K-THEORY AND IMMERSIONS OF SPATIAL POLYGON SPACES

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Abstract. For ℓ a generic n-tuple of positive numbers, $N(\ell)$ denotes the space of isometry classes of oriented n-gons in $\mathbb{R}^3$ with side lengths specified by ℓ. We determine the algebra $K(N(\ell))$ and use this to obtain nonimmersions of the $2(n-3)$-manifold $N(\ell)$ in Euclidean space for several families of ℓ. We also use obstruction theory to tell exactly when $N(\ell)$ immerses in $\mathbb{R}^{4n-14}$ for two families of ℓ's.

1. Introduction

If $\ell = (\ell_1, \dots, \ell_n)$ is an n-tuple of positive real numbers, $N(\ell)$ denotes the space of oriented n-gons in $\mathbb{R}^3$ with consecutive sides of the specified lengths, identified under translation and rotation of $\mathbb{R}^3$. Edges of the polygon are allowed to intersect. A more formal definition is

$$N(\ell) = \{(z_1, \dots, z_n) \in (S^2)^n : \sum_{i=1}^{n} \ell_i z_i = 0\}/SO(3).$$

See [8], [9], or [11]. It is clear from this definition that the diffeomorphism type of $N(\ell)$ is not affected by changing the order of the ℓ_i's.

Let $[\![n]\!] = \{1, \dots, n\}$. If there is no subset $S \subset [\![n]\!]$ for which $\sum_{i\in S} \ell_i = \sum_{i\notin S} \ell_i$, ℓ is said to be *generic*. If ℓ is generic, then $N(\ell)$ is a $2(n-3)$-manifold. ([8, p.285]) Throughout this paper, ℓ is always assumed to be generic. Generic spatial polygon spaces are classified in [9] by their genetic code, which is a collection of subsets of $[\![n]\!]$ determined by ℓ, which we will define at the beginning of Section 2. The manifolds $N(\ell)$ and $N(\ell')$ are diffeomorphic if and only if they have the same genetic code. All genetic codes for $n \leqslant 9$ are listed in [10]. For example, there are 134 diffeomorphism classes of nonempty spatial 7-gon spaces.

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Hausmann and Knutson determined the integral cohomology ring $H^*(N(\ell))$ in [8]. We give our interpretation of this result, in terms of the genetic code, in Theorem 2.1. Since $H^*(N(\ell))$ is generated by elements of $H^2(N(\ell))$ and is torsion-free, the Chern character effectively determines the complex K-theory algebra $K(N(\ell))$. However, a general statement of this seems too complicated to be useful. In Theorems 2.8, 4.2, and 3.1, we will give explicit results for the ring $K(N(\ell))$ when the genetic code of ℓ is $\{\{n,k\}\}$ or $\{\{n,k,1\}\}$, and in general modulo a certain ideal.

Our main goal is to obtain nonimmersion results for spatial polygon spaces in Euclidean space. We use an old result, which we state later as Theorem 2.11, which tells how K-theoretic Chern classes can yield lower bounds for the geometric dimension of stable vector bundles. This result is applied to the stable normal bundle of $N(\ell)$, using a result of [8] about the cohomology Chern classes of the tangent bundle of $N(\ell)$. We will obtain the following three nonimmersion results. Throughout the paper we let $m = n-3$, so that $N(\ell)$ is a $2m$-manifold. Also, $\alpha(m)$ denotes the number of 1's in the binary expansion of m.

Theorem 1.1. *If the genetic code of ℓ is $\{\{n,k\}\}$ with $k < n$, the $2m$-manifold $N(\ell)$ cannot be immersed in $\mathbb{R}^{4m-2\alpha(m)-1}$.*

This is the same as the standard result that can be proved for CP^m using complex K-theory. For $\alpha(m) \leqslant 8$, these results for CP^m are very close to optimal. (See [5].) Note that CP^m is a spatial polygon space with genetic code $\{\{n\}\}$.([7, Expl 2.6])

Theorem 1.2. *If the genetic code of ℓ is $\{\{n,k,1\}\}$ with $1 < k < n$, the $2m$-manifold $N(\ell)$ cannot be immersed in $\mathbb{R}^{4m-2\alpha(m)-3}$. If k is odd, it cannot be immersed in $\mathbb{R}^{4m-2\alpha(m)-1}$.*

The sets in the genetic code are called *genes*. In the following result, as always, n is the length of ℓ, and $m = n-3$.

Theorem 1.3. *Let $s+1$ be the size of the largest gene of ℓ. Let $M = \max(i - \nu\binom{m+i}{i} : i \leqslant m-s)$. Then the $2m$-manifold $N(\ell)$ cannot be immersed in $\mathbb{R}^{2m+2M-1}$.*

Here and throughout, $\nu\binom{n}{k}$ denotes the exponent of 2 in the binomial coefficient. In Table 1, we will tabulate some values for Theorem 1.3.

