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THE CONNECTIVE MORAVA K-THEORY OF THE SECOND MOD p EILENBERG-MACLANE SPACE

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ABSTRACT. We develop tools for computing the connective n-th Morava K-theory of spaces. Starting with a Universal Coefficient Theorem that computes the cohomology version from the homology version, we show that every step in the process of computing one is mirrored in the other and that this can be used to make computations. As our example, we compute the connective n-th Morava K-theory of the second mod p Eilenberg-MacLane space.

CONTENTS

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1. INTRODUCTION

Being able to compute is central to much of algebraic topology. Computing generalized (co)homology theories of basic spaces usually runs from difficult to impossible. One exception has been the extraordinary K-theories of Jack Morava, $K(n)_*(X)$. They have a Künneth isomorphism that makes them more tractable to computations than most.

There is a connective version of Morava K-theories, $k(n)_*(X)$, and in this paper we make some progress towards computing with this. In particular, we develop some tools that can be applied to this problem in general, and then we apply them to compute the nth connective Morava K-theory of the second mod p Eilenberg-MacLane space, $K_2 = K(\mathbf{Z}/p, 2)$, where $\mathbf{Z}/p$ is the integers modulo the prime p.

Anderson-Hodgkin [AH68] showed that $K(1)_*(K_2)$ was trivial. The third author searched, periodically over the decades, for the differentials in the Atiyah-Hirzebruch spectral sequence that would reduce the already small group $H_*(K_2; K(1)_*)$ to zero at $p = 2$. The differentials in the Atiyah-Hirzebruch spectral sequence are the same as those in the Adams spectral sequence, so this paper finally gives the third author great satisfaction. The project grew into this paper.

The main computation of the paper is to compute both $k(n)_*(K_2)$ and $k(n)^*(K_2)$ as $k(n)_*$ (and $k(n)^*$) modules. The $n = 1$ case is essential for the first and third authors' determination of $ku^*(K_2)$ and $ku_*(K_2)$ for all primes in [DW24b]. It was also useful in the first author's determination of $ko^*(K_2)$ and $ko_*(K_2)$ in [D], with the E_2 page of the Adams spectral sequence for this computed in [DW24a].

One of our main tools is obtained by combining results of Robinson and Lazarev for computational purposes.

Theorem 1.1. *For X a space of finite type with $K(n)_*(X)$ finitely generated over $K(n)_*$, there is a universal coefficient spectral sequence that collapses:*

$$\mathrm{Ext}^{s,t}_{k(n)_*}(k(n)_*(X), k(n)_*) \Rightarrow k(n)^{s+t}(X)$$

In [Rob87] Alan Robinson created the universal coefficient spectral sequence for homology theories satisfying certain hypotheses. These were shown to be satisfied by $k(n)_*$ by him in [Rob89] and later by Andrey Lazarev in [Laz01]. We will show the universal coefficient spectral sequence collapses in this case.

From this result, we derive the next important tool.

Theorem 1.2 (The Pairing). *For X a space of finite type with $K(n)_*(X)$ finitely generated over $K(n)_*$, there is a differential $d^r(\alpha) = v^r\beta$ in the Adams spectral sequence for $k(n)^*(X)$ if and only if there is a corresponding $d_r(\beta') = v^r\alpha'$ in the Adams spectral sequence for $k(n)_*(X)$, with $|\alpha| = |\alpha'|$ and $|\beta| = |\beta'|$.*

It is the interaction between $k(n)$ cohomology and homology from these two results that allows us to do our computation. Theorem 1.1 gives a duality of sorts between $k(n)_*(X)$ and $k(n)^*(X)$, but Theorem 1.2 goes even further and says that there is a duality every step of the way in the computation. In our case, we have that K_2 is an H-space so both Adams spectral sequences are multiplicative. Although this does not give us a Hopf algebra, there is enough similarity in the structure that we can make good use of it.

The plan of the paper is to state the results of the main computation in the next section. We set up some notation in Section 3. In Section 4 we compute the E_2 term of the Adams spectral sequence for $k(n)^*(K_2)$. In Section 5 we illustrate its behavior for $n = 2$. We give some necessary definitions and numbers in Section 6. In Section 7, we prove the two theorems in the introduction and establish some other preliminaries we need. All the hard work is done in Section 8 where the differentials are computed. The results for $k(n)_*(K_2)$ are all collected in Section 9 and the final section is devoted to describing the results at $p = 2$.

2. Statement of results

In this section we define only what we need to efficiently state the results of our main computation of $k(n)^*(K_2)$. Many details will be properly developed later.

2.1. **Basic notation.** All our cohomology and homology groups will be mod p. The connective nth Morava K-theory spectrum, $k(n)$, has $k(n)^* = \mathbf{Z}/p[v_n]$ with $|v_n| = -2(p^n - 1)$.

We let $P(x)$, $E(x)$, and $\Gamma(x)$ be the polynomial, exterior, and divided power algebras on x (which could be a single generator or a set of generators) over $\mathbf{Z}/p$. In addition, we need the truncated polynomial algebra, $T_k(x) := P(x)/(x^k)$, and its dual, $\Gamma_k(x)$.

The divided power algebra $\Gamma(x)$ for a single x is additively generated by elements $\gamma_i(x)$ for $i \geq 0$ (the divided powers of x) with $|\gamma_i(x)| = i|x|$ and

$$\gamma_i(x)\gamma_j(x) = \binom{i+j}{i}\gamma_{i+j}(x).$$

As an algebra, $\Gamma(x)$ is $T_p(\gamma_{p^m}(x) : m \geq 0)$. For $p = 2$, this is an exterior algebra.

For a rational number x, $\lfloor x \rfloor$, the *floor of* x, denotes the largest integer not exceeding x, and $\lceil x \rceil$, the *ceiling of* x, denotes the smallest integer not exceeded by x.

For a graded connected $\mathbf{Z}/p$-algebra A, we let $\overline{A}$ denote the augmentation ideal of A, its vector space of positive degree elements.

2.2. **The mod p cohomology of $K_2 := K(\mathbf{Z}/p, 2)$.** In what follows, *all tensor products are over $\mathbf{Z}/p$ unless otherwise stated*.

To compute with the Adams spectral sequence, we need (for p an odd prime)

$$\begin{aligned} &H^*K_2 = P(\iota_2) \otimes P(z_i : i > 0) \otimes E(u_i : i \geq 0) \\ &\text{with} \quad |\iota_2| = 2, \quad |z_i| = 2p^i + 2 \quad \text{and} \quad |u_i| = 2p^i + 1. \end{aligned} \tag{2.1}$$

Let $y_{n,j} = \iota_2^{p^j}$. In particular, $\iota_2 = y_{n,0}$ and $\iota_2^p = y_{n,0}^p = y_{n,1}$. In general, $y_{n,j}^p = y_{n,j+1}$ with $|y_{n,j}| = 2p^j$.

