20,47,83 |
| 21 | 2,4,11,22 | 53 | 1,2,6,12,28,54 | 85 | 2,4,11,20,44,86 |
| 22 | 2,6,10,23 | 54 | 1,2,6,14,27,55 | 86 | 2,4,11,22,43,87 |
| 23 | 2,6,13,25 | 55 | 1,2,6,14,30,57 | 87 | 2,4,11,22,46,89 |
| 24 | 1,2,7,13,25 | 56 | 1,4,5,15,30,57 | 88 | 2,6,10,23,46,89 |
| 25 | 1,2,7,10,29 | 57 | 1,4,5,15,27,61 | 89 | 2,6,10,23,43,93 |
| 26 | 1,2,6,12,30 | 58 | 1,4,5,14,29,62 | 90 | 2,6,10,22,45,94 |
| 27 | 1,2,6,14,30 | 59 | 1,4,5,14,31,62 | 91 | 2,6,10,22,47,94 |
| 28 | 1,4,5,15,30 | 60 | 1,5,6,13,32,62 | 92 | 2,6,12,21,48,94 |
| 29 | 1,4,5,14,31 | 61 | 1,5,6,13,31,63 | 93 | 2,6,12,21,47,95 |
| 30 | 1,5,6,13,32 | 62 | 1,5,8,13,30,64 | 94 | 2,6,13,22,46,96 |
| 31 | 1,5,10,13,32 | 63 | 1,5,10,13,32,64 | 95 | 2,6,13,25,47,96 |

Table 5. Exponents of 2 of summands of $E_2^{2,2k+1}(SU(2m))$, $k=0$

| $m$ | | $m$ | | $m$ | |
|---|---|---|---|---|---|
| | | 16 | 1,3,4,8,16 | 32 | 1,3,4,8,16,32 |
| 1 | 1,1 | 17 | 1,1,3,8,21 | 33 | 1,1,3,7,16,38 |
| 2 | 1,3 | 18 | 1,1,4,10,20 | 34 | 1,1,3,8,18,37 |
| 3 | 1,1,4 | 19 | 1,1,5,11,20 | 35 | 1,1,3,9,20,36 |
| 4 | 1,3,4 | 20 | 1,2,6,11,20 | 36 | 1,1,4,10,20,36 |
| 5 | 1,2,7 | 21 | 1,2,6,10,23 | 37 | 1,1,4,10,19,39 |
| 6 | 1,1,4,6 | 22 | 1,3,6,12,22 | 38 | 1,1,5,11,20,38 |
| 7 | 1,2,3,8 | 23 | 1,4,6,11,24 | 39 | 1,1,6,11,19,40 |
| 8 | 1,3,4,8 | 24 | 1,1,4,6,12,24 | 40 | 1,2,6,11,20,40 |
| 9 | 1,1,4,12 | 25 | 1,1,3,5,12,28 | 41 | 1,2,5,10,20,44 |
| 10 | 1,2,6,11 | 26 | 1,1,3,6,14,27 | 42 | 1,2,6,10,22,43 |
| 11 | 1,3,6,12 | 27 | 1,1,3,7,14,28 | 43 | 1,2,6,11,22,44 |
| 12 | 1,1,4,6,12 | 28 | 1,2,3,8,14,28 | 44 | 1,3,6,12,22,44 |
| 13 | 1,1,3,6,15 | 29 | 1,2,3,7,14,31 | 45 | 1,3,6,11,22,47 |
| 14 | 1,2,3,8,14 | 30 | 1,2,4,7,16,30 | 46 | 1,4,6,11,24,46 |
| 15 | 1,2,4,7,16 | 31 | 1,2,4,8,15,32 | 47 | 1,4,6,12,23,48 |

follows that $E_2^{s,t}(SU(n))=0$ if $t$ is even. This spectral sequence can also be obtained, using methods of [9], as the homotopy spectral sequence for the spectrum $\Phi_1 X$.

In [6, §11], a small chain complex for computing these Ext groups was developed. That chain complex was not used in an essential way in [6]; part of the significance of this paper is to demonstrate the utility of that chain complex.

The following definition is easily seen to be equivalent to [6, 11.1], e.g. using the Smith normal form.

**Definition 3.1.** *If $M$ is an integer matrix, $Q(M)$ is the torsion subgroup of the $\mathbf{Z}_2^\wedge$-module presented by $M$.*

As in [6, 11.2], an unstable 2-adic Adams module is said to be *algebraically spherically resolved* (ASR) if it can be built from various $QK^1(S^{2n_i+1})$ by short exact sequences. Our $QK^1(SU(n))$ is ASR. By [6, 11.9,11.3], we have the following useful result.

TABLE 6. Exponents of 2 of summands of $E_2^{2,2k+1}(SU(2m))$, $k=6$

| $m$ | | $m$ | | $m$ | |
|---|---|---|---|---|---|
| | | 16 | 1,3,6,6,16 | 32 | 1,3,6,6,16,32 |
| 1 | 1,1 | 17 | 1,1,6,7,18 | 33 | 1,1,6,6,17,34 |
| 2 | 1,3 | 18 | 1,1,7,8,18 | 34 | 1,1,6,7,18,34 |
| 3 | 1,1,3 | 19 | 1,1,6,8,20 | 35 | 1,1,6,8,16,36 |
| 4 | 1,3,6 | 20 | 1,2,6,10,23 | 36 | 1,1,7,8,18,39 |
| 5 | 1,2,7 | 21 | 1,2,4,11,24 | 37 | 1,1,5,8,19,40 |
| 6 | 1,1,3,6 | 22 | 1,3,4,12,23 | 38 | 1,1,6,8,20,39 |
| 7 | 1,2,3,6 | 23 | 1,3,5,12,23 | 39 | 1,1,6,9,19,40 |
| 8 | 1,3,6,6 | 24 | 1,1,3,6,14,23 | 40 | 1,2,6,10,22,39 |
| 9 | 1,1,7,8 | 25 | 1,1,3,4,16,24 | 41 | 1,2,4,10,23,41 |
| 10 | 1,2,6,10 | 26 | 1,1,3,5,15,26 | 42 | 1,2,4,11,23,42 |
| 11 | 1,3,4,12 | 27 | 1,1,3,6,13,28 | 43 | 1,2,4,12,21,44 |
| 12 | 1,1,3,6,14 | 28 | 1,2,3,6,15,30 | 44 | 1,3,4,12,23,46 |
| 13 | 1,1,3,5,16 | 29 | 1,2,3,6,14,32 | 45 | 1,3,4,12,22,48 |
| 14 | 1,2,3,6,15 | 30 | 1,2,4,6,15,31 | 46 | 1,3,5,12,23,47 |
| 15 | 1,2,4,6,15 | 31 | 1,2,5,6,15,31 | 47 | 1,3,6,12,23,47 |

**Theorem 3.2.** ([6]) *Let $N$ be an ASR unstable 2-adic Adams module with basis $B$. Thus $N/\operatorname{im}(\psi^2)$ is in $\mathcal{A}$. Let $\Psi^t$ and $\Theta_k$ denote the matrices of $\psi^t$ and $\psi^3-3^k$, respectively, with respect to $B$. Then*

$$\operatorname{Ext}_{\mathcal{A}}^{s,2k+1}(N/\operatorname{im}(\psi^2)) \approx Q(M_{s,k}),$$

*where*

$$M_{1,k} = \begin{pmatrix}\Psi^{-1}-(-1)^k & \Psi^2 & \Theta_k\end{pmatrix}$$

*and, for $s\geq 2$, with the last block of rows deleted if $s=2$,*

$$M_{s,k} = \begin{pmatrix} \Psi^{-1}+(-1)^{s+k} & \Psi^2 & \Theta_k & 0 \\ 0 & -\Psi^{-1}+(-1)^{s+k} & 0 & \Theta_k \\ 0 & 0 & -\Psi^{-1}+(-1)^{s+k} & -\Psi^2 \\ 0 & 0 & 0 & \Psi^{-1}+(-1)^{s+k} \end{pmatrix}.$$

One basis for $QK^1(SU(n))$ is given by $\{X,\ldots,X^{n-1}\}$ with $\psi^t(X)=(X-1)^t-1$ and $\psi^t$ acting multiplicatively. We remark that $QK^1(SU(n))$ does not admit a multiplication, but we use its isomorphism with $\widetilde{K}^0(CP^{n-1})$, which does. We will find it useful to use the alternate basis given in the following proposition.

**Proposition 3.3.** *For $\epsilon \in \{0,1\}$, there is a basis*

$$\{X, XY, \ldots, XY^{m-1}, Y, \ldots, Y^{m-1+\epsilon}\}$$

*for $QK^1(SU(2m+\epsilon))$ satisfying*

$$\psi^t(Y) = \begin{cases} Y & t=-1 \\ 4Y+Y^2 & t=2 \\ 9Y+6Y^2+Y^3 & t=3 \end{cases}$$

$$\psi^t(X) = \begin{cases} -X+Y & t=-1 \\ 2X+XY+Y & t=2 \\ 3X+4XY+3Y+XY^2+Y^2 & t=3 \end{cases}$$

*and multiplicativity.*

*Proof.* We let $Y = X^2/(1+X)$. The formulas follow from those for $\psi^t(X)$. For example,

$$\begin{aligned} \psi^{-1}(Y) &= \frac{((1+X)^{-1}-1)^2}{(1+X)^{-1}} = \frac{(1-(1+X))^2}{1+X} = \frac{X^2}{1+X} = Y \\ \psi^2(Y) &= \frac{((1+X)^2-1)^2}{(1+X)^2} = \frac{X^4+4X^2(1+X)}{(1+X)^2} = Y^2+4Y \\ \psi^3(X) &= 3X+3X^2+X^3 = 3X+3(XY+Y)+X^2Y+XY \\ &= 3X+4XY+3Y+(XY+Y)Y. \end{aligned}$$

■

Now we can easily prove the following result. As pointed out in the introduction, the proof of this result in [2] was very involved, occupying more than half of that paper.

**Theorem 3.4.** *If $n$ is odd, then $E_2^s(SU(n)) = 0$ for $s > 2$.*

*Proof.* Let $Q_m = SU(2m+1)/SU(2m-1)$. There are short exact sequences in $\mathcal{A}$

$$0 \to QK^1(Q_m)/\operatorname{im}(\psi^2) \to QK^1(SU(2m+1))/\operatorname{im}(\psi^2) \to QK^1(SU(2m-1))/\operatorname{im}(\psi^2) \to 0,$$

and so the theorem follows by induction on $m$, using the exact sequences in $\mathrm{Ext}_{\mathcal{A}}$ and the calculation in the following lemma. ■

**Lemma 3.5.** $\operatorname{Ext}^s_{\mathcal{A}}(QK^1(Q_m)/\operatorname{im}(\psi^2)) = 0$ *if* $s > 2$.

*Proof.* Using 3.3, we see that $QK^1(Q_m)$ has basis $\{XY^{m-1}, Y^m\}$ with matrices of $\psi^t$ given by

$$\Psi^{-1} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \Psi^2 = \begin{pmatrix} 2^{2m-1} & 0 \\ 2^{2m-2} & 2^{2m} \end{pmatrix}, \quad \Psi^3 = \begin{pmatrix} 3^{2m-1} & 0 \\ 3^{2m-1} & 3^{2m} \end{pmatrix}.$$

By Theorem 3.2, if $s+k$ is even and $s > 2$, then the desired Ext group is $Q(M)$, where $M$ is the following matrix:

$$\left(\begin{array}{cc|cc|cc|cc}
0 & 0 & 2^{2m-1} & 0 & 3^{2m-1}-3^k & 0 & 0 & 0 \\
1 & 2 & 2^{2m-2} & 2^{2m} & 3^{2m-1} & 3^{2m}-3^k & 0 & 0 \\
\hline
0 & 0 & 2 & 0 & 0 & 0 & 3^{2m-1}-3^k & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 3^{2m-1} & 3^{2m}-3^k \\
\hline
0 & 0 & 0 & 0 & 2 & 0 & -2^{2m-1} & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -2^{2m-2} & -2^{2m} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2
\end{array}\right)$$

One verifies that, with $R_i$ denoting the $i$th row,

$$\begin{aligned}
R_1 &= -2^{2m-1}R_4 - (3^{2m-1} - 3^k)R_6 + 2^{2m-2}(3^{2m-1} + 3^k)R_8 \\
R_3 &= -2R_4 + (3^{2m} - 3^k)R_8 \\
R_5 &= -2R_6 - 2^{2m}R_8.
\end{aligned}$$

After removing these dependent rows, the remaining matrix is of the form

$$M' = \begin{pmatrix} 1 & x & x & x & x & x & x & x \\ 0 & 0 & 1 & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 1 & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \end{pmatrix}.$$

For such a matrix, $Q(M') = 0$.

If $s+k$ is odd, the $2 \times 2$ matrices along the diagonal are negated and reversed. A similar and easier argument shows $Q(M) = 0$ in this case. ■

The portion of Theorem 1.2 regarding $v_1^{-1}\pi_{2k}(SU(n))$ and the order of the group $v_1^{-1}\pi_{2k-1}(SU(n))$ when $n$ is odd is now immediate from the following result, which is an easy adaptation to the prime 2 of [9, 9.2].

**Proposition 3.6.** *For all* $n \ge 2$, $E_2^{1,2k+1}(SU(n)) \approx \mathbf{Z}/2^{e(k,n)}$. *If* $n$ *is odd, the order of* $E_2^{2,2k+1}(SU(n))$ *equals* $2^{e(k,n)}$.

*Proof.* By [3, 1.1], $E_2^{1,2k+1} \approx QK^1(SU(n))/(\psi^2, \psi^t - t^k : t \text{ odd})$.[3] This is easily seen to be $\mathbf{Z}/2^{e(k,n)}$, using the same argument as was used by Bousfield in [9, §9] when localized at an odd prime, using the basis obtained from powers of the Hopf bundle $\xi$.

That $E_2^{2,t}(X)$ and $E_2^{1,t}(X)$ have the same order if $E_2^s(X) = 0$ for $s > 2$ follows from [4, 3.10], since one is the kernel and the other the cokernel of the same homomorphism. An additional possible contribution to $E_2^2(X)$ in [4, 3.10] is 0 because it is isomorphic to a summand of $E_2^s(X)$ for $s > 2$, and this is 0 when $X = SU(n)$ with $n$ odd. ■

We complete this section by proving the following result, which, with Theorem 3.4, implies the final part of Theorem 1.2(1). Here we have introduced the standard notation, $\mathrm{rk}_2(G)$, for the number of summands of a finite 2-group $G$.

**Proposition 3.7.** *If $m$ is a positive integer and $k$ is any integer, then*

$$\mathrm{rk}_2(E_2^{2,2k+1}(SU(2m+1))) = [\log_2(4m/3)].$$

*Proof.* Let $M$ denote the matrix of Theorem 3.2 with $N = QK^1(SU(2m+1))$ and $s = 2$, so that the last "row" is deleted. This is considered as a matrix over the 2-adics. Row and column operations do not change $Q(M)$, and can bring the matrix to diagonal form, with 1's and other 2-powers on the diagonal. The simplified matrix may also have rows and columns of all 0's. This is the Smith normal form. See, e.g. [1, 5.3.1]. Then our desired group $Q(M)$ is the direct sum of $\mathbf{Z}/2^e$ for those $2^e$ with $e > 0$ which appear on the diagonal. Clearly

$$\mathrm{rk}_2(Q(M)) = \mathrm{rank}(M) - \mathrm{rank}(M \bmod 2). \tag{3.8}$$

We first show $\mathrm{rank}(M) = 4m$. The rank may be computed over the rational numbers. Corresponding to the rational splitting of $SU(2m+1)$ as a product of spheres is an isomorphism of unstable Adams modules

$$QK^1(SU(2m+1); \mathbf{Q}) \approx \bigoplus_{i=1}^{2m} QK^1(S^{2i+1}; \mathbf{Q}).$$

[3]Actually, [3] showed the groups were Pontryagin dual, but we use that a finite group is isomorphic to its dual.

The result for rank$(M)$ follows from the observation that the rank of the matrix for $S^{2i+1}$ equals 2. Indeed, the matrix for $S^{2i+1}$ will be of one of the following two forms

$$\begin{pmatrix} 0 & 2^e & a & 0 \\ 0 & 2 & 0 & a \\ 0 & 0 & 2 & -2^e \end{pmatrix}, \qquad \begin{pmatrix} 2 & 2^e & a & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & -2^e \end{pmatrix},$$

with $a$ even, each of which has rank 2.

Now we work over $\mathbf{Z}/2$ and show that the rank of $M$ mod 2 is $4m - [\log_2(4m/3)]$. With (3.8), this will complete the proof.

We use the basis of Proposition 3.3, from which we obtain the following formulas for Adams operations mod 2:

$$\begin{aligned} \psi^{-1}(XY^{i-1}) &= XY^{i-1} + Y^i & \psi^{-1}(Y^i) &= Y^i \\ \psi^2(XY^{i-1}) &= XY^{2i-1} + Y^{2i-1} & \psi^2(Y^i) &= Y^{2i} \\ \psi^3(XY^{i-1}) &= XY^{i-1}(1+Y^2)^i + Y^i(1+Y)^{2i-1} & \psi^3(Y^i) &= Y^i(1+Y^2)^i \end{aligned} \quad . \tag{3.9}$$

The matrix $M$ with respect to this basis is given in (3.10). Each submatrix indicated by a single entry in (3.10) is $m \times m$, corresponding to one of the two halves of the basis in 3.3. Here $\Theta_i$ is a matrix of components of $\psi^3 - 1$, with $i = 1$ corresponding to the portion of either half of the basis to itself (which are equal for the two half bases), and $i = 2$ for the portion from the $XY^j$-half-basis to the $Y^j$-half-basis. Note that the meaning of the subscript of $\Theta$ is different than that in 3.2. The submatrices $\Psi^2$ are subscripted similarly.

$$\left(\begin{array}{cc|cc|cc|cc} 0 & 0 & \Psi^2_1 & 0 & \Theta_1 & 0 & 0 & 0 \\ I & 0 & \Psi^2_2 & \Psi^2_1 & \Theta_2 & \Theta_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \Theta_1 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & \Theta_2 & \Theta_1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \Psi^2_1 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & \Psi^2_2 & \Psi^2_1 \end{array}\right) \tag{3.10}$$

After pivoting on the three $I$'s, we obtain the matrix (3.11).

$$\left(\begin{array}{cc|cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 & \Psi_1^2\Theta_2+\Theta_1\Psi_2^2 & \Psi_1^2\Theta_1+\Theta_1\Psi_1^2 \\ I & 0 & 0 & \Psi_1^2 & 0 & \Theta_1 & \Psi_2^2\Theta_2+\Theta_2\Psi_2^2 & \Psi_2^2\Theta_1+\Theta_2\Psi_1^2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \Theta_1 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & \Theta_2 & \Theta_1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \Psi_1^2 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & \Psi_2^2 & \Psi_1^2 \end{array}\right) \tag{3.11}$$

We delete the rows and columns with the $I$'s, and the resulting columns of 0's. The $3m$ rows with the pivot 1's which we delete here all contribute to the rank of this mod 2 matrix. We also remove the last block of columns because the entry in the upper right corner of (3.11) is 0. This leaves the following $3m \times m$ matrix.

$$\begin{pmatrix} \Psi_1^2\Theta_2+\Theta_1\Psi_2^2 \\ \Theta_1 \\ \Psi_1^2 \end{pmatrix}.$$

The $m \times m$ matrix $\Psi_1^2$, mod 2, has a 1 in position $(2j, j)$ for $j \le [m/2]$ and 0's elsewhere. Similarly, the only nonzero entries of $\Psi_2^2$ are in $(2j-1, j)$. Both $\Theta_1$ and $\Theta_2$ are lower triangular, and $\Theta_1$ has 0's on the diagonal. This implies that $\Psi_1^2\Theta_2+\Theta_1\Psi_2^2$ is 0 in columns $> [m/2]$.

