SOLUTIONS TO 2015 LEHIGH CONTEST, ANNOTATED WITH THE NUMBER ANSWERED CORRECTLY BY THE 49 STUDENTS WITH AT LEAST 20 CORRECT ANSWERS

1. 10/9. [47] It is $\frac{1}{2}/\frac{9}{20}$.
2. 9. [46] Letting W and S denote the original amounts of water and syrup, we have $W = 2S$ and $W + 6 = 4S$. Thus $S = 3$ and $W = 6$.
3. 20. [48] If x is the number of questions on the test, then $4x = x + 60$, so $x = 20$.
4. 30. [48] The outer track has radius 60, and so the inner track will have radius 30.
5. 32/3. [49] Since 2 is 1/6 of $2 + 4 + 6$, then the answer is 1/6 of 64.
6. 81. [48] The top part that is broken is the hypotenuse of a right triangle with legs of length $3 \cdot 9$ and $4 \cdot 9$. So its length must be $5 \cdot 9$, which must be added to the 36.
7. $8 - \frac{4}{3}\pi$. [45] The volume between the two is $4^3 - \frac{4}{3}\pi 2^3$. The amount which is closest to one of the 8 vertices is 1/8 times this.
8. 36. [45] The equation is $0 = |x|^2 - 5|x| + 6 = (|x| - 2)(|x| - 3)$, so $x = \pm 2$ or ± 3.
9. 7. [49] We have $(2b + 4)^2 = 6b^2 + 4b + 2$. This simplifies to $2b^2 - 12b - 14 = 0$, which has roots 7 and -1.
10. 4/7. [48] Since $1+2+3+4+5+6 = 21$, the desired probability is $\frac{2}{21} + \frac{4}{21} + \frac{6}{21} = \frac{12}{21}$.
11. $10\sqrt{2}$. [43] $x^2 + x = 3x + 4$ simplifies to $x^2 - 2x - 4 = 0$, which has solutions $x = 1 \pm \sqrt{5}$. The difference in the x-values is $2\sqrt{5}$, and the difference in the y-values is 3 times as large. Thus the distance between the points is $\sqrt{4 \cdot 5(1 + 9)} = 10\sqrt{2}$.

12. 36. [43] Denote the desired angle by x, and the angle at B by y. The angles at D imply $y + (180 - 2x) = 180$, while equality of the base angles of the isosceles triangle ABC imply $y = x + (180 - 2y)$. We obtain $2x = x + 180 - 4x$ and so $x = 36$.

13. 17/6. [43] The sum is
$$\sum_{n\geq 0}((\tfrac{1}{4})^n + (\tfrac{1}{3})^n) = \frac{1}{1-\frac{1}{4}} + \frac{1}{1-\frac{1}{3}} = \tfrac{4}{3} + \tfrac{3}{2} = \tfrac{17}{6}.$$

14. 12 and 15. [46] If the two integers are A and B, and there are two A's and one B on each side, then there can be five A's and three B's, six and two, or four of each. However, the second and third cases cannot occur since the sum of all the entries must be odd. Thus we have $5A + 3B = 111$ and $2A + B = 42$, hence $A = 15$ and $B = 12$.

15. 4949. [46] Since $\log_b(c) = \frac{\log c}{\log b}$, we obtain $f(n) = \frac{\log n}{\log 2}$. Since $\log(2^k)/\log 2 = k$, the desired expression equals $2+3+\cdots+99 = 99 \cdot 50 - 1 = 4949$.

16. 10. [44] For $x < a$, f will have slope -3. For $a < x < 10$, f will have slope -1. For $10 < x < a + 10$, f will have slope 1. For $x > a + 10$, f will have slope 3. Thus its minimum occurs when $x = 10$, where the value is $(10 - a) + a = 10$.

17. 3. [43] A has 2^{26} subsets, and 1/8 of these are disjoint from B. Thus there are 2^{23} subsets of $A - B$, and so B has $26 - 23 = 3$ elements.

18. $-2/21!$. [16] We obtain
$$\frac{1}{21!}\sum_{k=1}^{21}(-2)^k\binom{21}{k} = \frac{1}{21!}((1-2)^{21} - 1) = \frac{-2}{21!}.$$

19. 5 and 845. [16] If such a circle has radius r, since it is tangent to the x-axis, its center is at (x, r). These satisfy $(x-1)^2 + (r-9)^2 = r^2$ and $(x-8)^2 + (r-8)^2 = r^2$. Subtracting yields $14x - 2r - 46 = 0$ or $r = 7x - 23$. Substitute this into the first

equation and obtain $x^2 - 128x + 496 = 0$, so $x = 4$ or 124, and then $r = 5$ or 845.