For $p = 2$, H^*K_2 has a similar description but with $u_i^2 = z_{i+1}$. We will say more about this in Section 10.

We define $w_{n,i} \in H^*K_2$ for $i \geq 0$ by

$$w_{n,i} := \begin{cases} u_n & \text{for } i = 0 \\ u_{n+i} - u_{n-i} z_i^{p^n - p^{n-i}} & \text{for } 0 < i \leq n \\ y_{n,i-n-1/2} z_i^{p^n - 1} & \\ \qquad = w_{n,i-n-1} y_{n,i-n-1}^{p-1} z_i^{p^n-1} & \text{for } i \geq n+1, \end{cases} \tag{2.2}$$

where

$$y_{n,i+1/2} := y_{n,i}^{p-1} w_{n,i} \qquad \text{for integers } i \geq 0. \tag{2.3}$$

In general, all our variables, such as n, i, j, k, s, are non-negative integers. The number $1/2$ arises often, and should be clear from context.

In Section 3 we will see that there is an Adams spectral sequence converging to $k(n)^*K_2$ for which the input is $k(n)^* \otimes H^*K_2$. It is indexed in such a way that

- the filtration and dimension of v_n are 1 and $-2(p^n - 1)$,
- the elements of H^*K_2 have their usual positive degrees and Adams filtration 0, and
- differentials d^r raise rather than lower degree by 1, while raising filtration by r.

In Section 4 we will see that the action of the Milnor primitive Q_n on H^*K_2 gives us $d^1(x) = v_n Q_n(x)$. From this we get

$$\begin{aligned} &d^1(y_{n,0}) = v_n u_n \\ &\text{and} \quad d^1(u_s) = \begin{cases} v_n z_{n-s}^{p^s} & \text{for } 0 \leq s < n \\ 0 & \text{for } s = n \\ v_n z_{s-n}^{p^n} & \text{for } s > n. \end{cases} \end{aligned} \tag{2.4}$$

This implies that for $w_{n,s}$ as in (2.2), $d^1(w_{n,s}) = 0$ for $0 \leq s \leq n$. We can regard $w_{n,s}$ for such s as a substitute for u_{n+s} that survives to E_2 .

In Section 8 for p odd and Section 10 for $p = 2$, we will see that there are higher Adams differentials

$$\begin{aligned} & d^{\rho_n(i)}(y_{n,i}) = v_n^{\rho_n(i)} w_{n,i} \\ \text{and} \qquad & d^{\rho_n(i+1/2)}(y_{n,i+1/2}) = v_n^{\rho_n(i+1/2)} z_{n+i+1} \end{aligned} \tag{2.5}$$

for integers $i \geq 0$, where the numbers $\rho_n(i)$ and $\rho_n(i+1/2)$ are given in Lemma 6.4. The latter are uniquely determined by the dimensions of the elements in question. For integers $0 \leq i \leq n$, they are

$$\rho_n(i) = p^i \qquad \text{and} \qquad \rho_n(i+1/2) = (p-1)p^i.$$

In particular $\rho_n(0) = 1$, so the first differential of (2.5) for $i = 0$ coincides with the first differential of (2.4).

2.3. **The effect of the Adams d^1.** *The additional d^1s of (2.4) make the passage from E_1 to E_2 more complicated than the passage to higher terms brought about by the higher differentials of (2.5).* We will outline these processes here in order to motivate the complicated expressions in Theorem 2.12, our main computational result.

In order to work out the implications of (2.4), the following additive isomorphisms and definitions for each positive n and $i \geq 0$ are convenient.

$$\begin{aligned} P(y_{n,i}) &\cong T_p(y_{n,i}) \otimes P(y_{n,i+1}), \\ E(u_s : s \geq 0) &\cong EE_n \otimes E(w_{n,0}) \otimes W_{n,0}, \text{ where} \\ EE_n &:= E(u_s : 0 \leq s < n) \otimes E(u_{2n+s} : s > 0) \qquad \text{and} \\ W_{n,i} &:= E(w_{n,i+s} : 1 \leq s \leq n) \quad \text{for } w_{n,i+s} \text{ as in (2.2),} \\ P(z_s : s > 0) &\cong L_n \otimes TZ_{n,0} \otimes PZ_n, \text{ where} \\ L_n &:= \bigotimes_{0<s<n} T_{p^{n-s}}(z_s), \\ TZ_{n,i} &:= T_{p^n}(z_{n+s} : s > i), \qquad \text{and} \\ PZ_n &:= P(z_s^{e_n(s)} : s > 0) \qquad \text{with } e_n(s) := \begin{cases} p^{n-s} & \text{for } 0 < s \leq n \\ p^n & \text{for } s > n. \end{cases} \end{aligned} \tag{2.6}$$

We will make use of these $W_{n,i}$ and $TZ_{n,i}$ for $i > 0$ later.

Remark 2.7. Although stated as additive isomorphisms, much of the algebra structure is preserved and we need it. For example, the additive isomorphism for $P(y_{n,i})$ comes from the multiplicative extension

$$P(y_{n,i+1}) \longrightarrow P(y_{n,i}) \longrightarrow T_p(y_{n,i}),$$

meaning that $P(y_{n,i})$ is a free module over the subring $P(y_{n,i+1})$ and

$$T_p(y_{n,i}) = P(y_{n,i}) \otimes_{P(y_{n,i+1})} \mathbf{Z}/p.$$

Here, if we have $d^r(y_{n,i}) \neq 0$, we have $d^r(y_{n,i}^p = y_{n,i+1}) = 0$, but this requires the multiplicative structure. Similarly, we have

$$PZ_n \longrightarrow P(z_s : s > 0) \longrightarrow L_n \otimes TZ_{n,0}.$$

Putting this all together, we have

$$H^*K_2 \cong T_p(y_{n,0}) \otimes P(y_{n,1}) \otimes EE_n \otimes E(w_{n,0}) \otimes W_{n,0} \otimes L_n \otimes TZ_{n,0} \otimes PZ_n. \tag{2.8}$$

We see in (4.2) that the d^1 of (2.4) on $k(n)^*$ tensored with (2.8) is confined to $k(n)^* \otimes D_1$ with

$$D_1 := T_p(y_{n,0}) \otimes E(w_{n,0}) \otimes EE_n \otimes PZ_n. \tag{2.9}$$

The d^1 homology of $k(n)^* \otimes T_p(y_{n,0}) \otimes E(w_{n,0})$ is just $k(n)^* \otimes E(y_{n,1/2}) \oplus T_{p-1}(y_{n,0}) \otimes \overline{E(w_{n,0})}$. This is illustrated in the top diagram of (8.3) for $j = 0$, where $\rho_n(0) = 1$. The d^1 homology of $k(n)^* \otimes EE_n \otimes PZ_n$ is $k(n)^*$ plus elementary v_n torsion elements. Combining these results is tricky so we just observe that the elementary v_n-torsion in (2.9) is the image of Q_n, i.e. $Q_n(D_1)$. If we remove the elements in $E(y_{n,1/2})$ from (2.9), what remains is free over $E(Q_n)$.