Existence of immersions is usually proved by obstruction theory. Recall that since $N(\ell)$ is a $2m$-manifold, it certainly immerses in $\mathbb{R}^{4m-1}$. In Section 5, we prove the following immersion result.

Theorem 1.4. *Let $n \geqslant 5$. If the genetic code of ℓ is $\{\{n,k\}\}$, then $N(\ell)$ can be immersed in $\mathbb{R}^{4m-2}$ if and only if it is not the case that m is a 2-power and k is even. If the genetic code of ℓ is $\{\{n,k,1\}\}$, then $N(\ell)$ can always be immersed in $\mathbb{R}^{4m-2}$.*

Remark 1.5. Optimal immersions*:* Combining Theorems 1.1, 1.2, and 1.4, we find that when $n-3$ is a 2-power, immersions of the $2m$-manifold $N_{n,k}$ in $\mathbb{R}^{4m-1}$ when k is even, and of both $N_{n,k}$ and $N_{n,k,1}$ in $\mathbb{R}^{4m-2}$ when k is odd are optimal.

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2. General results and proof of Theorem 1.1

In this section, we provide the background for all our proofs, and apply them to prove Theorem 1.1.

We assume without loss of generality that $\ell_1 \leqslant \cdots \leqslant \ell_n$. For a length vector ℓ, a subset $S \subseteq [\![n]\!]$ is called *short* if $\sum\limits_{i\in S}\ell_i < \sum\limits_{i\notin S}\ell_i$. A partial order is defined on the power set of $[\![n]\!]$, based on inclusions of sets and sizes of numbers.([9, §4]) The *genetic code* of ℓ is defined to be the set of maximal elements under this ordering in the set of all short subsets containing n. For example, the genetic code of $\ell = (\underbrace{1,\dots,1}_{k},\underbrace{2,\dots,2}_{n-k-1},2n-k-5)$ is $\{\{n,k\}\}$, and the genetic code of $\ell = (\frac{1}{2},\underbrace{1,\dots,1}_{k-1},\underbrace{2,\dots,2}_{n-k-1},2n-k-6)$ is $\{\{n,k,1\}\}$.

In [3], we defined a *gee* to be a gene without listing the n, and an element $\leqslant$ a gee is called a *subgee*. So the subgees are just the subsets T of $[\![n-1]\!]$ such that $T \cup \{n\}$ is short. Our interpretation in [3] of the theorem of [8] is as follows.

Theorem 2.1. *The integral cohomology ring $H^*(N(\ell))$ is generated by degree-2 elements R and V_i, $1 \leqslant i < n$, with relations*

a. *For a subset $S \subseteq [\![n-1]\!]$, let $V_S = \prod\limits_{i\in S} V_i$. Then $V_S = 0$ if S is not a subgee.*
b. *For all i, $RV_i + V_i^2 = 0$.*

c. *For each subgee T with $|T| \geqslant n-2-d$, there is a relation $\mathcal{R}_T$ in $H^{2d}(N(\ell))$:*

$$\sum_{S \not\subset T} R^{d-|S|} V_S = 0.$$

The following useful result is quickly deduced from [8].

Proposition 2.2. *The total Chern class of the tangent bundle $\tau(N(\ell))$ satisfies*

$$(1+R)c(\tau(N(\ell))) = \prod_{i=1}^{n-1}(1+2V_i+R).$$

Proof. Hausmann and Knudson utilize a space UP called the upper path space, and in [8, Remark 7.5c] note that

$$c(\tau(UP)) = (1+R)\prod_{i=1}^{n-1}(1+V_i+R)(1+V_i) \text{ and } (1+R)^2 c(\tau(N(\ell))) = c(\tau(UP)).$$

The result is immediate from this and the relation $V_i^2 + RV_i = 0$. ■

Note that $V_i = 0$ unless $\{i\}$ is a subgee of ℓ, which usually allows for cancellation of a factor $(1+R)$.

Corollary 2.3. *There are complex line bundles L_R and L_i, $1 \leqslant i \leqslant n-1$, over $N(\ell)$ such that $c_1(L_R) = R$, $c_1(L_i) = V_i$, and the complex bundles $L_R \oplus \tau(N(\ell))$ and $\bigoplus_{i=1}^{n-1} L_i^2 L_R$ are stably isomorphic.*

Proof. Since c_1 defines a bijection of complex line bundles over X with $H^2(X;\mathbb{Z})$ satisfying $c_1(L_1L_2) = c_1(L_1)+c_1(L_2)$, the two complex bundles in the corollary exist and have the same total Chern class by Proposition 2.2, and hence have the same Chern character. Since $ch : K(X)\otimes\mathbb{Q} \to H^*(X;\mathbb{Q})$ is an isomorphism, the stable bundles are equal in $K(X)$ mod torsion. Since $H^*(N(\ell))$ is confined to even dimensions, $K(N(\ell))$ has no torsion, and so the bundles are stably isomorphic. ■

If $V_i = 0$, then $L_i^2 L_R = L_R$.