We pivot on the 1's of $\Psi_1^2$, and then remove their rows and columns and the rows which are 0, leaving just the right half of $\Theta_1$, i.e. columns $j$ with $j > [m/2]$. We denote this matrix by $\Theta_R$. The $[m/2]$ rows with leading 1's which were removed also contribute to the rank.

We show in the next paragraph that the rank of $\Theta_R$ is $m-[m/2]-[\log_2(4m/3)]$, which, when combined with the ranks $3m$ and $[m/2]$ already removed, implies the claim about rank($M$ mod 2), and hence the proposition. This analysis is essentially equivalent to that of [7, 1.14] and [6, 5.6ff].

If the rows (omitting those with $i \le [m/2]$, since they are 0) and columns of $\Theta_R$ are ordered first by exponent of 2, and then in increasing order for those with the same exponent, we obtain a lower triangular matrix with 0's on the diagonal, and 1's on the subdiagonal except for the last column in each fixed-exponent grouping. This last column in each group is 0. Thus the rank of $\Theta_R$ equals its number of columns minus the number of integers $e$ such that there is an integer $j$ satisfying $[m/2] < j \le m$ and $\nu(j) = e$. This number is $[\log_2(4m/3)]$. To see this, note that if $2^t \le m < 3 \cdot 2^{t-1}$, then all integers $e \neq t-1$ satisfying $0 \leq e \leq t$ occur, while if $3 \cdot 2^{t-1} \leq m < 2^{t+1}$, then all integers $e$ satisfying $0 \leq e \leq t$ occur. ■

## 4. Relationship between $SU(n)$ and $Sp([n/2])$

In this section we show how the $E_2$-term of $SU(n)$ is related to that of $Sp([n/2])$ and $SU(n)/Sp([n/2])$. This is used in computing the summands of $v_1^{-1}\pi_*(SU(n))$ listed in Section 2.

In [5] and [7], Bendersky and the first author studied the exact sequence of UNSS for the fibration

$$Sp(m) \xrightarrow{i} SU(2m+\epsilon) \xrightarrow{p} H_{m,\epsilon} \tag{4.1}$$

and obtained relationships between some 1-line and 2-line groups of $Sp(m)$ and $SU(2m+\epsilon)$. Here $\epsilon = 0$ or 1, and $H_{m,\epsilon}$ is the group quotient defined by the fibration. We extend these results here, using the BTSS. These results accomplish two things: first, they shed light on the relationship between these spaces, and second, they give an easier way to compute some 2-line groups for $SU(n)$. A major ingredient in the proof of this proposition is our results, 3.7 and 6.8, about number of summands.

**Proposition 4.2.** *Let $i$ be as in (4.1).*

(1) *If $t \equiv 3 \mod 4$, there is a split short exact sequence*

$$0 \to \mathbf{Z}/2 \to E_2^{2,t}(Sp(m)) \xrightarrow{i_*} E_2^{2,t}(SU(2m+1)) \to 0.$$

(2) *If $t \equiv 1 \mod 4$, then*

$$E_2^{2,t}(H_{m,1}) \approx E_2^{2,t}(SU(2m+1)) \oplus \mathbf{Z}/2.$$

*The morphism $E_2^{2,t}(SU(2m+1)) \xrightarrow{p_*} E_2^{2,t}(H_{m,1})$ corresponds to multiplication by 2 under the above isomorphism.*

(3) *If $t \equiv 3 \mod 4$, then $E_2^{2,t}(SU(2m))$ is isomorphic to $E_2^{2,t}(Sp(m))$. The morphism $i_*$ has a split $\mathbf{Z}/2$ in its kernel and cokernel.*

*Proof.* The fibration (4.1) induces a short exact sequence in $QK^1(-)$. Indeed, under the description in 3.3, $QK^1(H_{m,\epsilon})$ corresponds to the subspace spanned by the $Y^i$'s, and $QK^1(Sp(m))$ to the quotient mod the $Y^i$'s. There is also a short exact sequence in $QK^1(-)/\operatorname{im}(\psi^2)$ and hence a long exact sequence in $E_2^{*,t}$.

Let $t \equiv 3 \mod 4$, and let $\ell = [\log_2(4m/3)]$. The beginning of this exact sequence is an isomorphism of cyclic groups $E_2^{1,t}(Sp(m)) \to E_2^{1,t}(SU(2m+1))$. This was proved for the UNSS in [5, 1.2], and the same proof works for the BTSS. Since $E_2^{s,t}(SU(2m+1)) = 0$ for $s > 2$,

$$E_2^{s,t}(H_{m,1}) \to E_2^{s+1,t}(Sp(m)) \tag{4.3}$$

is an isomorphism for $s \geq 3$, and by [6, p.26] these groups are $(\ell+1)\mathbf{Z}/2$, i.e. a direct sum of $(\ell+1)$ copies of $\mathbf{Z}/2$. By [4, 3.8], $E_2^{2,t}(H_{m,1})$ is also $(\ell+1)\mathbf{Z}/2$, and so (4.3) is also an isomorphism when $s = 2$. The short exact sequence claimed in part (1) now follows, using $E_2^{1,t}(H_{m,1}) = \mathbf{Z}/2$ by [5], and it is split because $E_2^{2,t}(Sp(m))$ has $(\ell+1)$ summands by [6, p.26,p.76], while $E_2^{2,t}(SU(2m+1))$ has $\ell$ summands by 3.7.

Now let $t \equiv 1 \mod 4$. This time the exact sequence is a bit more complicated, so we write it out.

$$\begin{aligned} 0 \quad &\to \quad E_2^{1,t}(Sp(m)) \to E_2^{1,t}(SU(2m+1)) \to E_2^{1,t}(H_{m,1}) \xrightarrow{\delta} E_2^{2,t}(Sp(m)) \\ &\to \quad E_2^{2,t}(SU(2m+1)) \xrightarrow{p_*} E_2^{2,t}(H_{m,1}) \to E_2^{3,t}(Sp(m)) \to 0. \end{aligned}$$

Using results and methods of [6], we have the following information about the groups in the exact sequence.

$$\begin{aligned} 0 \quad &\to \quad \mathbf{Z}/2 \to \text{Cyclic} \to \text{Cyclic} \xrightarrow{\delta} (\ell+1)\mathbf{Z}/2 \\ &\to \quad \ell \text{ summands} \xrightarrow{p_*} (\ell+1) \text{ summands} \to (\ell+1)\mathbf{Z}/2 \to 0. \end{aligned}$$

We observe that $\delta$ must hit one $\mathbf{Z}/2$. It follows that the two cyclic groups are of the same order with the morphism between them being $\cdot 2$, and the claim about $p_*$ is easily established by exactness properties.

For $SU(2m)$, the exact sequence is more complicated for two reasons. One is that the higher Ext groups of $SU(n)$ are not 0, and the other is that the space $H_{m,0}$ in (4.1) does not match up as nicely with $Sp(m)$ (in terms of Ext summands) as did $H_{m,1}$.

We consider only the case $t \equiv 3 \mod 4$. By [5, 1.2], $E_2^{1,t}(Sp(m)) \to E_2^{1,t}(SU(2m))$ is an isomorphism of cyclic groups. We analyze the resulting exact sequence,

$$0 \to E_2^{1,t}(H_{m,0}) \xrightarrow{\delta_1} E_2^{2,t}(Sp(m)) \to E_2^{2,t}(SU(2m)) \xrightarrow{p_*} E_2^{2,t}(H_{m,0}) \xrightarrow{\delta_2} E_2^{3,t}(Sp(m)) \xrightarrow{i_*} .$$

If $m$ is not of the form $3 \cdot 2^e$, then the sequence is

$$0 \to \mathbf{Z}/2 \xrightarrow{\delta_1} (\ell{+}1) \text{ summands} \to (\ell{+}1) \text{ summands} \xrightarrow{p_*} (\ell{+}1)\mathbf{Z}/2 \to (\ell{+}1)\mathbf{Z}/2 \xrightarrow{i_*}$$

with $i_*$ sending one $\mathbf{Z}/2$ across. This follows from 6.8, [6], and, for $i_*$, a discussion later in this proof. By pushing into the exact sequence of part (1), we find that the image of $\delta_1$ is a split $\mathbf{Z}/2$, from which we can also deduce that the only nontrivial component of $p_*$ is to send a split $\mathbf{Z}/2$ across, from which the claim of part (3) follows (provided $m \neq 3 \cdot 2^e$).

If $m = 3 \cdot 2^e$, $E_2^{2,t}(H_{m,0}) = \ell\mathbf{Z}/2$, but in this case, $i_*$ sends two $\mathbf{Z}/2$'s across, so that there is still a single $\mathbf{Z}/2$ in the image of $p_*$, and the above argument and conclusion apply.

We complete the proof by describing in a bit more detail the relevance of $m = 3 \cdot 2^e$, and the reason for the claim about $i_*$. There is a 6-term exact circle

$$\begin{tikzcd}
\eta_{\mathrm{od}}(Sp(m)) \arrow[r] & \mathbf{Z}_2 \oplus \mathbf{Z}_2 \arrow[r] & \eta_{\mathrm{od}}(H_{m,0}) \arrow[d, "\delta_2"] \\
\eta_{\mathrm{ev}}(H_{m,0}) \arrow[u, "\delta_1"] & \mathbf{Z}_2 \oplus \mathbf{Z}_2 \arrow[l] & \eta_{\mathrm{ev}}(Sp(m)), \arrow[l, "i_*"']
\end{tikzcd} \tag{4.4}$$

in which the groups $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ are the eta-towers of $SU(2m)$. This is the exact sequence in $E_2$ of the fibration (4.1) in filtration $> 2$. The $\eta_i$ notation is as in [6, pp 30-31]; $\eta_i(X) = E_2^{s,2(s+i)+1}(X)$ for $s \geq 3$ and depends only on the parity of $i$.

Using (3.9), $QK^1(H_{m,0}; \mathbf{Z}_2)$ and $QK^1(Sp(m-1); \mathbf{Z}_2)$ are isomorphic abelian groups with agreeing $\psi^2$ and $\psi^3$, but $\psi^{-1}$ is respectively 1 and $-1$. Using [4, 3.10], this implies that $\eta_{\mathrm{od}}(H_{m,0}) \approx \eta_{\mathrm{ev}}(Sp(m-1))$ and $\eta_{\mathrm{ev}}(H_{m,0}) \approx \eta_{\mathrm{od}}(Sp(m-1))$. The significance of $m = 3 \cdot 2^e$ is that for such $m$, $[\log_2(4(m-1)/3)] < [\log_2(4m/3)]$.

Using [6, 3.20] as a guide[4], and referring to [6, 5.6], one can check that, if $m \neq 3 \cdot 2^e$, both $\ker(\delta_i)$ and $\operatorname{coker}(\delta_i)$ have a single $\mathbf{Z}/2$. Here $\delta_i$ refers to the two labeled arrows in (4.4). If $m = 3 \cdot 2^e$, $\ker(\delta_2) = \mathbf{Z}_2$, $\operatorname{coker}(\delta_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$, $\ker(\delta_1) = 0$, and $\operatorname{coker}(\delta_1) = \mathbf{Z}_2$. The classes mapped nontrivially by $i_*$ are exactly $\operatorname{coker}(\delta_2)$. ■

The following corollary is immediate from 4.2 and is part of Theorem 1.4.

**Corollary 4.5.** *If $t \equiv 3 \bmod 4$, $E_2^{2,t}(SU(2m)) \approx E_2^{2,t}(SU(2m+1)) \oplus \mathbf{Z}/2$.*

[4]The listed eta towers of $\mathrm{Spin}(2n+1)$ in [6, 3.20] correspond to most of the eta towers of $Sp(n-1)$.

The relationship between $E_2^{2,t}(SU(2m))$ and $E_2^{2,t}(H_{m,1})$ when $t \equiv 1 \mod 4$ is more complicated and will not be described here.

We remark that the proof of part (2) above established the following result, which was conjectured in [5, 2.2], and was stated there as being equivalent to a certain statement about power series. We mention this primarily as an indication of the power of our new techniques.

**Proposition 4.6.** *If $t \equiv 1 \mod 4$, the cyclic groups $E_2^{1,t}(SU(2m+1))$ and $E_2^{1,t}(H_{m,1})$ are isomorphic.*

Computing $E_2^2(Sp(m))$ and $E_2^2(H_{m,\epsilon})$ is easier than computing $E_2^2(SU(2m+\epsilon))$ because $\psi^{-1}$ is $-1$ or 1, and not mixed. We obtain the following from parts (1) and (3) of Proposition 4.2, together with [6, 11.3]. This proposition underlies the calculation of the tables in Section 2 and proofs in Sections 5 and 7.

**Proposition 4.7.** (1) *Let $k$ be odd. Let $\Psi^2$ be an $m \times m$ matrix whose $j$th column is the coefficients of $Z^1, \cdots, Z^m$ in $(2Z+Z^2)(4Z+Z^2)^{j-1}$. Let $\Theta_k$ be an $m \times m$ matrix whose $j$th column is the coefficients of $Z^1, \cdots, Z^m$ in $(3Z+4Z^2+Z^3)(9Z+6Z^2+Z^3)^{j-1} - 3^k Z^j$. Then $E_2^{2,2k+1}(SU(2m+1))$ is isomorphic to the group $G_{m,k}$ presented by the stackmatrix $\begin{pmatrix} \Psi^2 \\ \Theta_k \end{pmatrix}$, and $E_2^{2,2k+1}(SU(2m)) \approx G_{m,k} \oplus \mathbf{Z}/2$.*

(2) *Let $k$ be even. Let $\Psi^{2\prime}$ be an $m \times m$ matrix whose $j$th column is the coefficients of $Y^1, \dots, Y^m$ in $(4Y+Y^2)^j$. Let $\Theta'_k$ be an $m \times m$ matrix whose $j$th column is the coefficients of $Y^1, \dots, Y^m$ in $(9Y+6Y^2+Y^3)^j$. Then $E_2^{2,2k+1}(SU(2m+1))$ is isomorphic to the group $G'_{m,k}$ presented by the stackmatrix $\begin{pmatrix} \Psi^{2\prime} \\ \Theta'_k \end{pmatrix}$.*

*Proof.* Using the classes in Proposition 3.3, the short exact sequence of unstable Adams modules

$$0 \to QK^1(H_{m,1}) \xrightarrow{p^*} QK^1(SU(2m+1)) \xrightarrow{i^*} QK^1(Sp(m)) \to 0$$

has $\mathrm{im}(p^*)$ the subgroup generated by the $Y^i$, and $\mathrm{coker}(p^*)$ the quotient mod the $Y^i$. Letting $Z^j$ denote $XY^{j-1}$ yields the basis for $QK^1(Sp(m))$ with Adams operations as described by the matrices of part (1).

Part (1) follows from parts (1) and (3) of Proposition 4.2 together with the first part of [6, 11.3] and [6, 5.6], which would yield $E_2^{2,2k+1}(Sp(m)) \approx G_{m,k} \oplus \mathbf{Z}/2$. Part (2) follows similarly, using [6, 11.9] since $\psi^{-1} = 1$, not $-1$, in $QK^1(H_{m,1})$. One may derive that, with $k$ even, $E_2^{2,2k+1}(H_{m,1}) \approx G'_{m,k} \oplus \mathbf{Z}/2$, which with 4.2(2) implies (2). ■

## 5. Summands of order 2 and 4

In this section we prove Theorem 1.2(2). To accomplish this, we show that, for any value of $k$ of the appropriate parity, the group $G_{m,k}$ or $G'_{m,k}$ presented by the matrix $\binom{\Psi^2}{\Theta_k}$ of Proposition 4.7 has summands of order 2 and 4 equal to those of Definition 1.3. We will describe in some detail the proof for the unprimed $G$, so $k$ is odd, and then outline the changes required for the groups $G'$.

Let $m = 2\ell + \epsilon$, with $\epsilon = 0$ or 1. We will postpone proofs of some propositions until later in the section.

**Proposition 5.1.** *Mod 8, $\Psi^2$ has 1 in $(2j, j)$, $2u$ in $(2j-1, j)$ and 0 elsewhere.*

Here, and throughout, $u$ denotes an odd integer (unit), whose value is not important. In this section, our matrices are always considered mod 8.

We pivot the stackmatrix of 4.7(1) on the 1's in $(2j, j)$ for $j = 1, \dots, \ell$. Thus we can remove the first $\ell$ columns of the stackmatrix, and the entire top half of the stackmatrix, except for a row with its only nonzero entry a 2 in column $\ell+1$ if $\epsilon = 1$. This is due to the 2 in $(2\ell+1, \ell+1)$ without a 1 just below it. We remark here that in simplifying a group presented by a matrix, after pivoting on an odd entry, its row and column may be removed without changing the group presented, and rows of 0's may also be removed. Also note that we label rows and columns by their original indices, not their position after various deletions and reorderings.

The matrix $\Theta_k$ has 0's on its diagonal (since $k$ is odd and hence $3^{2j-1} - 3^k \equiv 0$ mod 8), and 0's above. Thus (since the first $\ell$ columns have been deleted), the first $\ell$ rows of the remaining part of $\Theta_k$ are 0, and so we delete them. Remaining are rows and columns $\ell+1$ to $2\ell+\epsilon$, plus the one row from $\Psi^2$ if $\epsilon = 1$. We arrange these rows and columns by increasing 2-exponents and increasing value within a fixed exponent, and partition this matrix into odd (o) and even (e) entries. Thus, not including the extra row at the top if $\epsilon = 1$, the matrix can be denoted by blocks as

$$\begin{pmatrix} A_{\mathrm{oo}} & A_{\mathrm{oe}} \\ A_{\mathrm{eo}} & A_{\mathrm{ee}} \end{pmatrix}. \tag{5.2}$$

In the following proposition, whose proof appears later in the section, we introduce the notation of $\mathbf{e}_i$ for a row whose only nonzero entry is a 1 in (original) column $i$.

**Proposition 5.3.** *For the blocks just described,*

- **a:** *The submatrix $A_{oo}$ is lower triangular with 0's on its diagonal and units on its subdiagonal.*
- **b:** *The submatrix $A_{eo}$ contains just 0's and 4's. In the last column it is 0 except in row $2\ell$ if $m = 2\ell$ with $\ell$ odd.*
- **c:** *All entries of $A_{oe}$ are even. Its first row is 0 if $\ell$ is even, and is $2u\mathbf{e}_{\ell+1}$ if $\ell$ is odd.*
- **d:** *The submatrix $A_{ee}$ equals the entire matrix for $m = \ell$, with all row and column indices doubled.*

Starting from the top, pivot on the units on the subdiagonal of $A_{\mathrm{oo}}$. This can introduce new 4's in $A_{\mathrm{eo}}$, but they eventually get pivoted away, as we pivot on all odd columns except the last. The pivoting does not change $A_{\mathrm{ee}}$, since it is changed by 4 times a row with even entries (by (c)). Deleting rows and columns after each pivoting step, we will have removed all the odd columns except the last, and all the odd rows except the first. What remains is (by (d)) the matrix $\Theta_k$ for $\ell$, with (by (c)) a row $2u\mathbf{e}_{\ell+1}$ adjoined if $\ell$ is odd, and, if $m = 2\ell$ with $\ell$ odd, a column with a single 4 in row $2\ell$. If $m > 6$, this 4 will get pivoted away in the pivoting on $A_{\mathrm{ee}}$. The reason for the requirement $m > 6$ is so that $m - 2^{\nu(m)+1} > m/2$, for then position $(m, m - 2^{\nu(m)+1})$ has a pivot unit in row $m$. The extra row equals the row that had to be adjoined to $\Theta_k$ for $m = \ell$ when $\ell$ is odd, due to $\Psi^2$.