It is straightforward to compute the Poincaré series from this information. Although this does not give an explicit base, it isn't hard to filter it and get a basis for the associated graded version. For example, for $EE_n \otimes PZ_n$, we would have

$$\begin{aligned} &\bigoplus_{0<k\le n} E(u_i : 0 \le i < n-k) \otimes E(u_i : 2n < i) \otimes \overline{P(z_k^{e_n(k)})} \otimes P(z_i^{e_n(i)} : i > k) \\ &\oplus \bigoplus_{2n<k} E(u_i : i > k) \otimes \overline{P(z_{k-n}^{p^n})} \otimes P(z_i^{p^n} : i > k-n) \end{aligned} \tag{2.10}$$

We define

$$\begin{aligned} S_{n,0} &:= Q_n(D_1) \\ M_{n,i} &:= P(y_{n,i+1}) \otimes W_{n,i} \otimes L_n \otimes TZ_{n,i} \qquad \text{for } i \ge 0. \\ M_{n,i+1/2} &:= P(y_{n,i+1}) \otimes W_{n,i} \otimes L_n \otimes TZ_{n,i+1} \quad \text{for } i \ge 0, \\ S_{n,i} &:= T_{\rho_n(i)}(v_n) \otimes T_{p-1}(y_{n,i}) \otimes \overline{E(w_{n,i})} \\ &= T_{\rho_n(i)}(v_n) \otimes \left\{ w_{n,i} y_{n,i}^s : 0 \le s \le p-2 \right\} \quad \text{for } i > 0, \\ \text{and} \qquad S_{n,i+1/2} &:= T_{\rho_n(i+1/2)}(v_n) \otimes \overline{T_{p^n}(z_{n+i+1})} \\ &= T_{\rho_n(i+1/2)}(v_n) \otimes \left\{ z_{n+i+1}^s : 1 \le s < p^n \right\} \quad \text{for } i \ge 0. \end{aligned} \tag{2.11}$$

These will figure in Theorem 2.12. Using this notation, we have computed the elementary v_n-torsion as $S_{n,0} \otimes M_{n,0}$.

2.4. **The effect of higher Adams differentials.** The higher differentials of (2.5) are easier to deal with since each is nonzero on a single multiplicative generator. They are illustrated in the diagrams of (8.3) below.

The vector spaces $S_{n,i}$ and $S_{n,i+1/2}$ of (2.11) can also be written as

$$S_{n,i} = d^{\rho_n(i)}\left(\overline{T_p(y_{n,i})}\right)/v_n^{\rho_n(i)}$$

and $$S_{n,i+1/2} = d^{\rho_n(i+1/2)}\left(\overline{E(y_{n,i+1/2})}\otimes T_{p^n-1}(z_{n+i+1})\right)/v_n^{\rho_n(i+1/2)}.$$

We can now state the main computational result of this paper.

Theorem 2.12. *For an odd prime p, $k(n)^*(K_2)$ has the following three summands as a $k(n)^*$-module:*

(i) The $k(n)^*$ free summand, *$k(n)^*\otimes L_n$, for L_n as in (2.6).*

(ii) The higher torsion summand,

$$\bigoplus_{\ell>0}\left(M_{n,\ell/2}\otimes S_{n,\ell/2}\right),$$

for $M_{n,\ell/2}$ and $S_{n,\ell/2}$ as in (2.11).

(iii) The elementary torsion summand, *$S_{n,0}\otimes M_{n,0}$ as in (2.11).*

Remark 2.13. The v_n-torsion free summand. Inverting v_n kills all but the first summand of $k(n)^*K_2$, which becomes $K(n)^*(K_2)$, as described in [RW80, dual to Theorem 11.1]. This $k(n)^*$-free summand is all that appears in negative degrees, where it is finite in each degree. In addition, every positive degree is also finite.

Remark 2.14. The multiplicative structure. Theorem 2.12 describes a ring as well as a $k(n)^*$-module, but we can only show that the ring structure is that of the Adams E_∞-term. We cannot rule out nontrivial multiplicative extensions. For $n>2$, we cannot show by dimensional arguments that $v_n z_{n+i}^{p^n}=0$ for $i>0$. Let

$$\kappa_n = \prod_{0<i<n} z_{n-i}^{p^i-1},$$

the top class in L_n. We can show by induction on n that $|\kappa_n| = 2(p^n-1)(n-1)$ using the identity

$$\kappa_{n+1} = \kappa_n^p(z_1z_2\cdots z_n)^{p-1}.$$

We cannot rule out the multiplicative extension

$$v_n z_{n+i}^{p^n} = v_n^{n-1}\kappa_n z_{2n+i}$$

(note that $|v_n z_{n+i}^{p^n}| = |z_{2n+i}|$ and $|v_n^{n-1}\kappa_n| = 0$) for $n>2$ and $i>0$.

3. Our Adams spectral sequence notation

The $k(n)$ under consideration here is the the connective version of Morava's nth extraordinary K-theory $K(n)$. We have

$$\pi_* k(n) = P(v_n) \qquad \text{with } |v_n| = 2(p^n - 1)$$
$$\text{and} \qquad H^*(k(n)) = \mathcal{A}/\mathcal{A}(Q_n),$$

where $\mathcal{A}$ is the mod p Steenrod algebra and Q_n is the nth Milnor primitive.