Now we apply these results to $N_{n,k}$, which is defined to be $N(\ell)$ when the genetic code of ℓ is $\{\{n,k\}\}$. First we have the result for integral cohomology.

Proposition 2.4. *There is a ring isomorphism*

$$\begin{aligned} H^*(N_{n,k}) &= \mathbb{Z}[R, V_1, \ldots, V_k]/(V_iV_j = 0 \text{ if } i \neq j,\ V_i^2 = -RV_i, \\ &\quad V_1^m = \cdots = V_k^m, R^m = (-1)^m(k-1)V_i^m, R^{m+1} = 0 = V_i^{m+1}), \end{aligned} \tag{2.5}$$

with $|R| = |V_i| = 2$. *Bases for the nonzero groups* $\widetilde{H}^{2j}(N_{n,k})$ *are given by* $\{R^j, V_1^j, \ldots, V_k^j\}$, $1 \leqslant j < m$, *and* $\{V_1^m\}$.

Proof. The only nonempty subgees are $\{i\}$ with $1 \leqslant i \leqslant k$. Relations of type c only occur in grading m. So, using relation b, a basis in grading less than m consists of just the jth powers. In grading m, there is a relation $\mathcal{R}_i$ for each i from 1 to k of the form

$$R^m + \sum_{j \neq i} R^{m-1}V_j = 0,$$

or equivalently

$$R^m + (-1)^{m-1} \sum_{j \neq i} V_j^m = 0.$$

This system clearly reduces to the claim in (2.5). ■

Next we deduce the additive structure of $K(N_{n,k})$.

Definition 2.6. *For any* ℓ, *let* L_R *and* L_i *be the complex line bundles over* $N(\ell)$ *with* $c_1(L_R) = R$ *and* $c_1(L_i) = V_i$. *Let* $\alpha_i = [L_i - 1]$ *and* $\beta = [L_R - 1] \in \widetilde{K}(N(\ell))$.

Proposition 2.7. *The abelian group* $\widetilde{K}(N_{n,k})$ *is free with basis*

$$\{\alpha_1^j : 1 \leqslant j \leqslant m\} \cup \{\alpha_2^j, \ldots, \alpha_k^j, \beta^j : 1 \leqslant j < m\}.$$

Proof. We have $ch(\alpha_i^j) \equiv V_i^j$ and $ch(\beta^j) \equiv R^j$, where $\equiv$ means mod terms of higher degree. So the specified α_i^j and β^j are elements in $\widetilde{K}(N_{n,k})$ on which the first component of ch gives an isomorphism to a basis for $H^*(N_{n,k})$. Standard methods imply that for a space with only even-dimensional cohomology and no torsion, this implies the claim.

Indeed, let $K_{2j}(N_{n,k})$ denote the quotient of $K(N_{n,k})$ by elements trivial on the $(2j-1)$-skeleton. A result in [1], along with collapsing of the Atiyah-Hirzebruch spectral sequence, implies that

$$ch_j : K_{2j}(N_{n,k})/K_{2j+2}(N_{n,k}) \to H^{2j}(K_{n,k})$$

is an isomorphism. Now by downward induction the split short exact sequence

$$0 \to K_{2j+2}(N_{n,k}) \to K_{2j}(N_{n,k}) \to H^{2j}(N_{n,k}) \to 0$$

implies that each $K_{2j}(N_{n,k})$ is a free abelian group generated by powers $\geqslant j$.

■

Theorem 2.8. *The multiplicative relations in $K(N_{n,k})$ are $\alpha_i\alpha_j = 0$ for $i \neq j$, $\alpha_i\beta = -\alpha_i^2/(1+\alpha_i)$, $\alpha_1^m = \cdots = \alpha_k^m$, $\beta^m = (-1)^m(k-1)\alpha_i^m$, $\beta^{m+1} = 0 = \alpha_i^{m+1}$.*

Proof. Since ch is an isomorphism $K(X)\otimes\mathbb{Q} \to H^*(X;\mathbb{Q})$ and we have seen that bases correspond, it suffices to show that ch sends asserted relations to 0. Using the cohomology relations stated in (2.5), all are clear except the one for $\alpha_i\beta$. Letting $V = V_i$ and noting that when multiplied by V, R acts like $-V$, we have

$$ch(\alpha_i\beta) = (e^V-1)(e^R-1) = (e^V-1)(e^{-V}-1) = -(e^V-1)\tfrac{e^V-1}{e^V} = -\tfrac{(e^V-1)^2}{1+(e^V-1)} = ch(\tfrac{-a_i^2}{1+\alpha_i}).$$

■

Lemma 2.9. *For any i, $L_i^2 L_R - 1 = 2\alpha_i + \beta + \alpha_i\beta \in \widetilde{K}(N(\ell))$.*

Proof. Let $V = V_i$. Since R acts as $-V$ when multiplied by V, we have in $H^*(N(\ell))$

$$(2V+R)^j = R^j + ((2V-V)^j - (-V)^j) = \begin{cases} 2V^j + R^j & j \text{ odd} \\ R^j & j \text{ even.} \end{cases} \qquad (2.10)$$