Our desired group $L_m$ is the summands of order 2 and 4 in the group presented by (5.2) with the extra row if $m$ is odd. Let $\widetilde{L}_m$ denote the summands of order 2 and 4 in the group presented by just (5.2). Thus $\widetilde{L}_m = L_m$ if $m$ is even. The previous paragraph implies that if $m > 6$, then

$$\widetilde{L}_m \approx L_{[m/2]}. \tag{5.4}$$

Theorem 1.2(2) with $k$ odd follows immediately from (5.4) and the following two propositions, which will be proved below.

**Proposition 5.5.** *In $m>6$ and $m\neq 9$, then $\widetilde{L}_m \approx L_m$.*

**Proposition 5.6.** *$L_3 = L_1 = \mathbf{Z}/2$; $L_4 = L_2 = 0$; $L_9 = L_5 = \mathbf{Z}/4$; $L_6 = \mathbf{Z}/2 \oplus \mathbf{Z}/4$.*

Now we give the postponed proofs.

*Proof of Proposition 5.1.* Immediate from, mod 8,

$\psi^2(Z^j) = Z^j(2+Z)(4+Z)^{j-1} \equiv Z^{2j}+(2+4(j-1))Z^{2j-1} \equiv Z^{2j}+2uZ^{2j-1}.$

■

*Proof of Proposition 5.3.* **a:** The triangularity and diagonal are clear. The subdiagonal entry in column $j$ is the coefficient of $Z^2$ in $(3+4Z+Z^2)(1-Z)^{2(j-1)}$. Mod 2, this is $1+\binom{2(j-1)}{2}$, which is odd since $j$ is odd.

**b:** Column $2t+1$ of $A_{\text{eo}}$ contains the odd-power terms of $(3+4Z+Z^2)(1-Z)^{4t}$. Mod 4, this is $(3+Z^2)(1+2Z^2+Z^4)^t$, which has odd-power terms 0. The only possible nonzero entry in the last odd column occurs if $m=2\ell$, and is $4\ell$ mod 8 in $(2\ell, 2\ell-1)$.

**c:** We need the following lemma, which can be proved by induction on $j$.

**Lemma 5.7.** *Define polynomials by*

$$\begin{aligned} f_j(Z) &= (3+4Z+Z^2)(1-Z)^{2(2j-1)} \\ g_j(Z) &= (3+4Z^2+Z^4)(1-Z^2)^{2(j-1)}. \end{aligned}$$

*Then there is a polynomial $\phi_j$ such that, mod 8,*

$$f_j(Z) - g_j(Z) \equiv 2Z\phi_j(Z^2).$$

In column $2j$, the entries of $A_{\text{oe}}$ are the coefficients of odd powers of $Z$ in $f_j(Z)$. These are even, by the lemma.

If $\ell$ is odd, the first odd row has original index $\ell+2$. The only possible nonzero entry in this row is in original column $\ell+1$, and this is easily seen to be 2 mod 4.

**d:** The entry in position $(2t, 2j)$ of $A_{\text{ee}}$ is, mod 8, using Lemma 5.7,

$\operatorname{coef}(f_j(Z), Z^{2t-2j}) \equiv \operatorname{coef}(g_j(Z), Z^{2t-2j}) = \operatorname{coef}((3+4Z+Z^2)(1-Z)^{2(j-1)}, Z^{t-j}),$

which equals the entry in row $t$ and column $j$, which is part of the matrix corresponding to $\ell$. ■

*Proof of Proposition 5.5.* We will show that, if $m > 6$ and $m \neq 9$, the extra row $2\mathbf{e}_{\ell+1}$ in $L_m$ but not $\widetilde{L}_m$ is actually a linear combination of rows already in $\widetilde{L}_m$, and hence does not change the group presented. With $R_i$ denoting row $i$, this is accomplished by showing:

- If $\ell$ is odd, then $2\mathbf{e}_{\ell+1} = R_{\ell+2}$;
- If $\ell \equiv 2 \bmod 4$, then $2\mathbf{e}_{\ell+1} = 2R_{\ell+3} + 2R_{\ell+4}$;
- If $\ell \equiv 0 \bmod 8$, then $2\mathbf{e}_{\ell+1} = 2R_{\ell+3} + R_{\ell+4}$; and
- If $\ell \equiv 4 \bmod 8$, then $2\mathbf{e}_{\ell+1} = 2R_{\ell+3} + 4R_{\ell+6}$.

The requirement that $m > 6$ and $m \neq 9$ is due to the fact that if $\ell = 2$ the rows $R_{\ell+3}$ and $R_{\ell+4}$ are out of range, and similarly for $R_{\ell+6}$ if $\ell = 4$.

The proof of the four bulleted items involves analyzing binomial coefficients to give explicit formulas for the listed rows $R_i$. ■

*Proof of Proposition 5.6.* For $m = 1, 2, 3, 4, 5, 6$, and 9, the $\Theta_k$ matrices, augmented by a top row from the $\Psi$-matrix if $m$ is odd, are given in (5.8). The ordering of the rows and columns is:

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 9 |
|---|---|---|---|---|---|---|---|
| order | 1 | 2 | 3,2 | 3,4 | 3,5,4 | 5,6,4 | 5,7,9,6,8 |

The matrices for the 7 listed values of $m$ are

$$\begin{pmatrix} 2 \\ \hline 0 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ \hline 0 & 6 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ \hline 0 & 0 & 0 \\ u & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 \\ 4 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 6 & 0 \\ 6 & 7 & 0 & 2 & 2 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.8}$$

■

A very similar proof works when $k$ is even. The required changes in the argument are listed below.

(1) We use the matrix in Proposition 4.7(2).

(2) In Proposition 5.1, $\Psi^2$ mod 8 has 1 in $(2j, j)$, 4 in $(4j-3, 2j-1)$, and 0's elsewhere.

(3) After the pivoting which follows 5.1, we are left with the right half of $\Theta_k$ adorned with, if $m \equiv 2$ or 3 mod 8, an extra row $4\mathbf{e}_{\ell+1}$.

(4) Parts (a) and (d) of Proposition 5.3 are unchanged, while the new (b) says that $A_{\text{eo}}$ is even, and its last column is 0 unless $\epsilon = 0$, in which case it has $2u$ in row $2\ell$. The new (c) says that $A_{\text{oe}}$ has all 0's and 4's, and its first row is 0 unless $m \equiv 2, 3$ mod 8, in which case it is $4\mathbf{e}_{\ell+1}$.

(5) Propositions 5.5 and 5.6 are modified as in the primed portion of Definition 1.3.

(6) Lemma 5.7, used in the proof of 5.3, becomes: if $f_j(y) = y^{2j}((1-y)^{4j} - 1)$ and $g_j(y) = y^{2j}((1-y^2)^{2j} - 1)$, then $f_j(y) - g_j(y) \equiv 4y\phi_j(y^2)$ mod 8.

(7) The pivoting which follows 5.3 reduces from the matrix for $m = 2\ell+\epsilon$ to the matrix for $\ell$ if $m > 6$ and $m \neq 10$. Indeed, if $m > 6$ and $m \neq 10$, the extra row and column described above can be written as linear combinations of other rows or columns of the matrix. For example, if $m \equiv 2$ mod 8, with $C_j$ denoting column $j$, $C_{m-1} = 2C_{m-4} + 2C_{m-6}$, provided $m > 10$, but when $m = 10$, $C_{m-6}$ does not appear in the matrix.

(8) For $m = 4, 6$, and 10, the right half of the $\Theta_k$ matrices, mod 8, are as below, with rows and columns 3,4 for $m = 4$, 5,6,4 for $m = 6$, and 7,9,6,10,8 for $m = 10$. The extra row $4\mathbf{e}_{(m/2)+1}$ when $m = 10$ is not listed, as it equals row 7.

$$\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 6 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 4 & 0 & 0 \\ 3 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 6 & 7 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 \end{pmatrix}$$

## 6. Results and proofs for $SU(2m)$

In this section, we prove results about $v_1^{-1}\pi_*(SU(2m))$ which were stated or alluded to in Section 1.

Much of the proof of Theorem 1.4 was given in [2]. It involves the exact sequence in $E_2$ of the fibration

$$SU(2m-1) \to SU(2m) \to S^{4m-1}$$

and knowledge of the BTSS for $S^{4m-1}$, as given, for example, in [2, p.488]. Each group $E_2^{s,t}(SU(2m))$ with $t$ odd and $s > 2$ maps isomorphically to the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ in $S^{4m-1}$, and the $d_3$-differentials involving most of these elements correspond under this isomorphism. They are pictured in [2, p.488]. In our diagrams, we have not pictured most of the elements which are involved in nontrivial differentials, since they do not survive to homotopy classes, and their inclusion leads to a more cluttered diagram. For example, each group $E_2^{2,2k+1}(SU(2m))$ is $G_k \oplus \mathbf{Z}_2$, but in those cases in which the $\mathbf{Z}_2$ supports a nonzero differential, we do not include it in our charts. We record this result now.

**Proposition 6.1.** *There is an isomorphism*

$$E_2^{2,2k+1}(SU(2m)) \approx G_k \oplus \mathbf{Z}_2,$$

*where $G_k$ is a group of order $2^{e(k,2m)}$.*

*Proof.* If $k$ is odd, the result is immediate from 4.5 and 1.2.

Let $k$ be even. The exact sequence

$$\begin{aligned} 0 \quad \to \quad & E_2^{1,2k+1}(SU(2m-1)) \to E_2^{1,2k+1}(SU(2m)) \to E_2^{1,2k+1}(S^{4m-1}) \\ \to \quad & E_2^{2,2k+1}(SU(2m-1)) \to E_2^{2,2k+1}(SU(2m)) \to E_2^{2,2k+1}(S^{4m-1}) \to 0 \end{aligned}$$

has alternating sum of 2-exponents of orders of groups equal to 0, and the exponents of orders of the nonzero groups are, respectively,

$$e(k,2m-1), e(k,2m), 1, e(k,2m-1), \nu(|E_2^{2,2k+1}(SU(2m))|), 2.$$

This implies the claim about the order of $E_2^{2,2k+1}(SU(2m))$. That at least one of the summands is a $\mathbf{Z}_2$ follows from the exact sequence together with the fact (6.8 and 3.7) that

$$\mathrm{rk}_2(E_2^{2,2k+1}(SU(2m))) > \mathrm{rk}_2(E_2^{2,2k+1}(SU(2m-1))),$$

and $E_2^{2,2k+1}(S^{4m-1}) \approx \mathbf{Z}_2 \oplus \mathbf{Z}_2$. ■

Next we record part of the portion of Theorem 1.4 which deals with the exotic extensions in the spectral sequence. This refers to situations in which 2 times the element of order 2 in $E_\infty^{s,t}(SU(2m))$ (with $s = 1$ or 2) equals in $v_1^{-1}\pi_{t-s}(SU(2m))$ a nonzero element in $E_\infty^{s+2,t+2}(SU(2m))$.

**Proposition 6.2.** *The extension in the spectral sequence of $SU(2m)$ from $E_\infty^{s,2k+1}(SU(2m))$*

(1) *is trivial if $s = 1$;*

(2) *is trivial in $t - s = 8a - 1$ when $m$ is even;*

(3) *is nontrivial from the split[5] $\mathbf{Z}/2$ when $s = 2$ and $k = 4a - 1$ with $m$ odd or $k = 4a + 1$ with $m$ even, provided the target class is not hit by a $d_3$-differential. (See 6.5 and 6.7.);*

(4) *is nontrivial if $s = 2$ and either $k = 4a$ and $m$ odd or $k = 4a + 2$ and $m$ even, provided the target class is not hit by a $d_3$-differential. (See 6.5 and 6.7.)*

*Proof.* (1). There can be no extension from the 1-line because $E_2^{1,t}(SU(2m-1)) \to E_2^{1,t}(SU(2m))$ is injective but there is no possible extension in the spectral sequence for $SU(2m-1)$.

(2). The filtration-4 class maps to an element in $v_1^{-1}\pi_{8a-1}(S^{4m-1})$ which is not divisible by 2 (by [2, p.488]).

(3). Let $w \in E_\infty^{2,2k+1}(SU(2m))$ denote the split $\mathbf{Z}/2$, $y \in E_\infty^{4,2k+3}(SU(2m))$ be the putative target of the extension, and $z \in E_\infty^{2,2k-1}(SU(2m))$ satisfy $z\eta^2 = y$. Let $M$ denote the mod-2 Moore spectrum with bottom cell in dimension $2k - 3$, and map $M$ to $\Phi SU(2m))$ by extending $z$ (note that $2z = 0$ in $\pi_*(\Phi SU(2m))$). The $K$-based spectral sequence for $\pi_*(M)$ has elements $z'$, $y'$, and $w'$ in $(s,t) = (0, 2k-3)$, $(2, 2k+1)$, and $(0, 2k-1)$, respectively, satisfying $z'\eta^2 = y'$ and $2w' = y'$ in $\pi_*(M)$. Our map sends $z' \mapsto z$, $y' \mapsto y$, and $w' \mapsto w$. This can be seen by following into $S^{4m-1}$. The asserted extension in $SU(2m)$ follows by naturality.

[5] i.e., one which is not part of $G_k$

(4). By 6.5, the assumption that the target class is not hit by $d_3$ implies that there is an exact sequence

$$0 \to E_2^{1,2k+1}(S^{4m-1}) \to E_2^{2,2k+1}(SU(2m-1)) \to E_2^{2,2k+1}(SU(2m)) \to E_2^{2,2k+1}(S^{4m-1}) \to 0,$$

in which the groups are

$$0 \to \mathbf{Z}_2 \to G_k \to G'_k \oplus \mathbf{Z}_2 \to \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to 0,$$

in which the second $\mathbf{Z}_2$'s correspond, and $|G_k| = |G'_k|$. Thus[6] $G_k \to G'_k$ is

$$\mathbf{Z}/2^{e+1} \oplus \mathbf{Z}/2^f \oplus S \xrightarrow{(1,2,1)} \mathbf{Z}/2^e \oplus \mathbf{Z}/2^{f+1} \oplus S,$$

where $S$ denotes the other summands.[7] In order that the fibration yield an exact sequence of homotopy groups, it must be the case that there is an extension from the $\mathbf{Z}/2^e$ summand into the filtration-4 class whose image in $S^{4m-1}$ is hit by the differential from $E_2^{1,2k+1}(S^{4m-1})$. ■

The $d_3$-differentials from the 1-line were determined in [2, 1.1] using the following elementary result.

**Proposition 6.3.** *The differential*

$$d_3 : E_3^{1,2k+1}(X) \to E_3^{4,2k+3}(X)$$

*is nonzero for $X = SU(2m)$ if and only if it is nonzero for $X = S^{4m-1}$ and*

$$E_2^{1,2k+1}(SU(2m)) \to E_2^{1,2k+1}(S^{4m-1}) \tag{6.4}$$

*is surjective.*

*Proof.* This is immediate from the fact that $E_2^4(SU(2m)) \to E_2^4(S^{4m-1})$ is bijective. ■

The condition for surjectivity of (6.4) is obtained in [2, 1.1] from the exact sequence

$$0 \to E_2^{1,2k+1}(SU(2m-1)) \to E_2^{1,2k+1}(SU(2m)) \to E_2^{1,2k+1}(S^{4m-1})$$

as

[6] It can be shown that the $\mathbf{Z}_2$'s in the kernel and cokernel of $G_k \to G'_k$ are not split by an analysis similar to the proof of 4.2.

[7] It is possible that the kernel and cokernel elements of $G_k \to G'_k$ occur in the same summand, but this does not affect our conclusions.

**Corollary 6.5.** ([2, 1.1]) *The differential*

$$d_3 : E_3^{1,2k+1}(SU(2m)) \to E_3^{4,2k+3}(SU(2m))$$

*is nonzero if and only if one of the following conditions holds:*

- $k - 2m \equiv 1 \ mod\ 4$ *and* $e(k, 2m) = e(k, 2m-1) + 3$*;*
- $k - 2m \equiv 2 \ mod\ 4$ *and* $e(k, 2m) = e(k, 2m-1) + 1$*;*
- $k \equiv 1 \ mod\ 4$, $\nu(k-2m+1) \geq 2m-3$, *and* $e(k, 2m) = e(k, 2m-1) + 2m - 1$.

The following results shed some light on these rather intractable conditions. The conjecture will be discussed in Section 9, while the theorem will be proved at the end of this section.

**Conjecture 6.6.** *The third condition of Corollary 6.5 is satisfied if and only if* $m = 3$ *and* $k \equiv 13 \ mod\ 32$.

**Theorem 6.7.** *The first condition of Corollary 6.5 is never satisfied if* $k = 3,\ 5,\ 7,\ 9,$ *or* $13$. *When* $k = 11$*, it is satisfied if and only if* $m \equiv 71 \ mod\ 512$. *For* $k = 2, 4, 6, 8, 10$*, the second condition of Corollary 6.5 is satisfied (and hence* $d_3 \neq 0$ *on* $E_3^{1,2k+1}(SU(2m))$*) in exactly the following cases:*

$$\begin{cases} k = 2 & m \equiv 2\ (4) \\ k = 4 & m \text{ odd, } m \not\equiv 11\ (16) \\ k = 6 & m \equiv 2\ (4) \text{ or } 4\ (16) \\ k = 8 & m \text{ odd, } m \not\equiv 23\ (32) \text{ or } 69\ (128) \\ k = 10 & m \equiv 2\ (4) \text{ and } \not\equiv 198\ (256) \end{cases}$$

For fixed $n$, $e(k, n)$ is periodic in $k$, with period that increases with $n$. Thus Theorem 6.7 gives information about differentials for some larger values of $k$, too. However, the impact of the theorem is primarily qualitative; it suggests that nonzero differentials are rare when $k$ is odd, but rather frequent when $k$ is even.

The following result has been referred to several times.

**Proposition 6.8.** *If* $m$ *is a positive integer and* $k$ *is any integer, then*

$$\mathrm{rk}_2(E_2^{2,2k+1}(SU(2m))) = 1 + [\log_2(4m+3)].$$

*Proof.* The argument is like the proof of Proposition 3.7 with the modifications described below.

We now have $N = QK^1(SU(2m))$ determining the matrix $M$. We still have (3.8), and now rank$(M) = 4m - 2$ for the same reason. The 2-by-2 blocks in (3.10) now represent $m$ rows or columns in the first half and $(m-1)$ rows or columns in the second half. The "identity" matrices $I$ now have a column of 0's at the end, and the second $\Psi_1^2$ and $\Theta_1$ are restrictions of the first to the first $(m-1)$ rows and columns. After the first round of pivoting, a matrix which looks exactly like (3.11) is obtained. The third and fifth 0 in the first row could actually have as their last column the last column of $\Psi_1^2$ and $\Theta_1$, respectively, but these are 0. When we delete the rows with the $I$'s, we are now deleting just $(3m-3)$ rows. The $3m \times m$ matrix which remains is exactly as in the proof of 3.7, and so has rank $m - [\log_2(4m/3)]$. The final answer for $\mathrm{rk}_2(E_2^{2,2k+1}(SU(2m))$ is

$$4m - 2 - ((3m-3) + (m - [\log_2(4m/3)])).$$

■

*Proof of Theorem 6.7.* We give the proof when $k = 10$. The proof for other values of $k$ is performed similarly. The condition which must be satisfied is $2m \equiv 0 \bmod 4$, and $\nu(a(10, 2m-1))$ is strictly less than $\nu(a(10, j))$ for all $j \geq 2m$.