We have

$$k(n)^i X := [X, k(n)]_{-i} = \pi_{-i} F[X, k(n)] \qquad \text{and} \qquad k(n)_i X := \pi_i(k(n) \wedge X). \tag{3.1}$$

Given $\alpha \in k(n)_i X$ represented by a map $S^i \to k(n) \wedge X$, and $\beta \in k(n)^j X$ represented by a map $X \to \Sigma^j k(n)$, we get an element $\langle \alpha, \beta \rangle \in \pi_{i-j} k(n)$, which is the composite

$$S^i \xrightarrow{\quad\alpha\quad} k(n) \wedge X \xrightarrow{k(n)\wedge\beta} k(n) \wedge \Sigma^j k(n) \xrightarrow{\quad m\quad} \Sigma^j k(n),$$

where $m : k(n) \wedge k(n) \to k(n)$ is the multiplication in the ring spectrum $k(n)$.

The groups of (3.1) can be computed with the Adams spectral sequence as follows. In the above, $k(n)$ is the spectrum representing connective Morava K-theory and $F(X, Y)$ denotes the function spectrum for maps of spectra $X \to Y$. The ring structure on $k(n)$ allows us to extend the map $S^{|v_n|} \to k(n)$ representing v_n to the map v_n in the fiber sequence

$$\Sigma^{|v_n|} k(n) \xrightarrow{\ v_n\ } k(n) \xrightarrow{\ j\ } H/p \xrightarrow{\ \delta\ } \Sigma^{2p^n-1} k(n) \tag{3.2}$$

where H/p is the mod p Eilenberg-Mac Lane spectrum. The composite

$$H/p \xrightarrow{\ \delta\ } \Sigma^{2p^n-1} k(n) \xrightarrow{\ j\ } \Sigma^{2p^n-1} H/p \tag{3.3}$$

is the Milnor primitive operation operation Q_n.

With the maps in (3.2) we can construct the following *Adams diagram for* $k(n)$,

$$\begin{array}{ccccccccc} k(n) & \xleftarrow{\ v_n\ } & \Sigma^{|v_n|} k(n) & \xleftarrow{\ v_n\ } & \Sigma^{2|v_n|} k(n) & \xleftarrow{\ v_n\ } & \Sigma^{3|v_n|} k(n) & \longleftarrow & \cdots \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \\ H/p & & \Sigma^{|v_n|} H/p & & \Sigma^{2|v_n|} H/p & & \Sigma^{3|v_n|} H/p. & & \end{array} \tag{3.4}$$

Each fiber sequence

$$\Sigma^{(s+1)|v_n|} k(n) \to \Sigma^{s|v_n|} k(n) \to \Sigma^{s|v_n|} H/p$$

leads to a long exact sequence of homotopy groups. The same is true if we apply either the functor $F(X, -)$, the *cohomological case*, or $(X \wedge -)$, the *homological case*, to (3.4).

In each case these long exact sequences assemble into an exact couple (see [Rav86, §2.1]) leading to a spectral sequence $\{E_r^{s,t}\}$, where

- $E_1^{s,t}$ is either $\pi_{t-s}(F(X, \Sigma^{s|v_n|}H/p)) = H^{s(2p^n-1)-t}X$, the indicated mod p cohomology group of X, or $\pi_{t-s}(X \wedge \Sigma^{s|v_n|}H/p) = H_{t-s(2p^n-1)}X$, the indicated mod p homology group of X.
- $E_2^{s,t}$ is either $\mathrm{Ext}_{E(Q_n)}^{s,t}(\mathbf{Z}/p, H^*X)$ or $\mathrm{Ext}_{E(Q_n)}^{s,t}(H^*X, \mathbf{Z}/p)$. This can be derived from (3.3).
- $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$. The filtration index s is raised by r and the dimension index $t-s$ is lowered by one.
- $E_\infty^{s,t}$ is a certain subquotient of either $k(n)^{s-t}X$ or $k(n)_{t-s}X$.

These are the *classical Adams spectral sequences for* $k(n)^*X$ *and* $k(n)_*X$. It is common to depict them in a chart in which $E_r^{s,t}$ has Cartesian coordinate $(t-s, s)$. Thus d_r is an arrow lowering the first coordinate by 1 and raising the second by r, making it a line with slope $-r$.

The *Adams spectral sequence for* $k(n)^*(K_2)$ has

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(k(n)), H^*K_2) \cong \mathrm{Ext}_{E(Q_n)}^{s,t}(\mathbf{Z}/p, H^*K_2) \implies k(n)^{-(t-s)}(K_2).$$

We use the usual grading for the Adams spectral sequence so that $E_r^{s,t}$ is displayed with Cartesian coordinates $(t-s, s)$, but then we give the negative x-axis positive degrees, rewriting $E_r^{s,t}$ as $G_{s-t,s}^r$ in position $(s-t, s)$. We use d^r for our cohomology differentials. *In this depiction, differentials raise rather than lower the first coordinate by 1.*

We also have the *Adams spectral sequence for* $k(n)_*(K_2)$, and need to have distinct notation to clearly separate it from the cohomology notation. It has

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(k(n) \wedge K_2), \mathbf{Z}/p) \cong \mathrm{Ext}_{E(Q_n)}^{s,t}(H^*K_2, \mathbf{Z}/p) \implies k(n)_{t-s}(K_2).$$

We use the usual grading for the Adams spectral sequence so that $E_r^{s,t}$ is at the Cartesian $(t-s, s)$. Here we don't need the negative grading, but to distinguish this from the cohomology Adams spectral sequence, we write $E_r^{s,t}$ as $G_r^{t-s,s}$ in position $(t-s, s)$. Here we use d_r for the differential so we can keep track of which is which.

To summarize, in the cohomological case

$$G_{x,y}^r := E_r^{y,y-x} \qquad \text{with differentials } d^r : G_{x,y}^r \to G_{x+1,y+r}^r, \tag{3.5}$$

and in the homological case,

$$G_r^{x,y} := E_r^{y,y+x} \qquad \text{with differentials } d_r : G_r^{x,y} \to G_r^{x-1,y+r}. \tag{3.6}$$

We need the $E(Q_n)$-module structure of H^*X (which we will describe in the next section) in order to compute the E_2-terms. Any $E(Q_n)$-module M is the sum of a free module and $\mathbf{Z}/p$-vector space on which Q_n acts trivially. As a result, it is easy to compute the relevant Ext groups. We have

$$\left\{\begin{array}{ll} & \operatorname{Ext}_{E(Q_n)}^{*,*}(\mathbf{Z}/p, \mathbf{Z}/p) = P(v_n) \qquad \text{with } v_n \in \operatorname{Ext}^{1,2p^n-1}, \\ & \operatorname{Ext}_{E(Q_n)}^{s,t}(E(Q_n), \mathbf{Z}/p) = \begin{cases} \mathbf{Z}/p & \text{for } (s,t) = (0,0) \\ 0 & \text{otherwise,} \end{cases} \\ \text{and} & \operatorname{Ext}_{E(Q_n)}^{s,t}(\mathbf{Z}/p, E(Q_n)) = \begin{cases} \mathbf{Z}/p & \text{for } (s,t) = (1,2p^n-1) \\ 0 & \text{otherwise.} \end{cases} \end{array}\right. \tag{3.7}$$