20. $\frac{1}{2}\sqrt{3}$. [39] Let $T = \tan\theta$. The expression equals

$$\begin{aligned}\frac{T^{-2} - T^2}{2 + T^{-2} + T^2} &= \frac{(T^{-1} - T)(T^{-1} + T)}{(T^{-1} + T)^2} = \frac{1 - T^2}{1 + T^2} \\ &= \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta} = \cos(2\theta) = \cos(30^{\mathrm{o}}) = \tfrac{1}{2}\sqrt{3}.\end{aligned}$$

21. $5\frac{1}{2}$. [27] Let x, y, and z denote the number of hours required to traverse each of the three sides. Then $x+y+\frac{z}{4} = \frac{7}{2}$, $\frac{x}{4}+y+z = \frac{9}{2}$, and $y = x + \frac{1}{6}$. Substitute the last equation into the first two, and then subtract the second from 4 times the first, obtaining $x = 4/3$. Then obtain $y = 3/2$ and $z = 8/3$, so the total is $11/2$.

22. 1008. [30] If $x = n + f$, with n an integer and $0 \le f < 1$, then we want the smallest n such that $2nf + f^2 \ge 2015$. We should choose f very close to 1. Since $f < 1$, $n = 1007$ doesn't quite work, but $n = 1008$ and $2015/2016 < f < 1$ will work.

23. 3. [44] Note that $2^{12} \equiv 1 \bmod 13$ by Fermat's Little Theorem. Since $1000 = 83 \cdot 12 + 4$, $2^{1000} \equiv (2^{12})^{83} \cdot 2^4 \equiv 16 \equiv 3 \bmod 13$.

24. 9/2. [18] Let $r = \frac{1}{3}$ and let $S = 1 + 4r + 9r^2 + \cdots$ denote the desired sum. Then

$$\widetilde{S} := (1-r)S = S - rS = 1 + 3r^1 + 5r^2 + 7r^3 + \cdots.$$

Now

$$(1-r)^2 S = \widetilde{S} - r\widetilde{S} = 1 + 2r^1 + 2r^2 + \cdots = 1 + \tfrac{2r}{1-r}.$$

We obtain $\frac{4}{9}S = 2$.

25. 9. [21] The equation can be rewritten as $(x-4)(y-4) = 16$, $x, y \neq 0$. The solutions of this correspond to the ways of writing 16 as $2 \cdot 8$, $1 \cdot 16$, and $4 \cdot 4$. The first two of these yield four solutions each, corresponding to negating and reversing $(x-4)$ and $(y-4)$. The $4 \cdot 4$ factorization yields only $x - 4 = y - 4 = 4$,

since reversing does not change the values, and negating gives the excluded values $x = y = 0$.

Alternate solution: $x = y = 8$ works. Other solutions will come in pairs under reversal of x and y. If $\frac{1}{x} > \frac{1}{y}$, then the only other possibilities are $x = 1, 2, 3, 4, 5, 6, 7$. For $x = 2$, 3, 5, and 6, $\frac{1}{4} - \frac{1}{x} = -\frac{1}{4}$, $-\frac{1}{12}$, $\frac{1}{20}$, and $\frac{1}{12}$, respectively, while for $x = 1$, 4, and 7, $\frac{1}{4} - \frac{1}{x}$ is not of the form $1/y$ for an integer y.

26. $(-4, 1 - \sqrt{7})$. [34] We want s, d, and b so that there is an identity

$$\begin{aligned} & (x-s-d)(x-s)(x-s+d) \\ = \;& x^3 - 3sx^2 + (3s^2 - d^2)x - s(s^2 - d^2) \\ = \;& x^3 - 3x^2 + bx + 6. \end{aligned}$$

This yields $s = 1$ and $-s(s^2 - d^2) = 6$, so $d^2 = 7$ and $b = 3s^2 - d^2 = -4$. The roots are 1 and $1 \pm \sqrt{7}$.

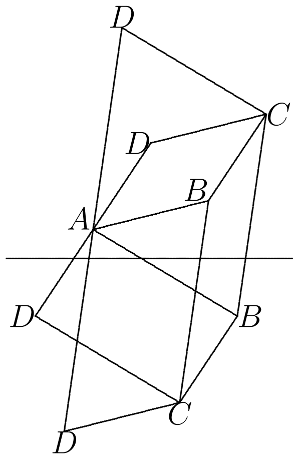
27. 179. [28] This prime p must be a divisor of $193499 - 180253 = 13246 = 2 \cdot 6623$ and of $180253 - 160921 = 19332 = 4 \cdot 4833$. Thus p must be a divisor of $6623 - 4833 = 1790 = 2 \cdot 5 \cdot 179$. Since our numbers are not divisible by 2 or 5, the answer must be 179. Indeed, the three numbers factor as $23 \cdot 47 \cdot 179$, $19 \cdot 53 \cdot 179$, and $29 \cdot 31 \cdot 179$.