Let $S(n, j)$ denote the Stirling number of the second kind, and let $j_t = j!/(j-t)!$, a product of $t$ consecutive integers. We have

$$\begin{aligned} \nu(a(10,j)) &= \sum_{i \text{ odd}} \binom{j}{i} i^{10} = \sum_{i \text{ odd}} \binom{j}{i} \sum_{t=0}^{10} S(10,t) i_t \\ &= \sum_{t=0}^{10} S(10,t) j_t \sum_{i \text{ odd}} \binom{j-t}{i-t} \\ &= \sum_{t=0}^{10} S(10,t) j_t 2^{j-t-1} = 2^{j-11} p(j), \end{aligned} \tag{6.9}$$

where $p(j) = \sum_{t=0}^{10} S(10,t) j_t 2^{10-t}$, a 10th degree polynomial in $j$ with integer coefficients. With hindsight, we use `Maple` to compute $p(8b + \Delta)$ for $395 \leq \Delta \leq 402$, a 10th degree polynomial in $b$. We obtain

$$p(8b + \Delta) = \sum_{i=0}^{10} u_i 2^{e_i} b^i, \tag{6.10}$$

with $u_i$ odd, $e_i > e_2$ if $i > 2$, and $e_0$, $e_1$, and $e_2$ given in Table 7.

TABLE 7. Exponents in (6.10)

| $\Delta$ | $e_0$ | $e_1$ | $e_2$ |
|---|---|---|---|
| 395 | 15 | 9 | 11 |
| 396 | 17 | 9 | 11 |
| 397 | 13 | 8 | 9 |
| 398 | 13 | 8 | 9 |
| 399 | 13 | 11 | 10 |
| 400 | 13 | 11 | 10 |
| 401 | 17 | 8 | 9 |
| 402 | 10 | 8 | 9 |

Since all mod 8 values have been included, this implies that $\nu(p(j)) \geq 8$ for all $j$. Using (6.9), we obtain that $\nu(a(10, 8b + \Delta)) - 8b - 384$ is given in Table 8.

TABLE 8. Values of $\nu(a(10, 8b + \Delta)) - 8b - 384$

| | $\nu(b)$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\Delta$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | $\geq 7$ |
| 395 | 9 | 10 | 11 | 12 | 13 | 14 | $\geq 16$ | 15 |
| 396 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | $\geq 17$ |
| 397 | 10 | 11 | 12 | 13 | 14 | $\geq 16$ | 15 | 15 |
| 398 | 11 | 12 | 13 | 14 | 15 | $\geq 17$ | 16 | 16 |
| 399 | 14 | $\geq 17$ | $\geq 18$ | 17 | 17 | 17 | 17 | 17 |
| 400 | 15 | $\geq 18$ | $\geq 19$ | 18 | 18 | 18 | 18 | 18 |
| 401 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | $\geq 21$ |
| 402 | 15 | 16 | $\geq 18$ | 17 | 17 | 17 | 17 | 17 |
| $> 402$ | $\geq 16$ | $\geq 16$ | $\geq 16$ | $\geq 16$ | $\geq 16$ | $\geq 16$ | $\geq 16$ | $\geq 16$ |

If $2m \equiv 0 \bmod 8$, then $2m - 1$ can be written as $8b + 399$, and a comparison of the 399 row of Table 8 with rows 401 and 402 shows that $\nu(a(10, 2m - 1))$ is not strictly less than both $\nu(a(10, 2m + 1))$ and $\nu(a(10, 2m + 2))$ in this case. Thus the condition of Theorem 6.7 is not satisfied when $k = 10$ and $2m \equiv 0 \bmod 8$.

If $2m \equiv 4 \bmod 8$, then $2m - 1$ can be written as $8b + 395$. Table 8 shows that if $\nu(b) < 6$, then $\nu(a(10, 8b + 395))$ is strictly less than $\nu(a(10, j))$ for all $j > 8b + 395$, while this is not the case if $\nu(b) \geq 6$, establishing the claim of the theorem when $k = 10$. ■

## 7. More differentials and extensions in $SU(2m)$

In this section, we prove some of the more difficult differentials and extensions in the spectral sequence for $SU(2m)$ which are part of Theorem 1.4.

**Theorem 7.1.** *In the spectral sequence for $SU(2m)$ pictured in Diagrams 1.5 and 1.6, the extension from $G_{4a+1}$ is nontrivial from the $\mathbf{Z}/8$ summand if $m = 2^e$ and from the $\mathbf{Z}/4$ summand if $m = 3 \cdot 2^e$, $e \geq 1$, and the differential from $G_{4a+1}$ does not emanate from a $\mathbf{Z}/2$ or $\mathbf{Z}/4$ summand. The differential is nontrivial from $\mathbf{Z}/8$ if $m = 2^e$.*

*Proof.* The non-differential from $\mathbf{Z}/2$ or $\mathbf{Z}/4$ follows easily from the fact ([2, p.488]) that on $E_2^{2,8a+3}(S^{4m-1}) \approx \mathbf{Z}/2^\nu \oplus \mathbf{Z}_2$ with $\nu \geq 3$, the nonzero $d_3$ arises from the larger summand. Let $p : SU(2m) \to S^{4m-1}$ and $\alpha \in E_2^{2,8a+3}(SU(2m))$. If $d_3(\alpha) \neq 0$, then, since $p_*$ is bijective in filtration $> 2$, we have $d_3(p_*(\alpha)) \neq 0$. Therefore $4p_*(\alpha) \neq 0$ and hence $4\alpha \neq 0$.

When $m = 2^e$ or $3 \cdot 2^e$, $e \geq 1$, we will compute part of the morphism

$$G_{4a+1} \oplus \mathbf{Z}_2 = E_2^{2,8a+3}(SU(2m)) \xrightarrow{p_*} E_2^{2,8a+3}(S^{4m-1}) = \mathbf{Z}/2^3 \oplus \mathbf{Z}_2. \tag{7.2}$$

The split $\mathbf{Z}_2$'s correspond under $p_*$. The filtration-4 class of the putative extension in $SU(2m)$ maps nontrivially to the class in $S^{4m-1}$ into which the $\mathbf{Z}/2^3$ extends. We will show that[8] when $m = 3 \cdot 2^e$, $G_{4a+1} = \mathbf{Z}_2 \oplus \mathbf{Z}_4 \oplus H$, where $H$ (higher) denotes summands of larger order, and (7.2) sends the $\mathbf{Z}_2$ summand trivially and the $\mathbf{Z}_4$ summand injectively. By naturality, this implies that the extension in $SU(2m)$ is from the $\mathbf{Z}_4$ summand. Similarly, we will show that when $m = 2^e$, the smallest summand of $G_{4a+1}$ is a $\mathbf{Z}/8$ and it maps isomorphically under (7.2), implying the extension from it, and the differential.

By Proposition 4.2, the commutative diagram

$$\begin{array}{ccc}
E_2^{2,8a+3}(Sp(m)) & \longrightarrow & E_2^{2,8a+3}(S^{4m-1}) \\
\big\downarrow & & \big\downarrow{\scriptstyle =} \\
E_2^{2,8a+3}(SU(2m)) & \longrightarrow & E_2^{2,8a+3}(S^{4m-1})
\end{array}$$

[8]Actually, we already showed the part about summands in 1.2 and 4.5.

shows that the morphism $G_{4a+1} \to \mathbf{Z}/8$ can be computed using Proposition 4.7(1). Our algorithm for computing $G_{4a+1}$ from the matrix of 4.7(1) is

- Find the least 2-divisible entry in the matrix. If there are several, choose the earliest one in the earliest column.
- Pivot on that entry and remove its row and column.
- Repeat.
- If the pivot entry is $2^e \cdot u$ with $u$ odd and $e \geq 1$, then a $\mathbf{Z}/2^e$ summand is obtained with generator $2^{-e}$ times the combination of generators with coefficients the entries of the row removed.

The image in the $\mathbf{Z}/8$ summand of $E_2^{2,8a+3}(S^{4m-1})$ of the generator just described is its entry $\alpha_m$ in the last column (the one with original label $m$). The $\mathbf{Z}/2^e$ summand maps injectively to $E_2^{2,8a+3}(S^{4m-1})$, and hence supports a nontrivial extension, if and only if $\nu(\alpha_m) + e = \nu(E_2^{2,8a+3}(S^{4m-1})) = 3$.

Theorem 7.1 follows from the following description of the result of the pivoting. ■

**Lemma 7.3.** *The result of the pivoting described above on the matrix of 4.7(1) with $k = 4a + 1$ is:*

- *If $m = 2^e$, after pivoting on all odd entries*[9]*, the obtained matrix has all entries divisible by 8, and has a row with $8u$ in the first column $(3 \cdot 2^{e-2})$ and $8u'$ in the last column $(m)$. Here and throughout, $u$ and its variants denote odd integers.*
- *Let $m = 3 \cdot 2^e$.*
  - *Pivoting on all odd entries removes all columns except $m - 2^e,\ m - 2^{e-1},\ \ldots,\ m - 2^0,\ m$.*
  - *There will be a row with a 2 in column $m - 2^e$ and a highly 2-divisible number in column $m$.*
  - *After pivoting on the 2 just described, all remaining entries are divisible by 4, and there is a row with $4u$ in column $m - 2^{e-1}$ and $8u'$ in column $m$.*

[9]and removing their row and column

*Proof.* We consider the mod 16 reduction of the matrix of 4.7(1) with $k = 4a+1$ and $m = 2^e$ or $3 \cdot 2^e$. In the rest of this proof, we are always working mod 16. Since

$$(2x + x^2)(4x + x^2)^{j-1} \equiv x^{2j} + 2ux^{2j-1} + 8(j-1)x^{2j-2},$$

where $u$, as always, denotes an odd number, the only nonzero entries in the $\Psi^2$-part are 1 in $(2j, j)$, $2u$ in $(2j-1, j)$, and $8(j-1)$ in $(2j-2, j)$. Pivoting on the 1's in $(2j, j)$, and removing pivot rows and columns and also rows of 0's leaves just the last $m/2$ rows and columns of the $\Theta_k$ part of the matrix, and these are not changed by this initial pivoting.

Let $M_{i,j}$, $m/2 < i, j \le m$, denote the entries of the remaining matrix $M$. We will prove the following about these entries.

**Proposition 7.4.** *If $M = \Theta_k$ of 4.7(1) with $k \equiv 1$ (4), then*

(1) *If $i < j$, then $M_{i,j} \equiv 0$ (16).*

(2) $M_{i,i} \equiv \begin{cases} 0\ (8) & \text{if } i \text{ odd} \\ 8\ (16) & \text{if } i \text{ even.} \end{cases}$

(3) *$M_{j+2^{\nu(j)+1},j}$ is odd.*

(4) *If $\nu(i) \ne \nu(j)$, then $M_{i,j}$ is even.*

(5) *If $\nu(i) > \nu(j)$, then $M_{i,j} \equiv 0$ (4).*

(6) *If $\nu(i) > \nu(j)+1$, then $M_{i,j} \equiv 0$ (8).*

(7) *If $e \ge t+3$ and $t \ge 0$, then $M_{2^e,2^e-2^t} \equiv 0$ (16) and $M_{3\cdot 2^e,3\cdot 2^e-2^t} \equiv 0$ (16).*

(8) *$M_{7\cdot 2^t,6\cdot 2^t} \equiv 2$ (4); $M_{6\cdot 2^t,5\cdot 2^t} \equiv 4$ (8); $M_{4\cdot 2^t,3\cdot 2^t} \equiv 8$ (16); $M_{5\cdot 2^t,4\cdot 2^t} \equiv 2$ (4).*

Arrange the rows and columns of $M$ by increasing 2-powers, and increasing value within a fixed 2-power. We pivot on the units in $(j+2, j)$ for odd $j$ in increasing order, removing the pivot row and column each time. We prove below that this pivoting preserves properties (1)-(8) of the rest of the matrix. Row $m/2+1$ and column $m-1$, which are not pivot rows or columns, will be 0 mod 16 after the pivoting, and so can be removed without changing $Q(M)$. This leaves the matrix with just even row and column indices from $m/2+2$ to $m$, and it satisfies all the properties (1)-(8) like the entire matrix for $m/2$, with indices doubled. Thus, if $m = 2^e$, by induction, the summands of order less than 16 in $Q(M)$ are the same as those of $Q(M_8)$, where $M_8$ is the mod 16 matrix in Table 9, which is the matrix which satisfies 7.4 when $m = 8$.

TABLE 9. Mod 16 matrix at end of induction, $m = 2^e$

| | 5 | 7 | 6 | 8 |
|---|---|---|---|---|
| 5 | $8a$ | 0 | 0 | 0 |
| 7 | $u$ | $8b$ | $2u'$ | 0 |
| 6 | 4 | 0 | 8 | 0 |
| 8 | $8c$ | 0 | 8 | 8 |

Here $a, b, c = 0$ or 1, and $u$ and $u'$ are odd. The specific values in (7,6), (6,5), and (8,6), respectively, are due to 7.4(8). Our proof below of the effects of pivoting shows that these specific congruences will not have changed during the pivoting. Pivoting on the $u$ in Table 9 reduces the matrix to one whose only nonzero elements are 8 in (8,6) and (8,8). This implies the case $m = 2^e$ of Lemma 7.3.

If $m = 3 \cdot 2^e$, the induction reduces to the case $m = 12$, still satisfying 7.4. After pivoting on the units in $(9, 7)$ and $(11, 9)$ and removing their rows and columns, and the 0-row 7, we obtain the matrix in Table 10, where $a$, $b$, and $c$ are integers, and $u$ and $u'$ are odd integers. This clearly leads to the claimed result.

TABLE 10. Mod 16 matrix at end of induction, $m = 3 \cdot 2^e$

| | 11 | 10 | 12 | 8 |
|---|---|---|---|---|
| 10 | 0 | $8a$ | 0 | $2u$ |
| 12 | $8b$ | $4u'$ | 8 | $2c$ |
| 8 | 0 | 0 | 0 | 8 |

Now we explain why pivoting does not change properties (1)-(8). Refer to Table 11. The only way that pivoting on a unit in $(t + 2, t)$, with $t$ odd, can change a congruence of $c := M_{i,j}$ is if, up to odd multiples, $M_{i,t} = 2^a$ and $M_{t+2,j} = 2^b$, both nonzero mod 16, and $c - 2^{a+b}/u$ no longer satisfies the congruence. Since $u$ is the only odd entry left in its row, we have $b \geq 1$, and we must have $j < t + 2$ by 7.4(1).

**Case 1:** If $i$ is odd, then $i > t + 2$, so $i > j$ and $\nu(j) > 0 = \nu(i)$, so the property of $c$ is that it is even, and this is not changed by this pivoting, since $b \geq 1$.

Table 11. Portion of matrix which can change congruence condition

$$\begin{array}{c|cc|} & t & j \\ \cline{2-3} t+2 & u & 2^b \\ i & 2^a & c \\ \cline{2-3} \end{array}$$

**Case 2:** If $i \equiv 2 \bmod 4$, then $a \geq 2$ by 7.4(5), and since $a < 4$, we must have $i > t$. It is possible that $i = j = t+1$. Then, since $a \geq 2$ and $b \geq 1$, $c$ is changed by 8, if at all. A change of 8 in $c$ is consistent with property (2) because the index $i$ will be divided by 2, and hence become odd, when we get to the next matrix to consider. Otherwise, $j < i$, for which there is no condition for $c$ if $\nu(i) = \nu(j)$, while if $\nu(j) > \nu(i)$, the condition is $c$ even, which is unaffected by the pivoting.

**Case 3:** If $\nu(i) > 1$, then $a \geq 3$, and since $b \geq 1$, $c$ will not be changed. ■

*Proof of Proposition 7.4.* Let $q_j(x) = (3+4x+x^2)(9+6x+x^2)^{j-1}$. Then $M_{i,j}$ is the coefficient of $x^{i-j}$ in $q_j(x)$ if $i > j$, while $M_{i,i} = 3^{2i-1} - 3$. We prove each of the 8 parts.

(1) Clearly, $M_{i,j} = 0$ if $i < j$. The proposition just states this to be true mod 16, because that is all that is maintained during pivoting, and all that is needed.

(2) Immediate from $\nu(3^{2i-1} - 3) = \nu(2i-2) + 2$.

(3) If $j = 2^e u$ with $u$ odd, then, mod 2, $q_j(x) \equiv (1+x^2)^j \equiv (1+x^{2^{e+1}})^u$, so the coefficient $u$ of $x^{2^{e+1}}$ is odd.

(4) As in the proof of (3) just completed, with $e = \nu(j)$, then, mod 2, $q_j(x)$ is a polynomial in $x^{2^{e+1}}$. For $i - j$ to be a multiple of $2^{e+1}$, we must have $\nu(i) = \nu(j)$.

(5) Mod 4, $q_j(x) \equiv (1+x)^{2j} - 2(1+x)^{2j-1}$. Note also that, mod 4, $(1+x)^{2^f} \equiv (1+x^{2^{f-1}})^2$ for $f \geq 2$. Thus, if $j = 2^e(2b+1)$, then

$$q_j(x) \equiv (1+x^{2^{e+1}})^{2b}((1+x^{2^e})^2 - 2\sum_{t=0}^{2^{e+1}-1} x^t).$$

The only term in the second factor of the form $x^{2^e u}$ with $u$ odd has coefficient $2-2$. Thus all terms in $q_j(x)$ of the form $x^{2^e u}$ have coefficient 0, and these are the terms $x^{i-j}$ with $\nu(i) > \nu(j)$.

(6) Let $j = 2^e(2b+1)$. We must show the coefficient of $x^{i-j}$ in $(3+4x+x^2)(1-x)^{2j-2}$ is 0 mod 8 if $i \equiv 0$ $(2^{e+2})$. This coefficient equals

$$\begin{cases} 3\binom{2^{e+3}t+2^{e+1}-2}{c2^{e+2}+3\cdot 2^e} - 4\binom{2^{e+3}t+2^{e+1}-2}{c2^{e+2}+3\cdot 2^e-1} + \binom{2^{e+3}t+2^{e+1}-2}{c2^{e+2}+3\cdot 2^e-2} & b=2t \\ 3\binom{2^{e+3}t+3\cdot 2^{e+1}-2}{c2^{e+2}+2^e} - 4\binom{2^{e+3}t+3\cdot 2^{e+1}-2}{c2^{e+2}+2^e-1} + \binom{2^{e+3}t+3\cdot 2^{e+1}-2}{c2^{e+2}+2^e-2} & b=2t+1. \end{cases} \tag{7.5}$$

We shall denote the three terms of (7.5) by $3C_1$, $-4C_2$, and $C_3$.

**Case 1:** $e=0$, $b=2t$. One easily shows that $C_1$ and $C_3$ are 0 mod 8 and $C_2$ is even, and hence (7.5) is 0 mod 8.

**Case 2:** $e=0$, $b=2t+1$. If $c=2d$, then $C_3 \equiv 0$ mod 8, while, mod 8, both $3C_1$ and $-4C_2$ are $4\binom{t}{d}$, and hence (7.5) is 0 mod 8. If $c=2d+1$, the conclusion is similar with $C_3$ and $3C_1$ interchanged.

**Case 3:** $e>0$, $b=2t$. Here we have $\nu(C_1) = \nu(C_3) = \nu\binom{4t}{2c+1} \ge 2$ and $\nu(C_2) \ge 1$, and hence (7.5) is 0 mod 8.