4. The Q_n homology of H^*K_2 and the Adams E_2 term

Following Tamanoi, [Tam99, Theorem 5.2], we have, at odd primes, $u_i = Q_i\iota_2$ and $z_i = Q_i u_0$ (in particular $z_0 = 0$), giving us H^*K_2 as in (2.1), where Q_i again is the ith Milnor primitive. Continuing to follow Tamanoi, we have

$$\begin{aligned} Q_n\iota_2 &= u_n \\ Q_n u_s &= \begin{cases} z_{n-s}^{p^s} & \text{for } 0 \le s < n \\ 0 & \text{for } s = n \\ z_{s-n}^{p^n} & \text{for } s > n \end{cases} \\ Q_n z_s &= 0. \end{aligned}$$

To compute the Q_n homology we filter H^*K_2 by powers of the ideal

$$\left(y_{n,0}, u_s, z_s^{e_n(s)} : s \ge 0\right),$$

where $z_0 = 0$ and $e_n(s)$ is as in (2.6). This means $z_s \in F^0$ and $z_s^{e_n(s)} \in F^1$ for $s > 0$, and $u_s \in F^1$ for $s \ge 0$. The associated bigraded object $E_0H^*K_2$ and its Q_n homology are indicated in the following diagram, which uses the notation of

(2.6).

$$
(4.1)\qquad
\begin{array}{ccc}
L_n \otimes TZ_{n,0} & \rightsquigarrow & L_n \otimes TZ_{n,0} \\
\otimes & & \otimes \\
P([y_{n,0}]) \otimes E([u_n]) & \rightsquigarrow & P([y_{n,1}]) \otimes E\left([y_{n,1/2}]\right) \\
\otimes & & \otimes \\
E([u_s] : 0 \le s < n) \otimes P\left(\left[z_s^{p^{n-s}}\right] : 0 < s \le n\right) & \rightsquigarrow & \mathbf{Z}/p \\
\otimes & & \otimes \\
E([u_{2n+s}] : s > 0) \otimes P\left(\left[z_{n+s}^{p^n}\right] : s > 0\right) & \rightsquigarrow & \mathbf{Z}/p \\
\otimes & & \otimes \\
E_0 W_{n,0} & \rightsquigarrow & E_0 W_{n,0},
\end{array}
$$

where elements enclosed in square brackets are in $E_0 H^* K_2$ (where they are indecomposable) corresponding to unbracketed elements in $H^* K_2$, which may be decomposable. The elements $[y_{n,1}]$ and $[y_{n,1/2}]$ are in F^p, and the other named elements below the top row are in F^1.

For the second row of (4.1) we have an additive isomorphism

$$P(y_{n,0}) \otimes E(u_n) \cong T_p(y_{n,0}) \otimes E(w_{n,0}) \otimes P(y_{n,1}),$$

and the behavior of the first two factors is illustrated in the upper diagram of (8.3) below for $j = 0$, where $\rho_n(0) = 1$. This can be done now with explicit computation, eliminating the need for the filtration on this part.

Using the notation of (2.6), we can consolidate the third and fourth rows of (4.1), and rewrite them as

$$EE_n \otimes PZ_n \rightsquigarrow \mathbf{Z}/p.$$

Because we end up with a trivial result, we can also eliminate the need for the filtration here as well.

Thus we can rewrite (4.1) as

$$\begin{array}{ccc}
L_n \otimes T_{p^n}(z_{n+i} : i > 0) & \rightsquigarrow & L_n \otimes T_{p^n}(z_{n+i} : i > 0) \\
\otimes & & \otimes \\
T_p(y_{n,0}) \otimes E(w_{n,0}) & \rightsquigarrow & E(y_{n,1/2}) \\
\otimes & & \otimes \\
P(y_{n,1}) & \rightsquigarrow & P(y_{n,1}) \\
\otimes & & \otimes \\
EE_n \otimes PZ_n & \rightsquigarrow & \mathbf{Z}/p \\
\otimes & & \otimes \\
W_{n,0} & \rightsquigarrow & W_{n,0}.
\end{array} \tag{4.2}$$

Theorem 4.3. *For $G^2_{*,*}$ as in (3.5), we have elements*

$$v_n \in G^2_{-2(p^n-1),1}, \qquad y_{n,1} \in G^2_{2p,0}, \qquad w_{n,i} \in G^2_{2p^{n+i}+1,0},$$
$$y_{n,1/2} \in G^2_{2(p^n-1)+2p+1,0}, \qquad \text{and } z_j \in G^2_{2(p^j+1),0}.$$

The E_2 term of the odd primary Adams spectral sequence for $k(n)^*(K_2)$ is

$$P(v_n) \otimes L_n \otimes P(y_{n,1}) \otimes E(y_{n,1/2}) \otimes W_{n,0} \otimes TZ_{n,0}$$

plus $S_{n,0} \otimes M_{n,0}$ from §2.3, the elements annihilated by v_n.

Proof. The Q_n homology of H^*K_2 gives us the trivial $E(Q_n)$-module part. The rest is free over $E(Q_n)$. The Ext groups for both kinds of modules are as in (3.7). The result follows. □

5. ILLUSTRATION FOR $n = 2$

In this section we will sometimes write our generators other than ι_2 with subscripts enclosed in parentheses indicating their dimensions. In the cohomological case we have $|v_2| = 2 - 2p^2$, which is -16 for $p = 3$.

Thus we have

$$\begin{aligned} H^*K_2 &= P(\iota_2) \otimes E(u_s : s \geq 0) \otimes P(z_s : s > 0) \\ &= P(\iota_2) \otimes E(u_{(3)}, u_{(7)}, u_{(19)}, \dots) \otimes P(z_{(8)}, z_{(20)}, z_{(56)}, \dots) \qquad \text{for } p = 3 \end{aligned}$$

with

$$(5.1)\qquad \begin{aligned} Q_2(\iota_2) &= u_2, \\ Q_2(u_s) &= \begin{cases} z_2 & \text{for } s=0 \\ z_1^p & \text{for } s=1 \\ 0 & \text{for } s=2 \\ z_{s-2}^{p^2} & \text{for } s\geq 3, \end{cases} \\ \text{and}\qquad Q_2(z_s) &= 0. \end{aligned}$$

The actions of Q_2 on the first five u_ss imply that

$$\begin{aligned} &Q_2(u_3 - z_1^{p^2-p}u_1) = 0 \\ \text{and}\qquad &Q_2(u_4 - z_2^{p^2-1}u_0) = 0, \end{aligned}$$

so as in (2.2) we define

$$w_{2,0} := u_2, \qquad w_{2,1} := u_3 - z_1^{p^2-p}u_1 \qquad \text{and} \qquad w_{2,2} := u_4 - z_2^{p^2-1}u_0,$$

with each being killed by Q_2.