28. $\frac{5}{17}\sqrt{34}$. [20] Arrange it so that one chord is the line $x = a$ and the other $y = a$ with $a > 0$. Then $4(\sqrt{1-a^2} - a) = \sqrt{1-a^2} + a$, and hence $3\sqrt{1-a^2} = 5a$. Thus $a^2 = \frac{9}{34}$ and the length is $2\sqrt{1 - \frac{9}{34}} = 10/\sqrt{34}$.

29. 3. [11] Note that $\displaystyle\sum_{i=1}^{6} \frac{1}{27i} = \frac{1}{27} \cdot \frac{49}{20}$. Since the least common multiple of the other denominators is of the form 3^2B, with B not divisible by 3, their sum is of the form $\frac{A}{9B}$ for some integer A. Thus the desired sum is of the form $\frac{A}{9B} + \frac{C}{27D}$, with none of B, C, and D divisible by 3. This equals $\frac{3AD+BC}{27BD}$, which, when reduced, has denominator divisible by 3^3.

30. 2, 4, 6, and 8. [8] Think of the line as being the x-axis and A at $(0, 1)$. Then B can be at y-value ± 2, and C at y-value ± 5. The y-component of D is $1 + y(C) - y(B) = 1 + \pm_1 5 - \pm_2 2$, where the two $\pm$ are independent. This can take the value 8, 4, -2, and -6, but we choose their absolute value.

The four possibilities are illustrated in the following diagram.



31. 64. [7] The general formula is that the number of odd coefficients in $(x + 1)^n$ is 2^a, where a is the number of 1's in the binary expansion of n. In this case $a = 6$, since $2000 = 16(64 + 32 + 16 + 8 + 4 + 1)$. Alternatively, working mod 2, we have

$$\begin{aligned} & (x+1)^{2000} \\ = \; & (x+1)^{1024}(x+1)^{512}(x+1)^{256}(x+1)^{128}(x+1)^{64}(x+1)^{16} \\ \equiv \; & (x^{1024}+1)(x^{512}+1)(x^{256}+1)(x^{128}+1)(x^{64}+1)(x^{16}+1), \end{aligned}$$

which has 2^6 terms when expanded. Here we have used that $(x + 1)^n \equiv x^n + 1 \bmod 2$ if n is a 2-power.

32. 90. [24] There must be exactly two diagonal moves, two horizontal moves, and two vertical moves. (This can be seen by solving $D + H + V = 6$, $D(1, 1) + H(1, 0) + V(0, 1) = (4, 4)$.) These can be done in any order. The answer is the number of ways of writing the six symbols D, D, H, H, V, and V in order.

There are $\binom{6}{2} = 15$ positions for the D's, and then $\binom{4}{2} = 6$ positions for the H's.

33. $(-1+\sqrt{33})/2$. [25] The left end of the rod starts at the point $(1,0)$. In $91/3$ seconds, it will move 121 full revolutions plus an additional $1/3$ revolution, so it will be at the point $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$. The right end will be at the point $(x,0)$ which satisfies $x > 0$ and $(x+\frac{1}{2})^2+\frac{3}{4} = 9$. Hence $x^2+x-8 = 0$, and so $x = (-1+\sqrt{33})/2$.

34. $-\frac{1}{2} \le x < \frac{45}{8}$, $x \neq 0$ or $[-\frac{1}{2}, 0) \cup (0, \frac{45}{8})$. [3] Multiplying numerator and denominator of the left hand side by $(1+\sqrt{1+2x})^2$ simplifies it to $(1+\sqrt{1+2x})^2 < 2x+9$ and then to $\sqrt{1+2x} < 7/2$. Hence $x < 45/8$. We must restrict to $x \ge -1/2$ so that the square root is defined, and exclude $x = 0$ to avoid dividing by 0.

35. 719. [17] Let $a \le b \le c$. By parity, and since $2^4+2^4+2^4-3$ is not prime, we must have $a = 2$ and $b > 2$. Thus $p = b^4+c^4+13$. If $b \neq 3$, then both b and c are ± 1 mod 6, and so p would be divisible by 3. Since 3^4+3^4+13 is not prime, we must have $b = 3$, $c > 3$ and $p = c^4+94$. If $c \neq 5$, then $c^4 \equiv 1$ mod 5, and so p would be divisible by 5. Thus $p = 5^4+94 = 719$, which is prime. (You need not verify its primality, because the problem said there was one prime, and you have eliminated all other possibilities.)