**Case 4:** $e>0$, $b=2t+1$. Again $4C_2 \equiv 0$ mod 8. We write $3C_1 + C_3$ as $P \cdot S$, where

$$\begin{aligned} P &= (2^{e+3}t+3\cdot 2^{e+1}-2)(2^{e+3}t+3\cdot 2^{e+1}-3)\binom{2^{e+3}t+3\cdot 2^{e+1}-4}{c2^{e+2}+2^e-2} \\ S &= \frac{3}{(c2^{e+2}+2^e)(c2^{e+2}+2^e-1)} + \frac{1}{(2^{e+3}t+5\cdot 2^e-c2^{e+2})(2^{e+3}t+5\cdot 2^e-c2^{e+2}-1)} \end{aligned}$$

The product $P$ has the same 2-exponent as $2\binom{2t+1}{c}\binom{2^{e+1}-4}{2^e-2}$, and hence $\nu(P) \ge e$. The sum $S$ has the same 2-exponent as

$$3(8t+5-4c)^2 + (4c+1)^2 - 2^{-e}(3(8t+5-4c)+4c+1).$$

Thus the mod 8 value of $P \cdot S$ is a multiple of $2^e(3+1)-(15-8c+1)$ which is 0.

(7) We work mod 16, and will show $\mathrm{coef}(q_{2^e-2^t}, x^{2^t}) \equiv 0$. The proof for $m = 3\cdot 2^e$ is virtually identical. We use $(9+6x+x^2)^2 \equiv$

$(1-x)^4$ and

$$(3+4x+x^2)(9+6x+x^2) \equiv 11+6x+4x^2+10x^3+x^4.$$

**Case 1:** $t=0$. The required coefficient is

$$\operatorname{coef}((3+4x+x^2)(1-x)^{2^{e+1}-4}, x) = 4-3(2^{e+1}-4) \equiv 0$$

since $e \ge 3$.

**Case 2:** $t=1$. The required coefficient is

$$\operatorname{coef}((11+6x+4x^2)(1-x)^{2^{e+1}-8}, x^2) = 4-6(2^{e+1}-8)+11\binom{2^{e+1}-8}{2} \equiv 4+11\cdot 36 \equiv 0,$$

since $e \ge 4$.

**Case 3:** $t=2$. The required coefficient is

$$\begin{aligned} & \operatorname{coef}((11+6x+4x^2+10x^3+x^4)(1-x)^{2^{e+1}-12}, x^4) \\ = \;& 1-10(2^{e+1}-12)+4\tbinom{2^{e+1}-12}{2}-6\tbinom{2^{e+1}-12}{3}+11\tbinom{2^{e+1}-12}{4} \\ \equiv \;& 1+8+8+8+11\tfrac{(2^{e+1}-12)(2^{e+1}-13)(2^{e+1}-14)(2^{e+1}-15)}{24} \\ \equiv \;& 8+\tfrac{1}{3}(3+11(2^{e-1}-3)3(2^e-7)) \equiv 8+\tfrac{1}{3}(3+5) \equiv 0, \end{aligned}$$

since $e \ge 5$.

**Case 4:** $t \ge 3$. The required coefficient is

$$\begin{aligned} & \operatorname{coef}((11+6x+4x^2+10x^3+x^4)(1-x)^{2^{e+1}-2^{t+1}-4}, x^{2^t}) \\ = \;& 11\tbinom{2^{e+1}-2^{t+1}-4}{2^t}-6\tbinom{2^{e+1}-2^{t+1}-4}{2^t-1}+4\tbinom{2^{e+1}-2^{t+1}-4}{2^t-2}-10\tbinom{2^{e+1}-2^{t+1}-4}{2^t-3}+\tbinom{2^{e+1}-2^{t+1}-4}{2^t-4}. \end{aligned}$$

One easily shows that the 2-exponent in each of the three middle terms (including their coefficient) is $t+1 \ge 4$. The sum of the first and last terms is analyzed similarly to Case 4 of (6). It is $P \cdot S$, where $P$ is a product of four terms with exponent sum $3+\nu\binom{2^{e+1}-2^{t+1}-8}{2^t-4} = t+1$. The sum $S$ is $11/p_1 + 1/p_2 = (11p_2+p_1)/(p_1p_2)$ with $\nu(p_1)=\nu(p_2)=t+1$, and $11p_2+p_1$ given by

$$11D(D-1)(D-2)(D-3)+2^t(2^t-1)(2^t-2)(2^t-3),$$

with $D = 2^{e+1} - 3 \cdot 2^t$. Then $P \cdot S$ becomes, mod 16,

$$\begin{aligned} & 11(2^{e+1-t}-3)(2^{e+1}-3\cdot 2^t-1)(2^e-3\cdot 2^{t-1}-1)(2^{e+1}-3\cdot 2^t-3) \\ & +(2^t-1)(2^{t-1}-1)(2^t-3) \qquad (7.6) \\ \equiv\ & (3\cdot 2^t+1)(3\cdot 2^{t-1}+1)(3\cdot 2^t+3)+(2^t-1)(2^{t-1}-1)(2^t-3) \\ \equiv\ & 12\cdot 2^{t-1}\equiv 0. \end{aligned}$$

(8) **Case 1:** As in the proof of (5), mod 4,

$$q_{6\cdot 2^t}(x) \equiv (1+x^{2^{t+2}})^2((1+x^{2^{t+1}})^2 - 2\sum_{\ell=0}^{2^{t+2}-1} x^\ell),$$

which has 2 as its coefficient of $x^t$. The proof for $M_{5\cdot 2^t, 4\cdot 2^t}$ is identical.

**Case 2:** Similarly to the proof of (6), mod 8, the desired coefficient is

$$3\binom{2^{t+3}+2^{t+1}-2}{2^t} + \binom{2^{t+3}+2^{t+1}-2}{2^t-2} = P \cdot S$$

with

$$\nu(P) = 1 + \nu\binom{2^{t+3}+2^{t+1}-4}{2^t-2} = t$$

and

$$\nu(S) + t = \nu(3(8+1)(2^{t+3}+2^t-1)+(2^t-1)) = \nu(244\cdot 2^t - 28) = 2.$$

**Case 3:** This is like Case 4 of (7), except now $e = t+2$. This changes the first term of (7.6) by 8 mod 16. ■

Next we present substantial evidence to support the following conjecture.

**Conjecture 7.7.** *If $m \neq 2^e$ or $3 \cdot 2^e$, $e \geq 0$, the extension from $G_{4a+1}$ is trivial in the spectral sequence for $SU(2m)$ pictured in Diagrams 1.5 and 1.6.*

Some of the evidence supporting Conjecture 7.7 is a calculation when $a = 0$ and $m \leq 50$. The algorithm is that of the proof of Theorem 7.1, and the results are presented in Table 12. We are computing the homomorphism

$$E_2^{2,3}(SU(2m)) \xrightarrow{p_*} E_2^{2,3}(S^{4m-1})$$

without listing a split $\mathbf{Z}_2$ in each which correspond under $p_*$. We list the 2-exponents of the summands of $E_2^{2,3}(SU(2m))$, which is the same as those of Table 3, the 2-exponent of the coefficient of the image of each generator in the main summand of $E_2^{2,3}(S^{4m-1})$, and then its 2-exponent. For example, ignoring the split $\mathbf{Z}/2$ in each,

$$E_2^{2,3}(SU(10)) \xrightarrow{p_*} E_2^{2,3}(S^{19})$$

is $\mathbf{Z}/2^2 \oplus \mathbf{Z}/2^7 \to \mathbf{Z}/2^5$ with $p_*(g_1) = 2^4 G$ and $p_*(g_2) = G$. Often the exponent of the image of a summand will be larger than that of the sphere's summand; this just means that the summand maps to 0, but the more specific information is provided by our algorithm.

The information of Table 12 provides information about both differentials and extensions in $SU(2m)$. The differential occurs from the smallest summand which maps to a generator; i.e., to have 0 as the 2-exponent of the coefficient of its image. The extension occurs from the first summand such that the sum of its 2-exponent plus that of its image equals that of the sphere-summand, for this implies that $p_*$ is injective.

Once we get a summand which maps to the generator, we do not compute the images of subsequent (larger) summands. The reason for this is that, if $g_1$ and $g_2$ are the generators of the respective summands, then we could rechoose $g_2' = g_2 - g_1$ to have the same order, but the opposite property with regard to whether it maps to a generator.

The result of the table is that in this range the extension occurs if and only if $m = 2^e$ or $3 \cdot 2^e$, while the differential occurs on a large summand unless $m = 2^e$. The biggest obstacle to proving the statement about nonexistence of extensions would seem to come when the sphere-summand is large.

Indeed, another verification of the conjecture points to a subtlety when the sphere-summand has its maximum possible value. We performed a verification for $SU(10)$ by computing

$$E_2^{2,8a+3}(SU(10)) \xrightarrow{p_*} E_2^{2,8a+3}(S^{19}) \tag{7.8}$$

to be (not including a split $\mathbf{Z}/2$ in each which correspond under $p_*$)

$$\mathbf{Z}/2^2 \oplus \mathbf{Z}/2^{\min(\nu(a-18)+6,12)} \xrightarrow{p_*} \mathbf{Z}/2^{\min(\nu(a-2)+4,9)}$$

TABLE 12. $E_2^{2,3}(SU(2m)) \xrightarrow{p_*} E_2^{2,3}(S^{4m-1})$

| $m$ | $SU(2m)$ | $\mathrm{im}(p_*)$ | $S^{4m-1}$ | | $m$ | $SU(2m)$ | $\mathrm{im}(p_*)$ | $S^{4m-1}$ |
|---|---|---|---|---|---|---|---|---|
| 3 | 1,4 | 4,0 | 4 | | | | | |
| 4 | 3,4 | 0,0 | 3 | | 26 | 1,2,7,12,29 | 8,4,6,1,0 | 3 |
| 5 | 2,7 | 4,0 | 5 | | 27 | 1,2,7,15,28 | 11,7,9,1,0 | 4 |
| 6 | 1,2,8 | 10,1,0 | 3 | | 28 | 1,4,6,16,28 | 9,1,1,0,x | 3 |
| 7 | 1,4,8 | 9,2,0 | 4 | | 29 | 1,4,6,15,31 | 14,6,6,6,0 | 5 |
| 8 | 3,4,8 | 0,x,x | 3 | | 30 | 1,4,8,14,32 | 13,3,1,1,0 | 3 |
| 9 | 2,3,12 | 9,7,0 | 6 | | 31 | 1,4,8,16,32 | 17,6,4,2,0 | 4 |
| 10 | 2,4,13 | 2,1,0 | 3 | | 32 | 3,4,8,16,32 | 0,x,x,x,x | 3 |
| 11 | 2,7,12 | 5,1,0 | 4 | | 33 | 3,4,6,14,38 | 11,11,12,10,0 | 8 |
| 12 | 1,2,8,12 | 22,1,0,x | 3 | | 34 | 3,3,6,16,39 | 8,5,5,1,0 | 3 |
| 13 | 1,2,7,15 | 10,6,7,0 | 5 | | 35 | 3,3,6,20,37 | 11,8,8,0,x | 4 |
| 14 | 1,4,6,16 | 9,1,1,0 | 3 | | 36 | 2,3,8,20,38 | 6,4,1,1,0 | 3 |
| 15 | 1,4,8,16 | 13,4,2,0 | 4 | | 37 | 2,3,8,21,39 | 11,9,6,3,0 | 5 |
| 16 | 3,4,8,16 | 0,x,x,x | 3 | | 38 | 2,3,10,20,40 | 8,6,1,4,0 | 3 |
| 17 | 3,3,6,21 | 12,9,9,0 | 7 | | 39 | 2,3,12,20,40 | 11,9,2,6,0 | 4 |
| 18 | 2,3,8,22 | 6,4,1,0 | 3 | | 40 | 2,4,12,21,40 | 2,1,3,0,x | 3 |
| 19 | 2,3,12,20 | 9,7,0,x | 4 | | 41 | 2,4,12,19,44 | 9,8,8,7,0 | 6 |
| 20 | 2,4,12,21 | 2,1,3,0 | 3 | | 42 | 2,4,12,20,45 | 5,4,2,1,0 | 3 |
| 21 | 2,4,12,23 | 7.6.4.0 | 5 | | 43 | 2,4,12,23,44 | 8,7,5,1,0 | 4 |
| 22 | 2,6,11,24 | 4,1,5,0 | 3 | | 44 | 2,6,11,24,44 | 4,1,5,0,x | 3 |
| 23 | 2,7,12,24 | 7,3,2,0 | 4 | | 45 | 2,6,11,23,47 | 9,6,12,7,0 | 5 |
| 24 | 1,2,8,12,24 | 46,1,0,x,x | 3 | | 46 | 2,7,12,22,48 | 6,2,1,1,0 | 3 |
| 25 | 1,2,8,10,28 | 13,8,7,8,0 | 6 | | 47 | 2,7,12,24,48 | 9,5,4,2,0 | 4 |
| | | | | | 48 | 1,2,8,12,24,48 | 94,1,0,x,x,x | 3 |

with the second summand mapping surjectively, while the order of the image of the first summand is

$$\begin{cases} 2^1 & \text{if } \nu(a-2) < 4 \\ 2^0 & \text{if } \nu(a-2) = 4 \\ 2^2 & \text{if } \nu(a-2) \geq 5. \end{cases}$$

If $\nu(a-2) < 5$, then $v_1^{-1}\pi_{8a+1}(S^{19}) \approx \mathbf{Z}/2^{\nu(a-2)+4}$ is obtained by a nontrivial extension from $E_2^{2,8a+3}(S^{19}) \approx \mathbf{Z}/2^{\nu(a-2)+4}$ to a class in filtration 4, together with a nontrivial $d_3$-differential on $E_2^{2,8a+3}(S^{19})$. We have just noted that for such $a$, neither summand of $E_2^{2,8a+3}(SU(10))$ maps injectively, and hence the extension in $SU(10)$ is trivial.

If $\nu(a-2)\ge 5$, there is not an extension in $v_1^{-1}\pi_{8a+1}(S^{19})$ since the filtration-4 class is hit by a $d_3$-differential. No deduction about the extension in $v_1^{-1}\pi_{8a+1}(SU(10))$ can be made from the exact sequences in $E_2$ and $v_1^{-1}\pi_*$ of the fibration $SU(9)\to SU(10)\to S^{19}$ in this case.

Instead, we use the method of [15] to prove the following result, which implies that the extension is trivial in this case.[10]

**Proposition 7.9.** *If $\nu(a-2)\ge 5$, then*

$$v_1^{-1}\pi_{8a+1}(SU(10))\approx \mathbf{Z}/2\oplus\mathbf{Z}/2^2\oplus\mathbf{Z}/2^9.$$

*Proof.* We use heavily the notation of [15], and note that results listed as conjectures there were subsequently proved in [10]. By [15, 2.3,3.1], for $SU(10)$ we have

$$\begin{aligned} Q_{\mathbf{H}} &= \langle\widetilde{\lambda}_1+\widetilde{\lambda}_9,\ \widetilde{\lambda}_2+\widetilde{\lambda}_8,\ \widetilde{\lambda}_3+\widetilde{\lambda}_7,\ \widetilde{\lambda}_4+\widetilde{\lambda}_6,\ \widetilde{\lambda}_5\rangle\\ Q_{\mathbf{R}} &= \langle\widetilde{\lambda}_1+\widetilde{\lambda}_9,\ \widetilde{\lambda}_2+\widetilde{\lambda}_8,\ \widetilde{\lambda}_3+\widetilde{\lambda}_7,\ \widetilde{\lambda}_4+\widetilde{\lambda}_6,\ 2\widetilde{\lambda}_5\rangle.\end{aligned}$$

By [15, 2.1,2.2],

$$v_1^{-1}\pi_{8a+1}(SU(10))^{\#}\approx\ker\Big(Q_{\mathbf{R}}/\lambda^2(Q_{\mathbf{H}})\xrightarrow{\theta}Q_{\mathbf{R}}/\lambda^2(Q_{\mathbf{H}})\Big),\tag{7.10}$$

where, by the argument at the beginning of the proof of [15, 4.2], $\theta=\lambda^3-3^{4a+1}$. We use the isomorphism $Q(SU(10))\to PK^1(SU(10))$ with $\widetilde{\lambda}_i\leftrightarrow B_i$. Let $y_1=B_1+B_9,\dots,y_4=B_4+B_6$ and $y_5=B_5$. Using ([15, 3.15]) for $\psi^k(B_i)$, we obtain the following matrices of $\psi^k$ on the basis $\langle y_1,\dots,y_5\rangle$.

$$\Psi^2=\begin{pmatrix}10&-120&252&-120&10\\-1&45&-210&211&-45\\0&-10&120&-262&120\\0&1&-46&255&-210\\0&0&20&-240&252\end{pmatrix}\qquad \Psi^3=\begin{pmatrix}55&-1452&6766&-8560&2850\\-10&615&-4750&9568&-4740\\1&-210&2905&-9802&6765\\0&56&-1662&9615&-8350\\0&-20&1230&-9480&8953\end{pmatrix}$$

[10] The $\mathbf{Z}/2^{10}$ summand in $E_2$ supports a nontrivial $d_3$-differential, which is what brought its order down to $2^9$.

We let $a = 2$ and $\Theta = \Psi^3 - 3^9$. We use the agreement up to sign of $\psi^k$ in $PK^1(G)$ and $\lambda^k$ in $Q(G)$. By [15, 5.9][11], our desired group is presented by the matrix obtained from $\binom{\Psi^2}{\Theta}$ by dividing the last row of $\Psi^2$ by 2. Several steps of pivoting in `Maple` show easily that this group is $\mathbf{Z}/2 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2^9$.

Changing $a$ by a multiple of 32 does not change the matrix mod $2^9$. Thus it certainly does not change the $\mathbf{Z}/2$ and $\mathbf{Z}/4$ summands. The total order of the group can be obtained by other methods, and so the result is valid for all $a$ as in the proposition. ■

Now we present a rather lengthy account of the part of the last part of Theorem 1.4 dealing with summands of order 2 in $E_2^{2,2k+1}(SU(2m))$ when $k$ is even, and also the method used to compute the summands of these groups in Tables 5 and 6.

When $k$ is even, $E_2^{2,2k+1}(SU(2m))$ cannot be computed from Theorem 4.7. Instead, we use the Small Complex matrix $M_{2,k}$ of 3.2 and compute $Q(M_{2,k})$. We use the basis of 3.3. The matrix has the following form. Here $\Psi = \Psi^2$ and $\Theta = \Theta_k$ of 3.2, and the subscripts of $\Psi$ and $\Theta$ here just refer to their position in the matrix. We emphasize that the meaning of the subscript of $\Theta$ is different here than in 3.2.

$$\left(\begin{array}{cc|cc|cc|cc}
0 & 0 & \Psi_1 & 0 & \Theta_1 & 0 & 0 & 0 \\
I' & 2I & \Psi_2 & \Psi_3 & \Theta_2 & \Theta_3 & 0 & 0 \\
\hline
0 & 0 & 2I & 0 & 0 & 0 & \Theta_1 & 0 \\
0 & 0 & -I' & 0 & 0 & 0 & \Theta_2 & \Theta_3 \\
\hline
0 & 0 & 0 & 0 & 2I & 0 & -\Psi_1 & 0 \\
0 & 0 & 0 & 0 & -I' & 0 & -\Psi_2 & -\Psi_3
\end{array}\right) \tag{7.11}$$

Each $2 \times 2$ block consists of a batch of $m$ rows followed by a batch of $m-1$ rows, with exactly the same configuration for the columns. Thus, for example $\Psi_1$ is $m \times m$, while $\Psi_2$ is $(m-1) \times m$. The matrix $I'$ is an identity matrix with a column of 0's at the end. We pivot on the three $I'$'s, removing their rows and columns. We also remove the second, fourth, and sixth blocks of columns, which will have become 0 after removal of the second block of rows. (Recall that columns of 0's do not contribute to $Q(M)$.)