The Adams E^1-term is

$$P(v_2)\otimes P(\iota_2)\otimes E(u_s : s\geq 0)\otimes P(z_s : s\geq 1)$$

$$\begin{array}{llll} \text{with} & v_2 \in G^2_{2-2p^2,1} & \iota_2 \in G^2_{2,0} & \\ & u_s \in G^2_{2p^s+1,0} & z_s \in G^2_{2p^s+2,0} & \\ \text{and} & d^1\iota_2 = v_2u_2 & d^1u_0 = v_2z_2 & \\ & d^1u_1 = v_2z_1^p & d^1u_s = v_2z_{s-2}^{p^2} & \text{for } s\geq 3. \end{array}$$

It follows that modulo v_2-torsion, the Adams E_2-term is

$$(5.2)\qquad \begin{aligned} &P(v_2)\otimes P(\iota_2^p)\otimes E(\iota_2^{p-1}u_2, w_{2,1}, w_{2,2})\otimes T_p(z_1)\otimes T_{p^2}(z_s : s\geq 3) \\ &\quad = k(2)^*\otimes P(y_{2,1})\otimes E(y_{2,1/2}, w_{2,1}, w_{2,2})\otimes L_2\otimes T_{p^2}(z_s : s\geq 3). \end{aligned}$$

Lemma 5.3. *For any prime and for all n, in the Adams spectral sequence for $k(n)^*K_2$,*

(i) every power of ι_2 supports a differential, and
(ii) z_s is a nontrivial permanent cycle for $s>0$,
(iii) some v_n-multiple of each z_s for $s>n$ is killed by a differential.

Lemma 7.5 below is a similar statement.

Proof. (i) There is a fiber sequence

$$K(\mathbf{Z},2)\to K_2\to K(\mathbf{Z},3)$$

for which the Serre spectral sequence collapses, that is

$$\begin{aligned} H^*K(\mathbf{Z},2) &= P(\iota_2) \\ H^*K(\mathbf{Z},3) &= E(u_s : s \geq 0) \otimes P(z_s : s \geq 1) \\ H^*K_2 &= H^*K(\mathbf{Z},2) \otimes H^*K(\mathbf{Z},3). \end{aligned}$$

This means that nothing in $P(\iota_2)$ can be hit by an Adams differential for any n. Thus (5.4) below implies that each power of ι_2 must support a nontrivial Adams differential. For $n = 2$, the action of Q_2 in $H^*K(\mathbf{Z},3)$ is given in (5.1).

(ii) We also have a p-local fiber sequence

$$K(\mathbf{Z},3) \to BP\langle 1\rangle_{2p+2} \to BP\langle 1\rangle_4$$

in which the second and third spaces have even dimensional cohomology. The generators $z_i \in H^*K(\mathbf{Z},3)$ are in the image of the map from $H^*BP\langle 1\rangle_{2p+2}$, so they map to permanent cycles in the Adams spectral sequences for both $k(n)^*K(\mathbf{Z},3)$ and $k(n)^*K_2$.

(iii) We know by [RW80, dual to Theorem 11.1] that

$$K(n)^*K_2 = K(n)^* \otimes \bigotimes_{0<s<n} T_{p^{n-s}}(z_s). \tag{5.4}$$

□

This means that our Adams E_∞-term must be congruent to

$$k(n)^* \otimes \bigotimes_{0<s<n} T_{p^{n-s}}(z_s)$$

modulo v_n-torsion.

It turns out that there is only one pattern of higher Adams differentials that leads to an answer meeting the conditions imposed by Lemma 5.3.

For $p = 3$, it begins as follows.

$$\begin{array}{llllll} [d^1(\iota_2) & = v_2 w_{(19)} & \in G^1_{3,1}] & d^2(\iota_2^2 w_{(19)}) & = v_2^2 z_{(56)} & \in G^2_{24,2} \\ d^3(\iota_2^3) & = v_2^3 w_{(55)} & \in G^3_{7,3} & d^6(\iota_2^6 w_{(55)}) & = v_2^6 z_{(164)} & \in G^6_{68,6} \\ d^9(\iota_2^9) & = v_2^9 w_{(163)} & \in G^9_{19,9} & d^{18}(\iota_2^{18} w_{(163)}) & = v_2^{18} z_{(488)} & \in G^{18}_{200,18} \end{array} \tag{5.5}$$

Remark 5.6. Standard notational abuse. In (5.5) we are abusing notation for higher differentials in the usual way. For example, the source of the d^2, is written as $\iota_2^2 u_{(19)}$. However it is not really a product in G^2 because ι_2 is no longer present there since it supported a d^1. Strictly speaking, $\iota_2^2 u_{(19)}$ is an abbreviation for the Massey product

$$\langle v_2^2,\, u_{(19)},\, u_{(19)},\, u_{(19)}\rangle \in G^2_{23,0}.$$

Similarly ι_2^3 is code for

$$\langle u_{(19)},\, v_2,\, v_2u_{(19)},\, v_2,\, u_{(19)}\rangle \in G^2_{6,0} = G^3_{6,0}.$$

An introduction to Massey products can be found in [Rav86, A1.4] (and in [Rav04, A1.4]), which is an introduction to Peter May's definitive paper on the subject [May69].