We obtain

$$\begin{aligned} ch(L_i^2 L_R - 1) &= \sum_{j\geqslant 1} \tfrac{(2V+R)^j}{j!} \\ &= 2(V + \tfrac{V^3}{3!} + \tfrac{V^5}{5!} + \cdots) + e^R - 1 \\ &= (e^V - 1) - (e^{-V} - 1) + e^R - 1 \\ &= (e^V - 1) + (e^V - 1)(1 + e^{-V} - 1) + e^R - 1 \\ &= 2(e^V - 1) + (e^V - 1)(e^{-V} - 1) + (e^R - 1) \\ &= ch(2\alpha_i + \alpha_i\beta + \beta). \end{aligned}$$

Hence we have equality in $K(N(\ell))$. ■

To obtain nonimmersions, we use the following result, which we quote from [2, Theorem 4.4], although earlier versions apparently exist.

Theorem 2.11. *For complex bundles θ over finite complexes, there are natural classes $\Gamma(\theta) \in K(X) \otimes \mathbb{Z}[\frac{1}{2}]$ satisfying $\Gamma(\theta_1 + \theta_2) = \Gamma(\theta_1)\Gamma(\theta_2)$ and $\Gamma(L) = 1 + \frac{L-1}{2}$ if L is a (complex) line bundle. If θ is a stably complex vector bundle over a finite complex X which, as a real bundle, is stably isomorphic to a bundle of (real) dimension $2s + 1$, then $2^s\Gamma(\theta) \in K(X)$.*

The components of $\Gamma(X)$ are just divided versions of the K-theoretic Chern classes.

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Let η be the normal bundle of an immersion of $N_{n,k}$ in $\mathbb{R}^t$ for some t. By Corollary 2.3, the tangent bundle of $N_{n,k}$ is stably isomorphic to

$\bigoplus_{i=1}^{k} L_i^2 L_R \oplus (n-2-k)L_R$, and hence, with $\Gamma(-)$ as in Theorem 2.11,

$$\begin{aligned}
\Gamma(\eta) &= \Gamma(\tau)^{-1} = \prod_{i=1}^{k} \Gamma(L_i^2 L_R)^{-1} \cdot \Gamma(L_R)^{-(m+1-k)} \\
&= \prod_{i=1}^{k} (1+\alpha_i+\tfrac{1}{2}\beta+\tfrac{1}{2}\alpha_i\beta)^{-1} \cdot (1+\tfrac{1}{2}\beta)^{-(m+1-k)} \\
&= \prod_{i=1}^{k} (1+\alpha_i)^{-1} \cdot (1+\tfrac{1}{2}\beta)^{-(m+1)} \qquad (2.12) \\
&= (1+\tfrac{1}{2}\beta)^{-(m+1)} + \textstyle\sum_{i,j} c_{i,j}\alpha_i^j,
\end{aligned}$$

using the relations in Theorem 2.8 in the last step. Here $c_{i,j}$ is an element of $\mathbb{Z}_{(2)}$ whose value does not matter. These terms are independent by Proposition 2.7, and one of the terms is $\binom{-m-1}{m-1}\beta^{m-1}/2^{m-1}$. The binomial coefficient here is $(-1)^{m-1}\binom{2m-1}{m-1} = (-1)^{m-1}\frac{1}{2}\binom{2m}{m}$, so its exponent of 2 is $\alpha(m)-1$. Thus $2^{m-\alpha(m)-1} \cdot \binom{-m-1}{m-1}\beta^{m-1}/2^{m-1}$ is not integral, and so by Theorem 2.11, η is not stably equivalent to a bundle of dimension $2(m-\alpha(m)-1)+1$. If a d-dimensional manifold immerses in $\mathbb{R}^{d+c}$, then its normal bundle is c-dimensional. Thus we obtain the theorem. ■

Exactly as we did for planar polygon spaces in [4], we can prove that the total Stiefel-Whitney class of the tangent bundle of $N(\ell)$ is $(1+R)^{m+1} \in H^*(N(\ell);\mathbb{Z}_2)$ with $|R| = 2$, where m, as throughout this paper, is 3 less than the length of ℓ. Similarly to [4, Corollary 1.5], for each m from 16 to 31, Stiefel-Whitney classes give a nonimmersion of $N_{n,k}$ in $\mathbb{R}^{61}$. For these values of m, Theorem 1.1 gives nonimmersions of $N_{n,k}$ in Euclidean spaces of the following dimensions: 61, 63, 67, 69, 75, 77, 81, 83, 91, 93, 97, 99, 105, 107, 111, and 113.

3. Proof of Theorem 1.3

We can expand these results easily if we restrict to cases in which relations of type c do not appear in $H^*(N(\ell))$. Let s denote the maximal size of a gee of ℓ. In grading $\leqslant 2(m-s)$, there are no type-c relations. In this range, a basis for $H^*(N(\ell))$ consists of all R^iV_S such that S is a subgee and $i+|S| \leqslant m-s$. This includes the empty subgee.