If $N$ is a matrix with $m-1$ rows, such as $\Theta_i$ or $\Psi_i$ with $i = 2$ or 3, let $\underline{N}$ denote the matrix obtained from $N$ by appending a row of 0's beneath. If $N$ is a matrix with

[11]There is a misprint in [15, 5.9]. The entries $2C_2$ and $\frac{1}{2}C_3$ should actually be $C_2$ and $C_3$. This misprint does not affect the validity of the applications made of this result in [15].

$m$ columns, such as $\Theta_1$ or $\Psi_1$, let $\widetilde{N}$ denote the matrix obtained from $N$ by deleting the last column, and let $N_\ell$ denote the last column of $N$.[12] What remains after the pivoting described above is the $(m+m+m)\times(1+1+m+(m-1))$ matrix in (7.12).

$$\begin{pmatrix} \Psi_{1,\ell} & \Theta_{1,\ell} & \widetilde{\Psi}_1\Theta_2-\widetilde{\Theta}_1\Psi_2 & \widetilde{\Psi}_1\Theta_3-\widetilde{\Theta}_1\Psi_3 \\ 2I_\ell & 0 & \Theta_1+2\underline{\Theta}_2 & 2\underline{\Theta}_3 \\ 0 & 2I_\ell & -\Psi_1-2\underline{\Psi}_2 & -2\underline{\Psi}_3 \end{pmatrix} \tag{7.12}$$

The columns here correspond, in (7.11), to the last column of the third and fifth of the 8 blocks, and all the columns of the last two blocks.

To obtain the summands in Tables 5 and 6, we perform on (7.12) the algorithm described in the proof of Theorem 7.1. Note that this matrix yields the split $\mathbf{Z}/2$ as well as the $G_k$ summand of Diagrams 1.5 and 1.6, unlike the algorithm based on 4.7(1) when $k$ is odd, which just produces the $G_k$ summand of $E_2^{2,2k+1}(SU(2m))$.

To prove the portion of Theorem 1.4 about $\mathbf{Z}/2$ summands in $G_{4a\pm2}$ and differentials from them, we prove the following theorem and two corollaries.

**Theorem 7.13.** *Let $M$ be the matrix (7.12) reduced mod 4 with $k$ even. Since $3^k$ has the same mod 4 value for all even $k$, $M$ is independent of even $k$. Let $M'$ be obtained from $M$ by pivoting on odd entries and removing their rows and columns. Then*

$$\mathrm{rk}_2(M')=\begin{cases} 2 & \text{if } m\in\{3\}\cup\bigcup_{e\ge0}(4\cdot2^e,5\cdot2^e)\cup[6\cdot2^e,7\cdot2^e) \\ 1 & \text{otherwise.}\end{cases} \tag{7.14}$$

*Here we have used open and closed interval notation. Considering only columns 1, 2, $(m+2)$ of $M$, $M'$ has a row $(2,0,2)$ if and only if $m=3\cdot2^e$ or $2^{e+1}$, $e\ge0$, and a row $(0,2,0)$ if and only if $m\ne2^e$.*

**Corollary 7.15.** *If $k$ is even, the number of $\mathbf{Z}/2$ summands in $E_2^{2,2k+1}(SU(2m))$ is as in (7.14). The image of these summands in $E_2^{2,2k+1}(S^{4m-1})=\mathbf{Z}_2\oplus\mathbf{Z}_2$ contains the unstable summand (first one in its box in [2, p.488]) if and only if $m=3\cdot2^e$ or $2^{e+1}$, $e\ge0$, and contains the stable summand (second one in its box) if and only if $m\ne2^e$.*

The following is part of Theorem 1.4.

[12] Note that $\ell$ stands for "last", not for an integer.

**Corollary 7.16.** *If $k \equiv 2$ mod 4, the group $G_k$ in Diagram 1.5 or 1.6 has a $\mathbf{Z}/2$ summand iff $m = 3$ or $4 \cdot 2^e < m < 5 \cdot 2^e$ or $6 \cdot 2^e \le m < 7 \cdot 2^e$ for some $e \ge 0$. The differential is nonzero on the $\mathbf{Z}/2$ summand iff $m = 3 \cdot 2^e$ for $e \ge 0$.*

*Proof.* By [2, p.488], if $k \equiv 2$ mod 4, the unstable summand of $E_2^{2,4k+1}(S^{4m-1})$ always supports a nonzero $d_3$-differential, while the stable summand does iff $m$ is even. Since the target of $d_3$ maps bijectively from $SU(2m)$ to $S^{4m-1}$, $d_3$ is nonzero on a $\mathbf{Z}/2$ summand of $E_2^{2,4k+1}(SU(2m))$ iff it is nonzero on its image in $S^{4m-1}$.

Recall that $E_2^{2,2k+1}(SU(2m)) = G_k \oplus \mathbf{Z}_2$, and Diagram 1.6 does not show the split $\mathbf{Z}_2$ which is supporting a nonzero $d_3$. By Corollary 7.15, the number of $\mathbf{Z}_2$ summands of $E_2^{2,2k+1}(SU(2m))$ which map to a summand of $E_2^{2,2k+1}(S^{4m-1})$ which supports a nonzero differential is 0 if $m$ is odd and $m > 3$, 2 if $m = 3 \cdot 2^e$, and 1 otherwise. Since one of these is accounted for by the split $\mathbf{Z}_2$ when $m$ is even, the result follows. ∎

*Proof of Corollary 7.15.* We begin by studying the computation of $E_2^{2,2k+1}(S^{4m-1})$ when $k$ is even, using 3.2. We obtain

$$E_2^{2,2k+1}(S^{4m-1}) \approx Q \begin{pmatrix} 0 & 2^{2m-1} & 2u & 0 \\ 0 & 2 & 0 & 2u \\ 0 & 0 & 2 & -2^{2m-1} \end{pmatrix}. \tag{7.17}$$

We claim the $\mathbf{Z}/2$ from the second row is the unstable class, while the $\mathbf{Z}/2$ from the first (or third) row is the stable one. To see this, we consider the morphism of small resolutions (see [6, 11.11]) induced by the surjection $\mathbf{Z}_2^\wedge/2^n \to \mathbf{Z}_2^\wedge/2^{n-1}$, corresponding to the double suspension homomorphism. One easily checks that the induced morphism of the groups $R_s$ of [6, 11.11] for $s \ge 2$ is multiplication by 2 on the second and fourth summands. This will yield the unstable summands, since they are the ones that map by $\cdot 2$ under double suspension, and it corresponds to the second row of the matrix above.

We compute the morphism $E_2^{2,2k+1}(SU(2m)) \to E_2^{2,2k+1}(S^{4m-1})$ by applying the algorithm of the proof of 7.1 to (7.11). We look at the entries of a row corresponding to a generator in the columns corresponding to the last column in first, third, fifth, and seventh blocks of (7.11). As the first will have all 0's, the ones of interest correspond to columns 1, 2, and $m+2$ of (7.12), and to the last three columns of (7.17). Now we combine the result of Theorem 7.13 about rows of the form (2,0,2) and (0,2,0) in $M'$

with the result of the preceding paragraph which characterizes 1/2 times these rows in the sphere as respectively unstable and stable. ■

*Proof of Theorem 7.13.* The proof is somewhat similar to that in Section 5. We first outline the proof, and then provide more details to fill in the outline.

Let $0_2$ denote a column with all 0's except for a 2 in its last entry. The matrix (7.12) can be written as

$$\begin{pmatrix} 0 & 0_2 & M_1 & M_2 \\ 0_2 & 0 & M_3 & M_4 \\ 0 & 0_2 & M_5 & M_6 \end{pmatrix}. \tag{7.18}$$

This involves an easy verification for the top left columns, and merely a new name for the other submatrices; $M_i$ has $m$ rows and $m$ (resp. $m-1$) columns if $i$ is odd (resp. even).

**Step 1:** The submatrix $M_5$ has $-1$ in position $(2j,j)$, $1 \le j \le [m/2]$, a 2 in $(m,[m/2]+1)$ if $m$ is odd, and 0's elsewhere. We pivot[13] on these odd entries. The only change in the first two columns of (7.18) occurs (due to the lowest $0_2$) if $m$ is even, in which case $M_1$ has an odd entry in its position $(m,m/2)$, and $M_3$ has an odd entry in column $m/2$ in row $\frac{m}{2}+2^{\nu(m)}$ if this is $\le m$, and possibly also in rows $\frac{m}{2}+c\cdot 2^{\nu(m)}$ for certain $c>1$. The pivoting on position $(m,m/2)$ of $M_5$ will add 2 in column 2 to the rows just mentioned. The obtained matrix now has the form

$$\begin{pmatrix} 0 & 0_2' & M_1' & M_2' \\ 0_2 & C_m & M_3' & M_4' \\ 0 & 0_2' & M_5' & M_6' \end{pmatrix}, \tag{7.19}$$

where the bottom block of rows has $m-[m/2]$ rows, and the $M'_{\mathrm{odd}}$ have $m-[m/2]$ columns. Here, also,

$$0_2' = \begin{cases} 0_2 & \text{if } m \text{ odd} \\ 0 & \text{if } m \text{ even.} \end{cases}$$

and the column $C_m = 0$ if $m$ is odd, while if $m$ is even, $C_m$ has a 2 in row $\frac{m}{2}+2^{\nu(m)}$ if this is $\le m$, and possibly some rows $\frac{m}{2}+c\cdot 2^{\nu(m)}$ with $c>1$. Here and throughout, we use the original row and column indices in each $M_i'$. For example, if $i$ is odd, the first column of $M_i'$ is labeled $m-[\frac{m}{2}]+1$.

[13] As usual, "pivoting" includes removal of the row and column of the pivot entry.

**Step 2.** The first odd entry in column $j$ of $M_3'$ occurs in row $j+2^{\nu(j)+1}$, if this is $\le m$. We pivot on these entries. Let

$$\begin{pmatrix} C_{1,1} & C_{1,2} & M_1'' & M_2'' \\ C_{2,1} & C_{2,2} & M_3'' & M_4'' \\ C_{3,1} & C_{3,2} & M_5'' & M_6'' \end{pmatrix}$$

be the obtained matrix. The only nonzero entries remaining are 2's in the positions described below, and this implies the theorem. The proof that this occurs, as well as some of the earlier steps in this proof, appears below. Note that the same $m$ can occur in two of the six listed possibilities.

- $m$ odd, $\ne 2^e-1$: last row of $C_{1,2}$ and $C_{3,2}$. Of course, one of these can be removed by pivoting.
- $m=2^e-1$: last row of $C_{1,2}$ and $C_{3,2}$, and last row of $M_1''$ and $M_5''$ in column $2^{e-1}$, which will be the first column in those submatrices. Here again, one of the two rows can be removed.
- $m=2^e$: last row of $C_{2,1}$, and last row, last column $(m)$ of $M_3'$.
- $m=3\cdot 2^e$, $e\ge 0$: last row of $C_{2,1}$; last row, first $(2^{e+1})$ and last $(m)$ column of $M_3''$; and last row, column $2^e$ of $M_4''$. (i.e. four 2's in the same row.)
- $m$ even, $\ne 2^e$: row $\frac{m}{2}+2^{\nu(m)}$ of $C_{2,2}$, and same row, column $m/2$ of $M_4''$.
- $m=a\cdot 2^{e+1}+\Delta$, $a=2$ or $3$, $0<\Delta<2^e$: row $a\cdot 2^{e+1}$ of $M_3''$, columns $a\cdot 2^{e+1}-2^i$ for $\Delta<2^i\le 2^e$.

Here we provide additional details of the proof of Theorem 7.13, a sketch of which was just completed. The details here are not meant as a self-contained account, but rather justification for statements of the above proof.

Using the basis of 3.3, mod 4, $\psi^2(Y)\equiv Y^2$, $\psi^3(Y)\equiv Y(1+Y)^2$, $\psi^2(X)\equiv X(2+Y)+Y$, and $\psi^3(X)\equiv X(-1+Y^2)+(-Y+Y^2)$. Hence $\Psi_1$, which is the matrix of $\psi^2$ from the $XY^j$-basis to itself, has 1 in $(2j,j)$, 2 in $(2j-1,j)$, and 0 elsewhere. Also, $\Psi_2$, which is the matrix of $\psi^2$ from the $XY^j$-basis to the $Y^j$-basis, has 1 in $(2j-1,j)$, and 0 elsewhere, while $\Psi_3$, which is the matrix of $\psi^2$ from the $Y^j$-basis to itself, has just 1's in $(2j,j)$. The matrices $\Theta_1$, $\Theta_2$, and $\Theta_3$ are those of $\psi^3-1$ with respect to the bases just described, and in column $j$ are the coefficients of $x^j((-1+x^2)(1+x)^{2(j-1)}-1)$, $x^j(-1+x)(1+x)^{2(j-1)}$, and $x^j((1+x)^{2j}-1)$, respectively.

The claims prior to Step 1 about the last column of $\Psi_1$ and $\Theta_1$ being 0 and $0_2$, respectively, are immediate from the above description. In Step 1, (a.) in $M_5 = -\Psi_1 - 2\underline{\Psi}_2$, the 2 in $(2j-1, j)$ from $-\Psi_1$ is cancelled by the $2\underline{\Psi}_2$ unless it is in the last row; (b.) the first odd entry of $M_3$ is that of $\Theta_1$, which occurs in the row of the first odd coefficient of $x^j((1+x)^{2j}-1)$, which is row $j+2^{\nu(2j)}$; and (c.) the claim about odd entries in column $m/2$ of $M_1$ is seen by noting that, mod 2, column $m/2$ of $\widetilde{\Psi}_1\Theta_2$ has a 1 in its last entry coming from column $m/2$ of $\Psi_1$, while column $m/2$ of $\widetilde{\Theta}_1\Psi_2$ equals column $m$ of $\Theta_1$, which is 0 mod 2.

**Step 2:** The pivoting of Step 1 did not change the remaining columns of $M_{\text{odd}}$, and so the odd entries in the remaining columns of $M_3'$ are as described in part (b) of the previous paragraph. We analyze the result of this pivoting.

**First column:** If $m = 2^e$ or $3 \cdot 2^e$, then $m \neq j + 2^{\nu(2j)}$ for $j \geq m/2$, and so the 2 in column 1 will not be involved in pivoting. For other $m$, there is such a $j$, there will be pivoting on position $(m, j)$ of $M_3'$, and the row with the 2 in the first column will be eliminated. The pivoting will create another 2 in column 1 if this column $j$ has another odd entry. This will not be the case in $M_3'$ since the pivot 1 under consideration is the last entry in $M_3'$ in its column, but the first odd in its column. Note that $M_1' = -\widetilde{\Theta}_1\Psi_2 + \widetilde{\Psi}_1\Theta_2$ with its first $[m/2]$ columns deleted. In these remaining columns, $\Psi_2$ is 0 and hence so is $\widetilde{\Theta}_1\Psi_2$, while $\widetilde{\Psi}_1\Theta_2$ is 0 since in these columns $\Theta_2$ is nonzero only in rows $> m/2$ but $\widetilde{\Psi}_1$ is 0 in columns $> m/2$. This establishes the claim that at the end of Step 2, column 1 has a 2 iff $m = 2^e$ or $3 \cdot 2^e$, and this occurs in the last row.

**Second column:** If $m$ is odd, there are no changes in the second column because the pivoting is on rows in the middle block, and the second column $C_m$ had all 0's there after the first step.

If $m$ is even, the column $C_m$ had, after the first step, 2's in rows $\frac{m}{2} + c \cdot 2^{\nu(m)}$ for $c = 1$ and perhaps some larger values of $c$. The columns of $M_3'$ occur only for $j > m/2$. For $c \geq 2$, $M_3'$ has a pivot 1 in $(\frac{m}{2} + c \cdot 2^{\nu(m)}, \frac{m}{2} + (c-1)2^{\nu(m)})$. This will cause removal of all these rows, leaving in the second column just the 2 in row $\frac{m}{2} + 2^{\nu(m)}$. This pivoting cannot cause additional 2's in column 2 in other rows, because $M_1'$, $M_5'$, and other rows of $M_3'$ have no odd entries. This observation for $M_1'$ is similar to that of (c.) several paragraphs above. This establishes the claim that at the end of Step 2, column 2 has a 2 in the last row of the first and third blocks if $m$ is odd, and in row $\frac{m}{2}+2^{\nu(m)}$ of the second block if $m$ is even.

**The submatrices** $M_i''$**,** $i=1$**,** $2$**,** $5$**, and** $6$: We first show that the only nonzero entry in $M_i'$ for these values of $i$ is a 2 in position $(m, \frac{m+1}{2})$ if $m$ is odd and $i=1$ or 5.

To do this, one easily checks that, mod 4, $M_6 \equiv 2M_5$ except in $(m, \frac{m}{2})$ if $m$ is even, where $M_5$ has 1 and $M_6$ has 0. We will show that the same is true of $M_2$ and $M_1$. Also, it is easily verified that these four matrices are 0 in columns $> [\frac{m+1}{2}]$. This implies that the pivoting in Step 1, which zeros the first $[\frac{m}{2}]$ columns of $M_1$ and $M_5$, does the same for those columns of $M_2$ and $M_6$, since they are twice as large.[14] The 2 in $(m, \frac{m+1}{2})$ of $M_5$ is from $-\Psi_1$, and the 2 in that position of $M_1$ comes from the 2 in position $(m, \frac{m+1}{2})$ of $\Psi_1$ times the 1 in $(\frac{m+1}{2}, \frac{m+1}{2})$ of $\Theta_2$. These 2's are not altered in the first step of pivoting.

Next we verify the claim of the preceding paragraph that $M_2 \equiv 2M_1$ with the one mentioned exception. We first note that, mod 2, the $j$th column of $\widetilde{\Theta}_1\Psi_2$ is the $(2j-1)$st column of $\Theta_1$, which is $x^{2j-1}((1+x^2)^{2j-1}-1)$. Also the $j$th column of $\widetilde{\Psi}_1\Theta_2$ is $x^{2j}(1+x^2)^{2j-1}$. Adding these yields $x^{2j-1}((1+x)^{4j-1}-1)$ as the generating polynomial for the $j$th column of $M_1$ mod 2, and doubling it generates $2M_1$ mod 4. Similarly, the $j$th column of $\widetilde{\Theta}_1\Psi_3$ is generated by the $(2j)$th column of $\widetilde{\Theta}_1$, which is $x^{2j}((3+x^2)(1+x)^{2(2j-1)}-1)$.[15] The even components of $\widetilde{\Psi}_1\Theta_3$ in column $j$ are those of $x^{2j}((1+x^2)^{2j}-1)$, while its odd components are those of $2x^{2j-1}((1+x^2)^{2j}-1)$. Adding and reducing mod 4, we find that $2x^{2j-1}((1+x)^{4j-1}-1)$ generates the $j$th column of $M_2$. Some care is required due to the tildes. This causes the one exception, which was described in the footnote.

Finally we observe that the 2 in $M_1'$ and $M_5'$ when $m$ is odd will be removed in the second step of pivoting, on an odd in $(\frac{m+1}{2}+2^{\nu(m+1)}, \frac{m+1}{2})$, provided $\frac{m+1}{2}+2^{\nu(m+1)} \le m$, which is true unless $m=2^e-1$. This establishes the claim that this latter case $(m=2^e-1)$ is the only time that $M_i''$ is nonzero when $i \in \{1,2,5,6\}$.

[14] The anomaly in column $\frac{m}{2}$ when $m$ is even presents no problem here, since both $M_2$ and $M_6$ have a similar 0.

[15] unless $m$ is even and $2j=m$, in which case the column is 0 due to the tilde, but the polynomial would have given 2.