Let

$$y_{2,s} := \iota_2^{p^s} \qquad \text{for } s \geq 0 \qquad \text{and} \qquad y_{2,s+1/2} := y_{2,s}^{p-1} w_{2,s} \qquad \text{for } s \geq 0.$$

Then for a general prime p, (5.5) reads

$$(5.7) \qquad \begin{aligned} d^1(y_{2,0}) &= v_2 w_{2,0} & d^{p-1}(y_{2,1/2}) &= v_2^{p-1} z_3 \\ d^p(y_{2,1}) &= v_2^p w_{2,1} & d^{p^2-p}(y_{2,3/2}) &= v_2^{p^2-p} z_4 \\ d^{p^2}(y_{2,2}) &= v_2^{p^2} w_{2,2} & d^{p^3-p^2}(y_{2,5/2}) &= v_2^{p^3-p^2} z_5 \end{aligned}$$

The first differential reflects the fact that

$$Q_2 y_{2,0} = Q_2 \iota_2 = u_2 = w_{2,0} \in H^* K_2.$$

We also have

$$Q_2 u_s = \begin{cases} z_{2-s}^{p^s} & \text{for } 0 \leq s \leq 1 \\ 0 & \text{for } s = 2 \\ z_{s-2}^{p^2} & \text{for } s \geq 3. \end{cases}$$

These lead to

$$E_2 \cong T_p(z_1) \otimes P(v_2, y_{2,1}) \otimes E(y_{2,1/2}, w_{2,1}, w_{2,2}) \otimes T_{p^2}(z_{2+s} : s > 0)$$

modulo v_2-torsion as in (5.2), where

$$y_{2,1/2} = y_{2,0}^{p-1} w_{2,0}.$$

For $p = 3$, this reads

$$E_2 \cong T_3(z_{(8)}) \otimes P(v_2, y_{(6)}) \otimes E(y_{(23)}, w_{(55)}, w_{(163)}) \otimes T_9(z_{(56)}, z_{(164)}, z_{(488)}, \dots).$$

Lemma 5.3(iii) requires a differential hitting a v_2-multiple of $z_{(56)}$. It cannot be supported by a v_2-torsion element since its target is torsion free. The only classes in low enough dimensions live in $T_p(z_1) \otimes P(y_{2,1}) \otimes E(y_{2,1/2})$. Since z_1 is a permanent cycle, we can restrict our attention to $P(y_{2,1}) \otimes E(y_{2,1/2})$, which is $P(y_{(6)}) \otimes E(y_{(23)})$ for $p = 3$. The only class in a dimension congruent to $|z_3| - 1$ mod $|v_2|$ (55 mod 16 for $p = 3$) is $y_{2,1/2}$, which is $y_{(23)}$ for $p = 3$.

This gives us the second differential listed in (5.7). It also gives

$$E_p \cong T_p(z_1) \otimes P(v_2, y_{2,1}) \otimes E(w_{2,1}, w_{2,2}, w_{2,3}) \otimes T_{p^2}(z_{2+s} : s > 1)$$

modulo v_2-torsion, which for $p = 3$ reads

$$E_3 \cong T_3(z_{(8)}) \otimes P(v_2, y_{(6)}) \otimes E(w_{(55)}, w_{(163)}, w_{(471)}) \otimes T_9(z_{(164)}, z_{(488)}, z_{(1460)}, \dots).$$

What happens next? Something has to kill $z_{(164)}$, but none of the listed lower dimensional generators are in the right dimension to do so. However if $d^3 y_{(6)} = v_2^3 w_{(55)}$, we would get a new generator $y_{2,3/2} = y_{2,1}^{p-1} w_{2,1}$, which is $y_{(67)} = y_{(6)}^2 w_{(55)}$ at $p = 3$. It is in the right dimension to kill $v_2^6 z_{(164)}$. Thus we get the next two differentials listed in (5.7) and we have (modulo v_2-torsion)

$$E_{p+1} = E_{p^2-p} \cong T_p(z_1) \otimes P(v_2, y_{2,2}) \otimes E(y_{2,3/2}, w_{2,2}, w_{2,3}) \otimes T_{p^2}(z_{2+i} : i > 1)$$

and $$E_{p^2-p+1} \cong T_p(z_1) \otimes P(v_2, y_{2,2}) \otimes E(w_{2,2}, w_{2,3}, w_{2,4}) \otimes T_{p^2}(z_{2+i} : i > 2)$$

with $w_{2,4} = y_{2,3/2} z_4^{p^2-1}$.

For $p = 3$, this reads

$$E_3 = E_6 \cong T_3(z_{(8)}) \otimes P(v_2, y_{(18)}) \otimes E(y_{(67)}, w_{(163)}, w_{(471)}) \otimes T_9(z_{(164)}, z_{(488)}, \dots)$$

and $$E_7 \cong T_3(z_{(8)}) \otimes P(v_2, y_{(18)}) \otimes E(w_{(163)}, w_{(471)}, w_{(1379)}) \otimes T_9(z_{(488)}, z_{(1460)}, \dots).$$

Note that at each stage we have the following factors:

- $k(2)^* \otimes T_p(z_1)$,
- the polynomial algebra generated by some p^kth power of ι_2,
- an exterior algebra on three generators (two ws plus a y or a third w) with each having a dimension congruent to 3 or $2p + 1$ mod $|v_2|$, and
- a truncated polynomial algebra of height p^2 on infinitely many z_is having dimensions alternately congruent to 4 and $2p + 2$ mod $|v_2|$.

The exterior generator with the y label is the only one in a position to kill the next z. *These phenomena persist throughout the spectral sequence and generalize to larger values of n.* The factor $T_p(z_1)$ generalizes to L_n as in (2.6). The dimension of each $y_{n,i}$ is congruent to 2 modulo $2p - 2$, while those of the exterior generators and the z_is are congruent to 3 and 4 respectively. Half the generators remove $y_{n,i}$ and $w_{n,i}$, replacing them with $y_{n,i+1}$ and $y_{n,i+1/2}$. The others remove z_{n+i+1} and replace $y_{n,i+1/2}$ by $w_{n,i+n+1}$.

We want to extend (5.7) further with differentials supported by higher powers of ι_2 in the left column and ones killing v_2-multiples of higher z_is in the right column.

Since $v_2 z_3^{p^2} = 0$ in G^2, the element

$$w_{2,3} := y_{2,0}^{p-1} w_{2,0} z_3^{p^2-1} = \langle v_2^{p-2},\ v_2 z_3,\ z_3^{p^2-1} \rangle \qquad \text{as in (2.2)}$$

is a d^{p-1}-cycle and hence a target for $y_{2,3}$. Thus we have

$$d^{p^3-p+2}y_{2,3} = v_2^{p^3-p+2}w_{2,3} \qquad \text{and} \qquad d^{p^4-p^3+p-2}(y_{2,3}^{p-1}w_{2,3}) = v_2^{p^4-p^3+p-2}z_6$$

with

$$y_{2,7/2} := y_{2,3}^{p-1}w_{2,3} = \langle v_2^{(p-1)(p^3-1)}, \underbrace{w_{2,3}, \ldots, w_{2,3}}_{p \text{ factors}}\rangle.$$

We denote the indices of the differentials on $y_{2,i}$ and $y_{2,i+1/2}$ by $\rho_2(i)$ and $\rho_2(i+1/2)$. Hence the ith row of (5.7) is

$$d^{\rho_2(i)}y_{2,i} = v_2^{\rho_2(i)}w_{2,i} \qquad \text{and} \qquad d^{\rho_2(i+1/2)}y_{2,i+1/2} = v_2^{\rho_2(i+1/2)}z_{i+n+1},$$

where

$$w_{2,i} = \begin{cases} u_2 & \text{for } i=0 \\ u_3 - z_1^{p^2-p}u_1 & \text{for } i=1 \\ u_4 - z_2^{p^2-1}u_0 & \text{for } i=2 \\ y_{2,i-3}^{p-1}w_{i-1}z_i^{p^2-1} = y_{2,i-5/2}z_i^{p^2-1} & \text{for } i \geq 3 \end{cases}$$

as in (2.2).