Using the notation and methods used earlier, we have the following result.

Theorem 3.1. *Let s equal the maximal size of a gee of ℓ, and k the largest size-1 subgee of ℓ. The ring $K(N(\ell))$ is generated by $\alpha_1, \dots, \alpha_k$, and β. The quotient, modulo $(m-s+1)$-fold (or more) products of these, is*

$$\mathbb{Z}[\alpha_1, \dots, \alpha_k, \beta]/(\alpha_i\beta + \tfrac{\alpha_i^2}{1+\alpha_i}, \textstyle\prod_{i\in T}\alpha_i\text{: } T \text{ not a subgee}).$$

A basis for this quotient consists of all $\beta^i \prod_{j\in S}\alpha_j$ such that S is a subgee and $i+|S| \leqslant m-s$.

Proof. Since $ch(\alpha_i^j) \equiv V_i^j$ and $ch(\beta^j) \equiv R^j$ mod higher-degree classes, the Atiyah-Hirzebruch argument in the proof of Proposition 2.7 implies that these stated classes generate. The relations are exactly those which give a cohomology relation when ch is applied. The first type of relation is obtained as in the proof of Theorem 2.8, and the second type follows from type-a relations in cohomology. For every type-c relation in cohomology, there is a unique K-theory relation mapping to it, and this relation will involve only $(m-s+1)$-fold (or more) products of the generators. Thus they will not affect the quotient of $K(N(\ell))$ being considered. ■

Proof of Theorem 1.3. Corollary 2.3, Lemma 2.9, and (2.12) all apply exactly as in the proof of Theorem 1.1, as does the argument about independence of pure β classes from those involving any α_i's. The only difference from the situation of the previous section is that all we can assert is that certain multiples of β^i are nonzero for $i \leqslant m-s$. Thus $\Gamma(\eta)$ contains independent terms $2^{-i}\binom{-m-1}{i}\beta^i$ for $i \leqslant m-s$. Note that $\binom{-m-1}{i} = \pm\binom{m+i}{i}$. If M is as in Theorem 1.3, then $\Gamma(\eta)$ contains an independent term $2^{-M}\beta^i$, so $2^{M-1}\Gamma(\eta)$ is not integral. Therefore by Theorem 2.11, the dimension of the normal bundle is greater than $2(M-1)+1$, and so the manifold does not immerse in this codimension. ■

The results implied by Stiefel-Whitney classes are exactly those in which an i which determines M has $\binom{m+i}{i}$ odd. In Table 1, we list the nonimmersion dimension implied by Theorem 1.3 for $2m$-dimensional $N(\ell)$ having largest gee size s. Those having value $\leqslant 61$ (except for the 55 in columns 7 and 8) are also implied by Stiefel-Whitney classes, but 61 is the largest that Stiefel-Whitney classes can imply in any of the tabulated cases.

Table 1. Nonimmersion dimensions implied by Theorem 1.3

m/s	1	2	3	4	5	6	7	8
16	61	59	57	55	53	51	49	47
17	63	61	61	57	57	53	53	49
18	67	65	61	61	61	59	55	53
19	69	67	67	61	61	61	61	55
20	75	73	71	69	63	61	61	61
21	77	75	75	71	71	63	63	61
22	81	79	75	75	75	73	65	63
23	83	81	81	75	75	75	75	65
24	91	89	87	85	83	81	79	77
25	93	91	91	87	87	83	83	79
26	97	95	91	91	91	89	85	83
27	99	97	97	91	91	91	91	85
28	105	103	101	99	95	93	91	89
29	107	105	105	101	101	95	95	91
30	111	109	105	105	105	103	97	95
31	113	111	111	105	105	105	105	97

4. Proof of Theorem 1.2

For $1 < k < n$, let $N_{n,k,1} = N(\ell)$ when the genetic code of ℓ is $\{\{n, k, 1\}\}$. In this section we determine the algebra $K(N_{n,k,1})$ and use it to prove the nonimmersion Theorem 1.2. Throughout we let $m = n - 3$. We begin with (integral) cohomology.

Theorem 4.1. *Let $N = N_{n,k,1}$. The algebra $H^*(N)$ is generated by degree-2 classes $R, V_1, \ldots, V_k$. In grading $2i$ with $2 \leqslant i \leqslant m-2$, the only relations are due to $RV_i = -V_i^2$, and $V_iV_j = 0$ if $2 \leqslant i < j$, so a basis is given by*

$$\{R^i, V_1^i, \ldots, V_k^i, V_1V_2^{i-1}, \ldots, V_1V_k^{i-1}\}.$$

A basis for $H^{2(m-1)}(N)$ is $\{V_1^{m-1}, V_k^{m-1}, V_1V_2^{m-2}, \ldots, V_1V_k^{m-2}\}$, with $V_2^{m-1} = \cdots = V_k^{m-1}$ and $R^{m-1} = (-1)^{m-1}(k-2)V_k^{m-1}$. In $H^{2m}(N) \approx \mathbb{Z}$, we have $R^m = V_2^m = \cdots = V_k^m = 0$, $V_1V_2^{m-1} = \cdots = V_1V_k^{m-1}$ a generator, and $V_1^m = (k-2)V_1V_k^{m-1}$. Finally $H^i(N) = 0$ if $i > 2m$.