**The submatrix** $M_4''$: Except in its last row, $M_4 \equiv 2M_3$. The first round of pivoting will eliminate the 2's in $M_4$ as it eliminates the odd entries in $M_3$. One exception is that, since $M_6$ lacks a 2 in $(m, \frac{m}{2})$ when $m$ is even, the pivoting on $(m, \frac{m}{2})$ of $M_5$ will not zero the 2 in $(\frac{m}{2} + c \cdot 2^{\nu(m)}, \frac{m}{2})$ in $M_4$ for various $c \geq 1$.[16] All except the one with $c = 1$ will be eliminated in Step 2 by the pivoting on $(\frac{m}{2} + 2^{\nu(m)}, \frac{m}{2})$ of $M_3'$. This accounts for the asserted 2 in $M_4''$ of the fifth type.

The other exception is due to the last row of $M_4$ being 0. If row $m$ of $M_3$ has an odd entry in column $j \leq [\frac{m}{2}]$,[17] the pivoting in Step 1 will create a 2 in $(m, j)$ in $M_4'$. However, if $m \neq 3 \cdot 2^e$, then $M_3'$ has a pivot 1 in row $m$, and so this row will be removed. This accounts for the asserted 2 in $M_4''$ of the fourth type when $m = 3 \cdot 2^e$, since here $j = 2^e$.

**The submatrix** $M_3''$: $M_3'$ is just $M_3$ with the first $[m/2]$ columns deleted. Because of the triangularity, we can delete the first $[m/2]$ rows also. It is convenient to order the rows by first considering $\nu(i)$, and ordering the columns similarly. For example, if $m = 17$, the order of the rows and columns is $9, 11, 13, 15, 17|10, 14|12|16$.

With this ordering of the rows and columns, we write $M_3'$ as

$$A_m = \begin{pmatrix} A_{\mathrm{oo}} & A_{\mathrm{oe}} \\ A_{\mathrm{eo}} & A_{\mathrm{ee}} \end{pmatrix},$$

where the double subscripts e and o refer to even and odd rows and columns, and the single subscript refers to the value of $m$. We claim that the matrix $A_{\mathrm{ee}}$ for $2m$ equals the whole matrix $A_m$. This is true because the entry in $(i, j)$ for $A_m$ equals

$$\begin{cases} \binom{2j}{i-j} - \delta_{i,j} & i \neq m \\ 3\binom{2(j-1)}{m-j} + \binom{2(j-1)}{m-j-2} & i = m. \end{cases}$$

Here we have taken into account the irregularity in the last row of $\underline{\Theta}_2$. It is not difficult to show that the mod 4 value of this expression is not changed when $i$, $j$, and $m$ are doubled.

One easily verifies that, mod 4, $A_{\mathrm{oo}}$ is lower triangular with 0's on the diagonal, except a 2 in the last diagonal entry if $m$ is odd, and units on the immediate subdiagonal, and that $A_{\mathrm{eo}}$ and $A_{\mathrm{oe}}$ are even, and $A_{\mathrm{oe}}$ is 0 in its first row. We let $R(A)$ denote the matrix obtained from $A$ by pivoting on odd entries (and then removing

[16]There is no such 2 if $m = 2^e$, since $\frac{m}{2} + 2^{\nu(m)} > m$.

[17]Note that this does not happen if $m = 2^e$.

pivot row and column), and also removing rows and columns of 0's, so that $Q(A)$ is now the group presented by $R(A)$. Let $D(-)$ double the row and column indices of a matrix, while keeping the entries the same. The observations of this paragraph imply that $R(A_{2m}) = D(R(A_m))$.

Our goal is to prove that $R(A_m)$ has only nonzero entries 2's in

$$\begin{cases} (m,m) & m = 2^e \\ (m, 2^{e+1}), (m,m) & m = 3\cdot 2^e \\ (a\cdot 2^{e+1}, a\cdot 2^{e+1} - 2^i), \Delta < 2^i \le 2^e & m = a\cdot 2^{e+1} + \Delta \end{cases}$$

with $a = 2$ or 3 and $0 < \Delta < 2^e$. As this property is preserved by halving, it suffices to consider odd $m$. Also, the cases $m = 4$ and 6 are easily verified to start the induction in the first cases.

We will prove $R(A_{4k+3}) = D(R(A_{2k+1}))$ and $R(A_{4k+1}) = D(R(A_{2k+1}))$ unless $k = 2^e$ or $3\cdot 2^e$, in which case $R(A_{4k+1})$ has an extra 2 in position $(4k, 4k-2)$. With the easily-verified $R(A_3) = 0$ and $R(A_5) = 0$, this will yield an inductive proof for all $m$.

We now prove the claim of the preceding paragraph. Each subblock of constant $\nu(-)$ has the form described above for $A_{\mathrm{oo}}$. After pivoting on the odd subdiagonal entries in all subblocks of constant $\nu(-)$, all that remains are the last entry in each subblock as column indices, and the first entry in each subblock as row indices. For example, if $m = 15$, the column indices remaining are $15, 14, 12, 8$, while the row indices are $7, 10, 12, 8$. We call this matrix $A'_m$.

It is easy to see that for $A'_{4k+3}$, the column indices are $4k+3$ followed by twice the column indices of $A'_{2k+1}$, while the row indices are $2k+1$ followed by twice the row indices of $A'_{2k+1}$. Moreover, the entries in the first row and first column are 0 by triangularity, and the other entries equal those of the corresponding entries of $A'_{2k+1}$. This proves the first claim.

Similarly, the column indices of $A'_{4k+1}$ are $4k+1$, then $4k-2$, then 4 times the column indices after the first of $A'_{2k+1}$, while the row indices are $2k+1$ followed by twice the row indices of $A'_{2k+1}$. The first row and first column of $A'_{4k+1}$ are 0, as is the first column of $A'_{2k+1}$. The second column of $A'_{4k+1}$ will have a 2 in $(4k, 4k-2)$ if it has a row $4k$; i.e. if $4k - 2^{\nu(4k)+1} < 2k+1$. This is the case iff $k = 2^e$ or $3\cdot 2^e$. This proves the second claim, and hence establishes our claim for $M'_3$. ∎

## 8. Results for $SU(n)$ when $n \le 13$

In this section, we present complete results of $v_1^{-1}\pi_*(SU(n))$ for $n \le 13$. This serves to illustrate our methods and confirm their efficacy. We begin with the easier situation in which $n$ is odd. There is some overlap here with [2, 5.5].

**Theorem 8.1.** *$v_1^{-1}\pi_{2k}(SU(n)) = \mathbf{Z}/2^{e(k,n)}$, where*

$$\begin{aligned}
e(k,3) &= \min(2, 1+\nu(k))\\
e(k,5) &= \begin{cases} \min(6, 2+\nu(k-12)) & k \text{ even}\\ \min(4, 2+\nu(k-3)) & k \text{ odd}\end{cases}\\
e(k,7) &= \begin{cases} \min(8, 5+\nu(k-6)) & k \text{ even}\\ \min(8, 3+\nu(k-13)) & k \text{ odd}\end{cases}\\
e(k,9) &= \begin{cases} \min(12, 5+\nu(k-72)) & k \equiv 0\ (4)\\ \min(9, 5+\nu(k-6)) & k \equiv 2\ (4)\\ \min(8, 5+\nu(k-5)) & k \equiv 1\ (4)\\ \min(11, 5+\nu(k-7)) & k \equiv 3\ (4)\end{cases}\\
e(k,11) &= \begin{cases} \min(13, 7+\nu(k-40)) & k \equiv 0\ (4)\\ \min(15, 7+\nu(k-74)) & k \equiv 2\ (4)\\ \min(14, 6+\nu(k-73)) & k \equiv 1\ (4)\\ \min(11, 6+\nu(k-7)) & k \equiv 3\ (4)\end{cases}\\
e(k,13) &= \begin{cases} \min(14, 9+\nu(k-24)) & k \equiv 0\ (8)\\ \min(18, 9+\nu(k-12)) & k \equiv 4\ (8)\\ \min(17, 8+\nu(k-458)) & k \equiv 2\ (4)\\ \min(14, 8+\nu(k-9)) & k \equiv 1\ (4)\\ \min(15, 9+\nu(k-11)) & k \equiv 3\ (4)\end{cases}
\end{aligned}$$

$v_1^{-1}\pi_{2k-1}(SU(n)) \approx v_1^{-1}\pi_{2k}(SU(n))$ *for* $n=3$ *and* $5$. *Also,*

$$v_1^{-1}\pi_{2k-1}(SU(7)) = \begin{cases} \mathbf{Z}/4 \oplus \mathbf{Z}/2^{e(k,7)-2} & k \text{ even} \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2^{e(k,7)-1} & k \text{ odd} \end{cases}$$

$$v_1^{-1}\pi_{2k-1}(SU(9)) = \begin{cases} \mathbf{Z}/2 \oplus \mathbf{Z}/2^{e(k,9)-1} & k \text{ even} \\ \mathbf{Z}/8 \oplus \mathbf{Z}/2^{e(k,9)-3} & k \text{ odd} \end{cases}$$

$$v_1^{-1}\pi_{2k-1}(SU(11)) = \begin{cases} \mathbf{Z}/16 \oplus \mathbf{Z}/2^{e(k,11)-4} & k \equiv 0\ (4) \\ \mathbf{Z}/8 \oplus \mathbf{Z}/2^{e(k,11)-3} & k \equiv 2\ (4) \\ \mathbf{Z}/4 \oplus \mathbf{Z}/2^{e(k,11)-2} & k \text{ odd} \end{cases}$$

$$v_1^{-1}\pi_{2k-1}(SU(13)) = \begin{cases} \mathbf{Z}/2 \oplus \mathbf{Z}/8 \oplus \mathbf{Z}/2^{e(k,13)-4} & k \text{ even} \\ \mathbf{Z}/2 \oplus \mathbf{Z}/4 \oplus \mathbf{Z}/2^{e(k,13)-3} & k \text{ odd} \end{cases}$$

*Proof.* For the values of $e(k,n)$, we use `Maple` to compute $\min(\nu(a(k,n)),\dots,\nu(a(k,n+8)))$ for various families of values of $k$, using observed periodicities to focus the values of $k$. The choice of $n+8$ as the largest $j$ in $a(k,j)$ to consider is rather arbitrary. In practice, the minimum almost always occurs for $j \le n+3$. We can use that $\nu(a(k,j)) \ge j-\alpha(j)$ ([6, 8.1]), with $\alpha(j)$ the number of 1's in the binary expansion of $j$, to guarantee that we are not overlooking a smaller value of $\nu(a(k,j))$ which occurs for a larger value of $j$. We can also use the periodicity in $k$ of $e(k,n)$ ([2, p.492]) to obtain information for infinitely many $k$ with a finite number of calculations.

For the groups $v_1^{-1}\pi_{2k-1}(SU(2m+1))$, we use the algorithm described in the proof of Theorem 7.1, observing patterns in the small summands, and using 3.6 for the order. The summands of order 2 and 4 are proved in Section 5, and those of order 8 or 16 could be proved by the same method. The number of summands was proved in 3.7. ■

The homotopy results for $SU(2m)$ are much more complicated. We begin by recording the values of $e(k,2m)$ and $G_k$, which are a first step toward writing $v_1^{-1}\pi_*(SU(2m))$.

**Theorem 8.2.** *If $k$ is odd, then $e(k,2m)=e(k,2m+1)$. For even $k$, we have*

$$\begin{aligned}
e(k,4) &= 3\\
e(k,6) &= \min(6,3+\nu(k-4))\\
e(k,8) &= \begin{cases} 7 & k\equiv 0\ (4)\\ \min(9,5+\nu(k-6)) & k\equiv 2\ (4)\end{cases}\\
e(k,10) &= \begin{cases} \min(12,6+\nu(k-8)) & k\equiv 0\ (4)\\ \min(9,6+\nu(k-6)) & k\equiv 2\ (4)\end{cases}\\
e(k,12) &= \begin{cases} \min(13,8+\nu(k-8)) & k\equiv 0\ (4)\\ \min(15,8+\nu(k-74)) & k\equiv 2\ (4)\end{cases}
\end{aligned}$$

*Also, $E_2^{2,2k+1}(SU(2m))\approx G_k\oplus\mathbf{Z}_2$, where, if $k$ is odd, $G_k$ equals the groups $v_1^{-1}\pi_{2k-1}(SU(2m+1))$ given in the second part of 8.1, while, if $k$ is even, then*

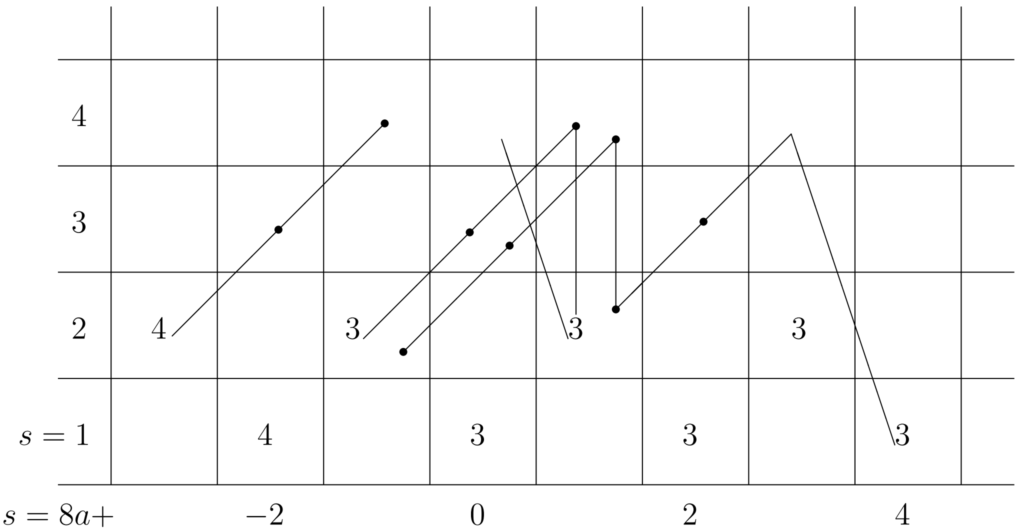
$$G_k=\begin{cases}\mathbf{Z}/2^{e(k,4)} & m=2\\ \mathbf{Z}/2\oplus\mathbf{Z}/2^{e(k,6)-1} & m=3\\ \mathbf{Z}/8\oplus\mathbf{Z}/2^{e(k,8)-3} & m=4\\ \mathbf{Z}/4\oplus\mathbf{Z}/2^{e(k,10)-2} & m=5\\ \mathbf{Z}/2\oplus\mathbf{Z}/8\oplus\mathbf{Z}/2^{e(k,12)-4} & m=6, k\equiv 2\ (4)\\ \mathbf{Z}/2\oplus\mathbf{Z}/16\oplus\mathbf{Z}/2^{e(k,12)-5} & m=6, k\equiv 0\ (4)\end{cases}$$

*Proof.* The first sentence was proved in [5, 1.2]. The last part is calculated using the algorithm which produced Tables 5 and 6 and is described just after (7.12). ■

Diagrams 8.3, 8.4, 8.5, 8.6, and 8.7 present the results for $SU(4)$, $SU(6)$, $SU(8)$, $SU(10)$, and $SU(12)$. Complete explicit formulas for all these homotopy groups can be read off from these charts together with Theorem 8.2. For the most part, these are Diagrams 1.5 and 1.6 with the specific information of 8.2 inserted, along with information about differentials and extensions given in 6.2, 6.5, 7.1, and 7.16. More delicate arguments for specific differentials and extensions are described later in this section.

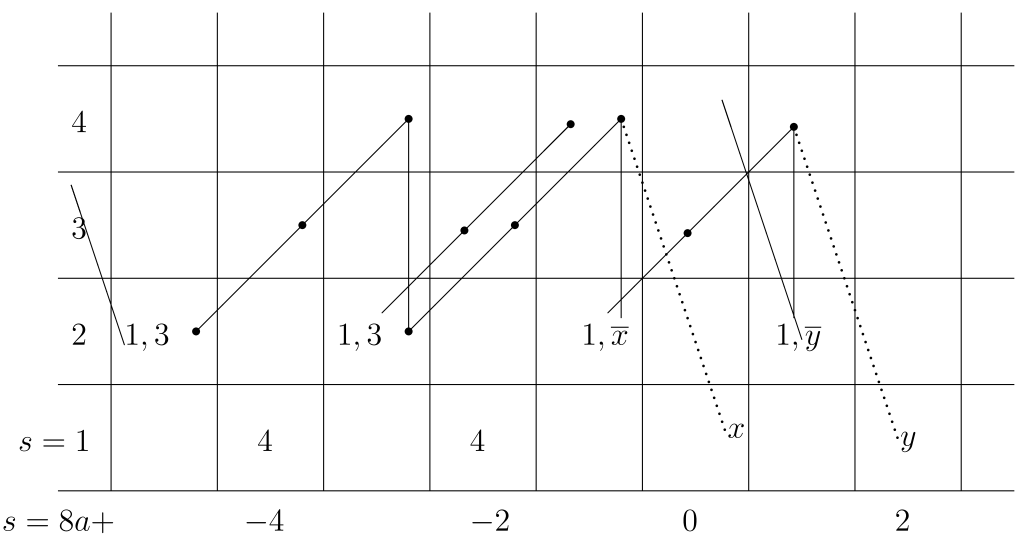
In these charts an integer $e$ or letter $e$ refers to a $\mathbf{Z}/2^e$ summand. If an extension is not depicted, it is trivial. As in Diagrams 1.5 and 1.6, many elements involved in differentials are not pictured.

**Diagram 8.3.** $v_1^{-1}\pi_*(SU(4))$




In the following diagram $x = e(4a, 6)$, as given in 8.2, $y = e(4a + 1, 6)$, $\overline{x} = x - 1$, and $\overline{y} = y - 1$. We have $5 \leq x \leq 6$ and $5 \leq y \leq 8$.

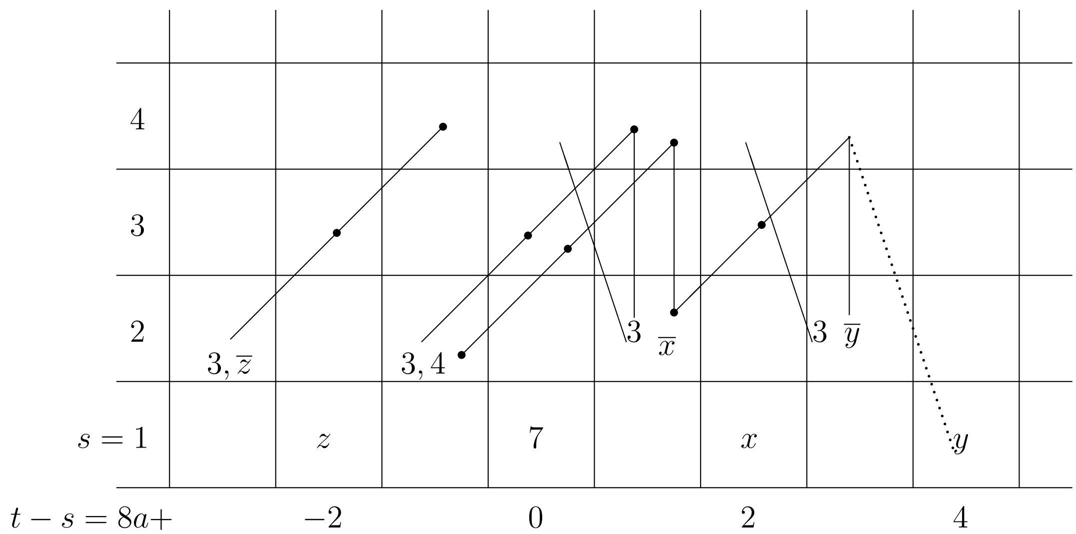
**Diagram 8.4.** $v_1^{-1}\pi_*(SU(6))$




In Diagram 8.4, the $d_3$-differential from $x$ is nonzero iff $a \not\equiv 3 \bmod 4$. The $d_3$-differential from $y$ is nonzero iff $a \equiv 3 \bmod 8$. The extension in $8a + 1$ goes from $\overline{y}$ if $a \not\equiv 3 \bmod 4$, and from the 1 if $a \equiv 7 \bmod 8$. Note that differentials and extensions are emanating from specific summands, but we do not specify on which summand $\eta$ is acting.