36. 45. [4] For $n \ge 3$, let F_n denote the number of faces of the polyhedron which have n sides. Since each edge intersects 2 faces, the number of edges $E = \frac{3}{2}F_3 + \frac{4}{2}F_4 + \frac{5}{2}F_5 + \cdots$. Since $V - E + F = 2$, we have

$$13 - \sum_{n\ge 3} \tfrac{n}{2} F_n + \sum_{n \ge 3} F_n = 2,$$

which simplifies to

$$22 = F_3 + 2F_4 + 3F_5 + \cdots .$$

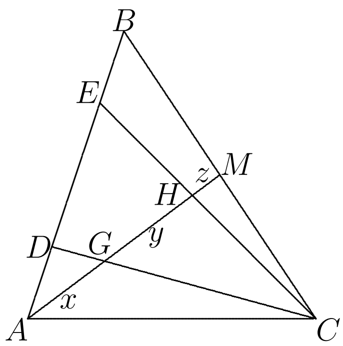
The number I of interior diagonals is $\binom{V}{2} - E - P$, where $P = 2F_4 + 5F_5 + 9F_6 + \cdots$ is the number of diagonals in the faces. We obtain

$$I = 78 - \tfrac{3}{2}F_3 - 4F_4 - \tfrac{15}{2}F_5 - \cdots .$$

The maximum value of I occurs when $F_3 = 22$ and $F_n = 0$ for $n > 3$. This is because the ratios $4/2$, $\frac{15}{2}/3$, etc., are greater than $3/2$. The optimal case can be realized by a pyramid sitting above and below a $(V-2)$-gon.

37. $\frac{2}{3}\pi\sqrt{3}$. [1] Make an affine transformation sending the 3-4-5 triangle to an equilateral triangle. Our ellipse will be mapped to the ellipse of maximum area inside the equilateral triangle, which, by symmetry, is the inscribed circle, and ratios of areas are preserved. Thus our desired area equals $A_T \cdot A_C/A_E$, where $A_T = 6$ is the area of the 3-4-5 triangle, and $A_C/A_E = \pi(s\sqrt{3}/6)^2/(s^2\sqrt{3}/4) = \pi\sqrt{3}/9$ is the ratio of the area of the inscribed circle to that of the equilateral triangle.

38. 14:16:5. [8] Let $x : y : z$ denote the desired ratio. Apply Menelaus' Theorem to triangle ABM with transversal CE, obtaining $\frac{x+y}{z} \cdot \frac{1}{2} \cdot \frac{1}{3} = 1$, hence $x + y = 6z$. Now apply Menelaus' Theorem to the same triangle with transversal CD, obtaining $\frac{x}{y+z} \cdot \frac{1}{2} \cdot \frac{3}{1} = 1$, hence $x = \frac{2}{3}(y+z)$. Setting $z = 1$ and solving, one obtains $x = \frac{14}{5}$ and $y = \frac{16}{5}$.



39. 10. [6] Let E_i denote the event that the first of three consecutive H's occurs for the first time on the ith toss. Then $P(n) = \sum_{i=1}^{n-2} \Pr(E_i)$, and

$$\Pr(E_i) = \begin{cases} 1/8 & i = 1 \\ \frac{1}{16}(1 - P(i-2)) & i \geq 2. \end{cases}$$

The latter is true because, for E_i to occur, the H on the ith toss must be immediately preceded by T, and the tosses before that cannot have contained three straight H's. Note that $P(i-2) = 0$ if $i < 5$. We obtain $P(3) = \frac{1}{8}$, $P(4) = \frac{3}{16}$, $P(5) = \frac{4}{16}$, $P(6) = \frac{5}{16}$, $P(7) = \frac{6}{16} - \frac{1}{16} \cdot \frac{1}{8}$, $P(8) = \frac{7}{16} - \frac{1}{16}(\frac{1}{8} + \frac{3}{16})$, $P(9) = \frac{8}{16} - \frac{1}{16}(\frac{1}{8} + \frac{3}{16} + \frac{4}{16})$, and $P(10) = \frac{9}{16} - \frac{1}{16}(\frac{1}{8} + \frac{3}{16} + \frac{4}{16} + \frac{5}{16})$.

40. $39\sqrt{39}/35$. [7] In the diagram below, the crease is XY, with triangle AXY congruent to triangle DXY. Let a, b, and x be the indicated side lengths. From the Law of Cosines in CYD, we obtain

$$a^2 = 9^2 + (12-a)^2 - 2 \cdot 9(12-a)\tfrac{1}{2},$$

implying $a = 117/15$. Similarly triangle BDX implies $b = 117/21$. Now triangle AXY says

$$\begin{aligned} x^2 &= a^2 + b^2 - 2ab \cdot \tfrac{1}{2} = 117^2\left(\tfrac{1}{15^2} + \tfrac{1}{21^2} - \tfrac{1}{15 \cdot 21}\right) \\ &= \frac{117^2(21^2 + 15^2 - 15 \cdot 21)}{15^2 21^2} = \frac{39^2 \cdot 39}{35^2}. \end{aligned}$$