Proof. This follows easily from Theorem 2.1. The type-c relations in $H^{2(m-1)}(N)$ are $R^{m-1}+(-1)^{m-2}\sum_{j\geqslant 2, j\neq i} V_j^{m-1}=0$. One can solve this system by row-reduction, but it is easier just to verify that the relations stated in the theorem are consistent with these. Similarly, in $H^{2m}(N)$, one can check that the relations stated in the theorem are consistent with the type-c relations $\mathcal{R}_{1,i}$, $\mathcal{R}_1$, and $\mathcal{R}_i$, $i>1$:

$$R^m+(-1)^{m-1}\sum_{j\geqslant 2, j\neq i} V_j^m=0, \qquad R^m+(-1)^{m-1}\sum_{j=2}^{k} V_j^m=0,$$

and

$$R^m+(-1)^{m-1}\sum_{j\neq i} V_j^m+(-1)^{m-2}\sum_{j\geqslant 2, j\neq i} V_1V_j^{m-1}=0.$$

■

Theorem 4.2. *The algebra $K(N_{n,k,1})$ is generated by classes $\alpha_1,\ldots,\alpha_k,\beta$ with relations $\alpha_i\beta=-\alpha_i^2/(1+\alpha_i)$, $\alpha_i\alpha_j=0$ if $2\leqslant i<j$, $\alpha_2^{m-1}=\cdots=\alpha_k^{m-1}$, and $\beta^{m-1}=(-1)^{m-1}(k-2)\alpha_k^{m-1}$. A basis consists of*

$$\{1,\alpha_i^j\ (1\leqslant i\leqslant k,\ 1\leqslant j\leqslant m-2),\beta,\ldots,\beta^{m-2},\alpha_1^{m-1},\alpha_k^{m-1},$$
$$\alpha_1\alpha_i^j(2\leqslant i\leqslant k,\ 1\leqslant j\leqslant m-2),\ \alpha_1\alpha_k^{m-1}\}.$$

Proof. The generators are as defined previously, and the relations are easily established by applying ch and using the cohomology relations. The stated K-theory relations imply the following additional relations: $\alpha_1^t\alpha_i^j=\alpha_1\alpha_i^{j+t-1}$, $\alpha_i^m=0$ $(2\leqslant i\leqslant k)$, $\beta^m=0$, and $\alpha_1^m=(k-2)\alpha_1\alpha_k^{m-1}$. These relations enable reduction of everything to terms in our asserted basis, and the listed monomials are linearly independent since applying ch to them yields linearly independent cohomology classes. ■

Remark 4.3. Although in the cases considered here the K-theory relations have corresponded exactly to the cohomology relations (except for the $\alpha_i\beta$ relation), this will not continue. We have studied the case in which the genetic code is $\{\{n,k,2\}\}$. In this case some K-theory relations contain an additional term. The complicated details cause us not to include it in this paper.

Proof of Theorem 1.2. The first part follows from Theorem 1.3, using $i = m-2$ if m is even, and $i = m-3$ if m is odd. Standard techniques imply that $\nu\binom{2m-2}{m-2} = \alpha(m)-1$ if m is even, and $\nu\binom{2m-3}{m-3} = \alpha(m)-2$ if m is odd. In both cases $i-\nu\binom{m+i}{i} = m-\alpha(m)-1$.

We might hope to use $2^{-(m-1)}\binom{2m-1}{m-1}\beta^{m-1} = 2^{\alpha(m)-m}u\beta^{m-1}$ with u odd to show that $2^{m-\alpha(m)-1}\Gamma(\eta)$ is not integral and deduce a nonimmersion in $\mathbb{R}^{4m-2\alpha(m)-1}$. If k is even, the relation $\beta^{m-1} = (-1)^{m-1}(k-2)\alpha_k^{m-1}$ prevents this from working. We will show that it works when k is odd, but since β^{m-1} is not independent of α_k^{m-1}, we must consider other terms. We have, similarly to (2.12),

$$\begin{aligned}
\Gamma(\eta) &= \prod_{i=1}^{k}((1+\alpha_i)(1+\tfrac{1}{2}\beta))^{-1}\cdot(1+\tfrac{1}{2}\beta)^{-(m-k+1)}\\
&= (1+\alpha_1)^{-1}(1+\alpha_2+\cdots+\alpha_k)^{-1}(1+\tfrac{1}{2}\beta)^{-(m+1)}\\
&= (1+\alpha_2+\cdots+\alpha_k)^{-1}\big((1+\tfrac{1}{2}\beta)^{-(m+1)} - \tfrac{\alpha_1}{1+\alpha_1}(1-\tfrac{\alpha_1}{2(1+\alpha_1)})^{-(m+1)}\big)\\
&= \big(1+\sum_{t>0,j\geqslant 2}\alpha_j^t\big)\big((1+\tfrac{1}{2}\beta)^{-(m+1)} - \alpha_1(1+\alpha_1)^m(1+\tfrac{1}{2}\alpha_1)^{-(m+1)}\big).
\end{aligned}$$

The desired term, $2^{-(m-1)}\binom{-m-1}{m-1}\beta^{m-1}$, is apparent, but we must consider possible cancelling terms.