In Diagram 8.5, $x = e(4a+1, 8)$, $y = e(4a+2, 8)$, $z = e(4a-1, 8)$, $\overline{x} = x-3$, $\overline{y} = y-3$, and $\overline{z} = z-3$. We have $7 \le x \le 9$, $7 \le y \le 8$, and $7 \le z \le 11$.

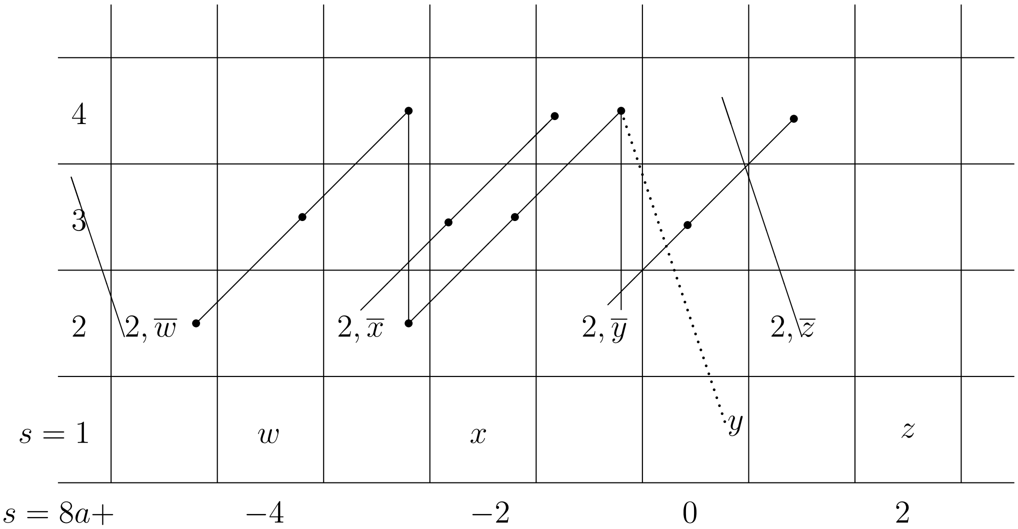
**Diagram 8.5.** $v_1^{-1}\pi_*(SU(8))$




The differential on $y$ in Diagram 8.5 is nonzero if and only if $a \equiv 1 \bmod 4$.

In Diagram 8.6, $w = e(4a-2, 10)$, $x = e(4a-1, 10)$, $y = e(4a, 10)$, $z = e(4a+1, 10)$, $\overline{w} = w-2$, $\overline{x} = x-2$, $\overline{y} = y-2$, and $\overline{z} = z-2$. We have $8 \le w \le 9$, $8 \le x \le 11$, $8 \le y \le 12$, and $8 \le z \le 14$.

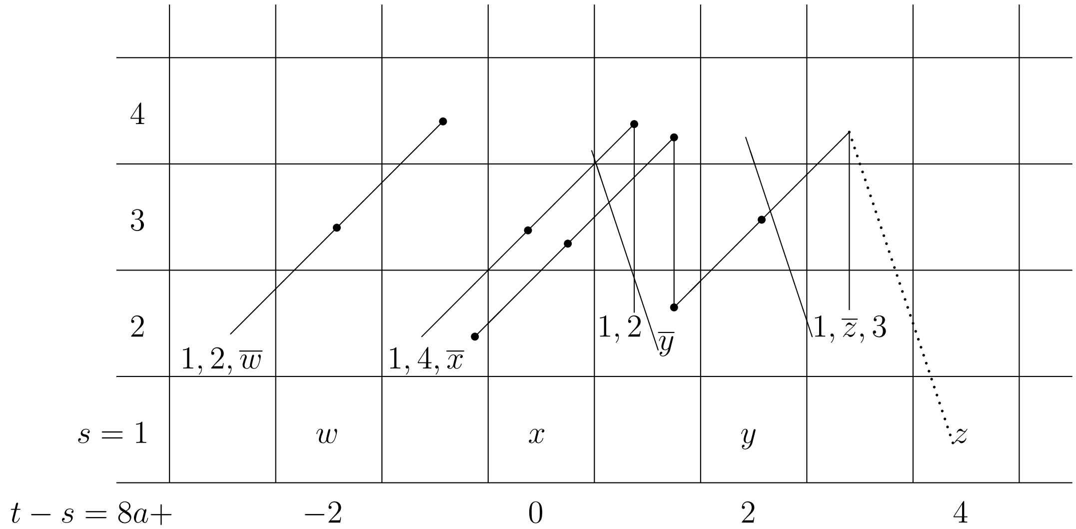
**Diagram 8.6.** $v_1^{-1}\pi_*(SU(10))$




In Diagram 8.6, the differential from $y$ is nonzero iff $a \not\equiv 18$ mod 32.

In Diagram 8.7, $w = e(4a-1, 12)$, $x = e(4a, 12)$, $y = e(4a+1, 12)$, $z = e(4a+2, 12)$, $\overline{w} = w-3$, $\overline{x} = x-5$, $\overline{y} = y-3$, and $\overline{z} = z-4$. We have $11 \le w \le 15$, $10 \le x \le 13$, $10 \le y \le 14$, and $10 \le z \le 15$. In this diagram, the differential from $z$ is nonzero iff $a \not\equiv 18$ mod 64. Extensions are as specifically indicated, from $\overline{y}$ in 8.6 and from 2 and $\overline{z}$ in 8.7. Also, differentials are from the summands specifically indicated, 2 and $\overline{z}$ in 8.6 and 1 in 8.7.

**Diagram 8.7.** $v_1^{-1}\pi_*(SU(12))$




Here we provide some detailed arguments for the above charts.

**Case 1:** $SU(6)$**:**

(1) The differential from $y$ is not pictured in Diagram 1.5 (which excludes $SU(6)$) because it is conjectured (see 6.6 and 9.7) that $m = 3$ is the only time this differential is nonzero. When $m = 3$, it is easily verified to be nonzero iff $a \equiv 3 \bmod 8$, using 6.5, 8.1, and 8.2.

(2) The differential from $x$ was already noted in [2, p.493].

(3) The differential from the 1 in $1, 3$ is Corollary 7.16.

(4) The differential from $\overline{y}$ in $1, \overline{y}$ is Theorem 7.1.

(5) That the extension in $8a-1$ is from $\overline{x}$ is proved similarly to Proposition 6.2(4). Here $E_2^{2,2k+1}(SU(5)) \to E_2^{2,2k+1}(SU(6))$ sends $\mathbf{Z}/2^6$ onto the $\mathbf{Z}/2^5$ summand, and so $2^5 \cdot$ gen is in the image from $E_2^{1,2k+1}(S^{11})$. The element of $E_2^{1,2k+1}(S^{11})$ supports a differential, and consideration of the homotopy exact sequence implies the extension in $SU(6)$.

(6) To compute the extension in $8a+1$, we compute $E_2^{2,2k+1}(SU(6)) \to E_2^{2,2k+1}(S^{11})$ as in Section 7, obtaining that the $\mathbf{Z}_2$ injects if $k \equiv 5$

mod 8 but not if $k \equiv 1$ mod 8, while the large summand is an isomorphism of $\mathbf{Z}/2^4$ if $k \equiv 1$ mod 8, an isomorphism of $\mathbf{Z}/2^5$ if $k \equiv 5$ mod 16, and a surjection from a larger summand onto $\mathbf{Z}/2^5$ if $k \equiv 13$ mod 16. The extension follows from this information using the argument just after the algorithm in the proof of 7.1. When $k \equiv 5$ mod 16, so that both summands inject, the extension is from the larger summand, by rechoosing the generator of smaller order.

**Case 2:** $SU(8)$: The only part of the proof that is not immediate from results cited just prior to Diagram 8.3 is the differential and extension from $G_{4a+2} \approx \mathbf{Z}/2^3 \oplus \mathbf{Z}/2^{\overline{y}}$. If $a \not\equiv 1$ mod 4, there is an exact sequence (with $t = 8a+5$)

$$0 \to E_2^{1,t}(S^{15}) \to E_2^{2,t}(SU(7)) \to E_2^{2,t}(SU(8)) \to E_2^{2,t}(S^{15}) \to 0$$

which is, ignoring a split $\mathbf{Z}/2$ in $E_2^{2,t}(SU(8))$ and $E_2^{2,t}(S^{15})$,

$$0 \to \mathbf{Z}/2 \to \mathbf{Z}/4 \oplus \mathbf{Z}/2^{e+1} \to \mathbf{Z}/8 \oplus \mathbf{Z}/2^e \to \mathbf{Z}/2 \to 0$$

with $e = 4$ or 5. The $\mathbf{Z}/8$ must map onto the $\mathbf{Z}/2$ and hence support the differential (since the $\mathbf{Z}/2$ does in $S^{15}$), and the extension must be from the $\mathbf{Z}/2^e$ by the argument in the proof of 6.2(4).

If $a \equiv 1$ mod 4, there is an exact sequence

$$0 \to E_2^{2,t}(SU(7)) \to E_2^{2,t}(SU(8)) \to E_2^{2,t}(S^{15}) \to 0$$

which is, again ignoring a $\mathbf{Z}/2$ in the latter two groups,

$$0 \to \mathbf{Z}/4 \oplus \mathbf{Z}/2^6 \to \mathbf{Z}/8 \oplus \mathbf{Z}/2^6 \to \mathbf{Z}/2 \to 0$$

and so again the $\mathbf{Z}/8$ must map onto the $\mathbf{Z}/2$ and hence support the differential. There is no extension in this case.

**Case 3:** $SU(10)$:

(1) That there is no extension in $8a+1$ was discussed in detail in Proposition 7.9 and the paragraphs preceding it.

(2) That the differential from $G_{4a-2}$ is from the $\mathbf{Z}/4$ is proved similarly to the proof of Corollary 7.15. When the pivoting algorithm is applied to (7.12) for $SU(10)$, the $\mathbf{Z}/2$ comes from a row with 2 in the second column, and then the $\mathbf{Z}/4$ comes from a row with

4's in columns 1 and 7. The image of these in the matrix (7.17) for $S^{19}$ is the row (0 0 2 0) for the $\mathbf{Z}/2$, and this is the stable generator, which does not support a differential, while for the $\mathbf{Z}/4$ the image is the unstable generator (0 2 0 2), which does support a differential.

(3) For the extension from the big summand in $G_{4a}$ when $a \equiv 18 \bmod 32$, Theorems 8.1 and 8.2 imply that there is an exact sequence (with $t = 8a + 1$)

$$0 \to E_2^{1,t}(S^{19}) \to E_2^{2,t}(SU(9)) \to E_2^{2,t}(SU(10)) \to E_2^{2,t}(S^{19}) \to 0$$

which is

$$0 \to \mathbf{Z}/2 \to \mathbf{Z}/2 \oplus \mathbf{Z}/2^{11} \to \mathbf{Z}/4 \oplus \mathbf{Z}/2^{10} \oplus \mathbf{Z}/2 \to \mathbf{Z}/2 \oplus \mathbf{Z}/2 \to 0.$$

By the proof of 6.2(4), this implies that the extension is from the $\mathbf{Z}/2^{10}$.

**Case 4:** $SU(12)$: Everything here follows by methods used in the three previous cases, using Theorem 7.1, Corollary 7.16, and Theorems 8.1 and 8.2.

## 9. Combinatorial conjectures

In this section we present two combinatorial conjectures which have implications about the $v_1$-periodic homotopy groups of $SU(n)$. Both involve the numbers $a(k,j)$ and $e(k,n)$ which appear in Definition 1.1.

A particularly attractive conjecture is one involving large $v_1$-periodic homotopy groups, because they give estimates for large actual homotopy groups. The $p$-exponent of a space $X$, denoted $\exp_p(X)$, is defined to be the largest $e$ such that $\pi_*(X)$ has an element of order $p^e$. Since $v_1$-periodic homotopy groups are, for spheres and compact Lie groups, direct summands of actual homotopy groups, computations of $v_1^{-1}\pi_*(X)$ lead to lower bounds for $\exp_p(X)$. Lower bounds for $\exp_p(SU(n))$ when $p$ is odd were obtained by the first author, using a different method, in [14].

The following conjecture about 2-divisibility is based on extensive computation. It is conjecturally sharp for $2^e < n \le 2^e + 6$ ($e \ge 4$) and $2^e + 8 < n \le 2^e + 12$ ($e \ge 5$) and other similar ranges. It has also been verified in many cases that the largest value of $e(k,n)$ occurs when $k = 2^L + n - 1$.

**Conjecture 9.1.** *If $L > n + \nu_2([n/2]!)$, then*

$$e(2^L + n - 1, n) \geq n - 1 + \nu_2([n/2]!).$$

Alternatively,

$$\nu(a(2^L + n - 1, j)) \geq n - 1 + \nu([n/2]!) \text{ for all } j \geq n.$$

As is well-known, the $\nu([n/2]!)$ here can be replaced by $\sum_{t\geq 2}[n/2^t]$, which shows some similarity to the odd-primary result of [14], which, we emphasize, was derived in an entirely different manner. Another well-known expression for $\nu(m!)$ is $m - \alpha(m)$, where $\alpha(m)$ is the number of 1's in the binary expansion of $m$.

The significance is given by the following elementary result.

**Proposition 9.2.** *If Conjecture 9.1 is true for $n$, then* $\exp_2(SU(n)) \geq n - 1 + \nu([n/2]!)$.

*Proof.* The conjecture implies that, for $k = 2^L + n - 1$, $E_2^{1,2k+1}(SU(n))$ contains an element of order $2^e$ with $e = n-1+\nu([n/2]!)$. The same will be true of $v_1^{-1}\pi_{2k}(SU(n))$, which is clear when $n$ is odd, while if $n$ is even, we use Diagrams 1.5 and 1.6 to see that there is no differential from the relevant 1-line group. Finally, as observed at the beginning of the section, this implies the same for some actual homotopy group. ∎

We point out one reduction of Conjecture 9.1.

**Proposition 9.3.** *If it is true that*

$$\nu\left(\sum \binom{j}{2\ell}\ell^{n-1}\right) \geq \nu([n/2]!) \textit{ for all } j \geq n, \tag{9.4}$$

*then Conjecture 9.1 is true.*

*Proof.* Let $j \geq n$. We write

$$a(2^L + n - 1, j) = \sum_{\text{odd } i} \binom{j}{i} i^{n-1}(i^{2^L} - 1) + \sum_{\text{odd } i} \binom{j}{i} i^{n-1}.$$

We note that, for odd $i$, $\nu(i^{2^L}-1) \geq L+1$, and so, by our assumption $L > n+\nu([n/2]!)$, the terms in the first sum will not affect whether or not the RHS is divisible by $2^{n-1+\nu([n/2]!)}$. Thus we omit the first sum. We also note that $\sum_{\text{all } i}(-1)^i\binom{j}{i}i^{n-1} = 0$, since $j \geq n$. One way to see this is that it equals $j!S(n-1,j)$, where $S(-,-)$ denotes a Stirling number of the second kind, and this is 0 since $n-1<j$. Consequently

$$\nu(a(2^L+n-1,j)) = \nu\left(\sum_{\text{even } i} \tbinom{j}{i} i^{n-1}\right) = n-1+H, \tag{9.5}$$

where $H$ is the left hand side of (9.4). Here we have written $i = 2\ell$ and factored $2^{n-1}$ out of $(2\ell)^{n-1}$. If $H \geq \nu([n/2]!)$, then (9.5) becomes 9.1. ∎

Conjecture 6.6 for $n(=2m) \geq 18$ is implied by similar conjectures.

**Conjecture 9.6.** *Let $n \equiv 2$ mod 4.*

(1) *If $n \geq 18$, then* $\nu\left(\sum \tbinom{n}{2\ell}\ell^{n-1}\right) < n-3-[\log_2(n-2)]$.

(2) *If $n \geq 6$, and $A \geq 1$, then*

$$\nu\left(\sum_{\text{odd } i} \tbinom{n}{i}(i^{2^{n-3}A}-1)i^{n-1}\right) \geq 2n-4-[\log_2(n-2)].$$

**Proposition 9.7.** *Conjecture 9.6 for $n$ implies Conjecture 6.6 for $2m = n$.*

*Proof.* One observes from the definitions that Conjecture 6.6 is implied by the statement that, if $n \equiv 2$ mod 4 and $A \geq 1$, then

$$\nu(a(n-1+2^{n-3}A,n)) - e(n-1+2^{n-3}A,n-1) < n-1. \tag{9.8}$$

We will show

$$e(n-1+2^{n-3}A,n-1) = \nu((n-1)!). \tag{9.9}$$

As above, we write

$$a(n-1+2^{n-3}A,n) = \sum_{\text{odd } i} \tbinom{n}{i}(i^{2^{n-3}A}-1)i^{n-1} + 2^{n-1}\sum \tbinom{n}{2\ell}\ell^{n-1}.$$

By Conjecture 9.6 (assumed), the divisibility of this sum of sums is determined by its second sum, and its 2-exponent is less than $2n-4-[\log_2(n-2)]$. The desired conclusion (9.8) follows from the easily-verified fact that

$$(2n-4-[\log_2(n-2)]) - \nu((n-1)!) \leq n-1.$$

We complete the proof by proving (9.9). This will be accomplished by proving that if $j \geq n-1$, then $\nu(a(n-1+2^{n-3}A,j) \geq \nu((n-1)!)$ with equality if $j = n-1$.

Similarly to the proof of 9.3, we have

$$a(n-1+2^{n-3}A,j)=\sum_{\text{odd } i}\binom{j}{i}(i^{2^{n-3}A}-1)i^{n-1}+2^{n-1}\sum_{\ell}\binom{j}{2\ell}\ell^{n-1}+j!S(n-1,j). \tag{9.10}$$

The last term equals $(n-1)!$ if $j=n-1$, and 0 if $j>n-1$. In the first sum, $(i^{2^{n-3}A}-1)$ is divisible by $2^{n-2}$, and $n-2\ge\nu((n-1)!)$ with equality if and only if $n-1$ is a 2-power. The other part is clearly divisible by $2^{n-1}$.

We conclude that if $j>n-1$, all terms of (9.10) have 2-exponent $\ge\nu((n-1)!)$, while if $j=n-1$ and $n-1$ is not a 2-power, (9.10) is $(n-1)!$ plus something which is more highly 2-divisible. If $j=n-1=2^e$, then all the binomial coefficients $\binom{j}{i}$ in the sum are even, and so the sum is again more highly 2-divisible than is $(n-1)!$.

■

Conjecture 6.6 is also true for $n=6$, 10, and 14, and the above argument works when $n=10$. However, 9.6(1) and (9.8) fail when $n=6$ and 14. If $n=6$, the result of Conjecture 6.6 is easily verified using 8.2. For $n=14$, one can easily check that $\nu(a(13+2^{11}A,18))=21$ and so

$$e(13+2^{11}A,14)-e(13+2^{11}A,13)\le 21-10,$$

which implies 6.6 when $n=14$.

The reader should note that Conjectures (9.4) and 9.6(1) deal with bounds on the 2-exponent of the same sum. Our data suggest that (9.4) tends to be sharper.

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