The terms α_1^t and $\alpha_1\alpha_j^t$ are independent of β^{m-1} and so need not be considered, but we must consider the coefficients of α_j^{m-1} for $2 \leqslant j \leqslant k$, since these terms are related to β^{m-1}. Since (as already used) β acts as $-\alpha_j/(1+\alpha_j)$ when multiplied by α_j, we obtain as possible cancelling terms

$$\sum_{t>0,j\geqslant 2}\alpha_j^t\big(1-\tfrac{\alpha_j}{2(1+\alpha_j)}\big)^{-(m+1)} = \sum_{t>0,j\geqslant 2}\alpha_j^t(1+\alpha_j)^{m+1}/(1+\tfrac{1}{2}\alpha_j)^{m+1}. \quad (4.4)$$

We will prove in Proposition 4.5 that the 2-exponent of the coefficient of x^i in $(1+x)^{m+1}/(1+\frac{1}{2}x)^{m+1}$ is $\geqslant \alpha(m)-m$ for $i \leqslant m-1$. Thus the coefficient of α_j^{m-1} in the RHS of (4.4) has 2-exponent $\geqslant \alpha(m)-m$. Since all these α_j^{m-1} are equal and have the same coefficient, and there is an even number of them, this RHS contains the term $2^v\alpha_k^{m-1}$ with $v > \alpha(m)-m$. Since β^{m-1} is an odd multiple of α_k^{m-1}, the combination of all terms is an odd multiple of $2^{\alpha(m)-m}$ times the basis element α_k^{m-1}, and so $2^{m-\alpha(m)-1}\Gamma(\eta)$ is not integral, from which the result follows. ■

Proposition 4.5. *For $i \leqslant m-1$, $\nu([x^i]:((1+2x)/(1+x))^{m+1}) \geqslant i+\alpha(m)-m$.*

Proof. The desired coefficient equals $\sum\limits_j 2^j\binom{m+1}{j}\binom{-(m+1)}{i-j}$. A formula of Gould ([6, (1.71)] is $\sum 2^j\binom{x}{j}\binom{y}{i-j} = \sum\binom{x}{j}\binom{x+y-j}{i-j}$, and so our coefficient equals

$$\begin{aligned}&\sum_j \binom{m+1}{j}\binom{-j}{i-j} = \pm\sum(-1)^j\binom{m+1}{j}\binom{i-1}{i-j}\\ = \quad &\pm([x^i] : (1-x)^{m+1}(1+x)^{i-1}) = \pm([x^i] : (1-x^2)^{i-1}(1-x)^{m+2-i}).\end{aligned}$$

The evaluation of this coefficient varies slightly with the parity of m and i. We consider here the case $m = 2r$ and $i = 2r - 2t$, with $t > 0$. Other parities lead to very slight modifications.

Up to sign, the coefficient is

$$\sum_{k=r-2t-1}^{r-t}(-1)^k\binom{2r-2t-1}{k}\binom{2t+2}{2r-2t-2k} = \binom{2r}{r}\sum_{k=r-2t-1}^{r-t}\tfrac{(r\cdots(k+1))\cdot(r\cdots(2r-2t-k))}{(2r)\cdots(2r-2t)}\cdot c_k,$$

where $c_k = (-1)^k\binom{2t+2}{2r-2t-2k}$ is an integer whose value is not relevant. For each value of k, either $r - t \geqslant k + 1$ or $r - t \geqslant 2r - 2t - k$. Thus one of the two products in the numerator of the fraction contains the product $r\cdots(r-t)$. We use this to cancel all but the 2 in each of the even factors in the denominator, leaving 2^{t+1} times an odd number in the denominator. Thus every term T in the sum has $\nu(T) \geqslant -(t+1)$. Since $\nu\binom{2r}{r} = \alpha(r) = \alpha(m)$, we obtain that the 2-exponent in our expression is $\geqslant \alpha(m) - t - 1 \geqslant \alpha(m) - 2t = \alpha(m) - (m - i)$, as claimed. ■

Our proof shows that Proposition 4.5 can be strengthened quite a bit, but we settle for what we need.

5. Obstruction theory: proof of Theorem 1.4

We adapt [12, Theorem 5.1] to our situation as follows.

Theorem 5.1. *The $2m$-manifold $N(\ell)$ can be immersed in $\mathbb{R}^{4m-2}$ if and only if*

a. *m is even and there exists $y \in H^{2m-2}(N(\ell);\mathbb{Z}_2)$ such that*

$$i_*(\mathrm{Sq}^2\, y + w_2(\eta)y) = \rho_4(c_m(\eta)) \in H^{2m}(N(\ell);\mathbb{Z}_4),\ \textit{or}$$

b. *m is odd and $\rho_2(c_m(\eta)) = 0 \in H^{2m}(N(\ell);\mathbb{Z}_2)$.*

Here $c_m(\eta) \in H^{2m}(N(\ell);\mathbb{Z})$ is the Chern class of the stable normal bundle η, and i_*, ρ_4, and ρ_2 are induced by coefficient homomorphisms $\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_4$, $\mathbb{Z} \twoheadrightarrow \mathbb{Z}_4$, and $\mathbb{Z} \twoheadrightarrow \mathbb{Z}_2$.

Proof of Theorem 1.4. By Proposition 2.2, the total Chern class of the normal bundle η for any $N(\ell)$ is

$$\prod_{i=1}^{k}(1+2V_i+R)^{-1}\cdot(1+R)^{-(m+1-k)}. \tag{5.2}$$

Using (2.10), we have

$$\begin{aligned}(1+2V_i+R)^{-1} &= \sum_{j\geqslant 0}(-1)^j(2V_i+R)^j\\ &= \sum_{j\geqslant 0}(-1)^jR^j + 2\sum_{j\text{ odd}}(-1)^jV_i^j\\ &= (1+R)^{-1} - 2V_i(1-V_i^2)^{-1}.\end{aligned}$$

Now we specialize to $N_{n,k}$ and $N_{n,k,1}$. Since $V_iV_j = 0$ for $i \neq j$ in $H^*(N_{n,k})$, and R acts as $-V_i$ when multiplied by V_i, we obtain in $H^*(N_{n,k})$

$$\begin{aligned}\prod_{i=1}^{k}(1+2V_i+R)^{-1} &= (1+R)^{-k} - 2(1+R)^{-(k-1)}\sum_{i=1}^{k}V_i(1-V_i^2)^{-1}\\ &= (1+R)^{-k} - 2\sum_{i=1}^{k}V_i(1-V_i^2)^{-1}(1-V_i)^{-(k-1)}.\end{aligned}$$

In $H^*(N_{n,k,1})$, there are additional terms since $V_1V_j \neq 0$, but they will be divisible by 4. Thus, from (5.2), in both $N_{n,k}$ and $N_{n,k,1}$,

$$c_m(\eta) \equiv [(1+R)^{-(m+1)} - 2\sum_{i=1}^{k}V_i(1-V_i^2)^{-1}(1-V_i)^{-m}]_{2m} \mod 4, \tag{5.3}$$

where $[-]_{2m}$ denotes the component in grading $2m$. (Recall $|R| = |V_i| = 2$.) If m is odd, the coefficient of R^m in $(1+R)^{-(m+1)}$ is even, and so Theorem 5.1(b) implies the immersion.

Now we restrict to even values of m. Mod 2, we have, with $|V| = 2$,

$$[V(1-V^2)^{-1}(1-V)^{-m}]_{2m} \equiv V[(1+V)^{-(m+2)}]_{2(m-1)} = \tbinom{-(m+2)}{m-1}V^m \equiv \tbinom{2m}{m-1}V^m.$$

This coefficient is even when m is even, and so the second part of (5.3) is 0 mod 4.

Since $\nu\binom{-(m+1)}{m} = \nu\binom{2m}{m} = \alpha(m)$, we find that, mod 4, $c_m(\eta)$ equals $2R^m$ if m is a 2-power, and is 0 otherwise, so we obtain the immersion when m is not a 2-power. By Theorem 4.1, $R^m = 0$ in $H^{2m}(N_{n,k,1})$ and so we obtain the immersion in this case. By Proposition 2.4, $2R^m \neq 0 \in H^{2m}(N_{n,k};\mathbb{Z}_4)$ iff k is even, and so we obtain the immersion when m is a 2-power and k is odd. Now we evaluate the indeterminacy, $\mathrm{Sq}^2\, y + w_2(\eta)y$, when m is a 2-power and k is even.

Note that $H^{2m}(N_{n,k};\mathbb{Z}_2) \approx \mathbb{Z}_2$, so we just say whether terms are 0. We have $\mathrm{Sq}^2(V_i^{m-1}) \neq 0$ since m is even, and $\mathrm{Sq}^2(R^{m-1}) \neq 0$ since m is even and k is even. Also, $w_2(\eta) = (m+1)R$, so $w_2(\eta)V_i^{m-1} \neq 0$ since m is even, and $w_2(\eta)R^{m-1} \neq 0$ since m is even and k is even. So the indeterminacy is 0 in all cases, and we obtain the nonimmersion for $N_{n,k}$ when m is a 2-power and k is even. ■

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