The following is a special case of Lemma 6.4 below.

Proposition 5.8. *The indices $\rho_2(i)$ and $\rho_2(i+1/2)$ for integers $i \geq 0$ are*

$$\rho_2(i) = \begin{cases} p^i & \text{for } 0 \leq i \leq 2 \\ p^i - p^{i-2} + 1 + \rho_2(i-3) & \text{for } i \geq 3 \end{cases}$$

and $$\rho_2(i+1/2) = p^{i+1} - \rho_2(i).$$

6. Numbers and definitions

In this section we give some definitions and compute some numbers we need. We already have elements $y_{n,i}$, z_i and $w_{n,i}$ and $y_{n,i+1/2}$ with

$$\begin{aligned} |y_{n,i}| &= 2p^i, \\ |z_i| &= 2(p^i+1), \\ |y_{n,i+1/2}| &= |y_{n,i}^{p-1}w_{n,i}| = 2p^i(p-1) + |w_{n,i}|, \\ \text{and} \quad |w_{n,i+n+1}| &= |y_{n,i+1/2}z_{n+i+1}^{p^n-1}| = |y_{n,i}^{p-1}w_{n,i}z_{n+i+1}^{p^n-1}| \\ &= 2p^i(p-1) + |w_{n,i}| + 2(p^n-1)(p^{n+i+1}+1) \\ &= 2p^i(p^{2n+1} - p^{n+1} + p - 1) + 2(p^n-1) + |w_{n,i}| \\ &= 2p^i(p^{n+1}c_n + c_1) + 2c_n + |w_{n,i}|, \\ &\qquad \text{where } c_k := p^k - 1. \end{aligned} \tag{6.1}$$

Regarding these as functions of i, we will see that each one satisfies a recursive formula similar to that for $|w_{n,i}|$. To study such functions, we need some notation.

Definition 6.2.

(i) For a fixed positive integer n, let $i_1 = \lfloor i/(n+1) \rfloor$ and $i_0 = i - (n+1)i_1$ (the reduction of i modulo $n+1$) for any integer i.

(ii) Similarly let $i'_1 = \lfloor i/n \rfloor$ and $i'_0 = i - ni'_1$.

(iii) Let

$$\begin{aligned} g_n(i) := \frac{p^i - p^{i_0}}{p^{n+1}-1} &= \begin{cases} 0 & \text{for } 0 \le i \le n \\ p^{i-n-1} + g_n(i-n-1) & \text{for } i \ge n+1 \end{cases} \\ &= p^{i-(n+1)} + p^{i-2(n+1)} + p^{i-3(n+1)} + \cdots + p^{i_0}. \end{aligned}$$

It follows that i can be written uniquely as

$$i = \begin{cases} i_0 + (n+1)i_1 & \text{with } 0 \le i_0 \le n \\ i'_0 + ni'_1 & \text{with } 0 \le i'_0 < n \end{cases}$$

Lemma 6.3. *For a fixed positive integer n, suppose we have an integer valued function $f_n(i)$ defined for integers $i \ge 0$ and satisfying the recursive equation*

$$f_n(i+n+1) = ap^i + f_n(i) + b \qquad \text{for constants } a \text{ and } b.$$

Then, with notation as in Definition 6.2,

$$\begin{aligned} f_n(i) &= ag_n(i) + b\lfloor i/(n+1) \rfloor + f_n(i_0) = ag_n(i) + f_n(i_0) + bi_1 \\ &\equiv a\left(\frac{p^{i'_0} - p^{i_0}}{p-1}\right) + f_n(i_0) + b\lfloor i/(n+1) \rfloor \qquad \bmod (p^n - 1), \end{aligned}$$

and the latter expression is an integer.

Proof. Iterating the recursion relation gives

$$\begin{aligned} f_n(i) &= ap^{i-(n+1)} + f_n(i-(n+1)) + b \\ &= a\left(p^{i-(n+1)} + p^{i-2(n+1)}\right) + f_n(i-2(n+1)) + 2b \\ &\vdots \\ &= a\left(p^{i-(n+1)} + p^{i-2(n+1)} + \cdots + p^{i_0}\right) + f_n(i_0) + i_1 b \\ &= ag_n(i) + f_n(i_0) + \lfloor i/(n+1) \rfloor. \end{aligned}$$

The congruence modulo $(p^n - 1)$ follows from the fact that $p^n \equiv 1$. □

Lemma 6.4. *The values of a, b, $f_n(i_0)$ and $f_n(i) \bmod 2(p^n-1)$ for some functions of interest are shown in the following table, where again $c_k := p^k - 1$.*

$f_n(i)$	a	b	$f_n(i_0)$	$f_n(i) \bmod 2c_n$
$g_n(i)$	1	0	0	
$\lfloor i/(n+1)\rfloor$	0	1	0	
$\|y_{n,i}\| = 2p^i$	$2c_{n+1}$	0	$2p^{i_0}$	$2p^{i_0'}$
$\|u_i\| = 2p^i+1$	$2c_{n+1}$	0	$2p^{i_0}+1$	$2p^{i_0'}+1$
$\|z_i\| = 2p^i+2$	$2c_{n+1}$	0	$2p^{i_0}+2$	$2p^{i_0'}+2$
$\rho_n(i)$	pc_n	1	p^{i_0}	
$\rho_n(i+1/2)$	$p^{n+1}c_1$	-1	$p^{i_0}c_1$	
$\|w_{n,i}\|$	$2(c_1+p^{n+1}c_n)$	$2c_n$	$2p^{n+i_0}+1$	$2p^{i_0'}+1$
$\|y_{n,i+1/2}\|$	$2p^{n+1}(c_1+c_n)$	$2c_n$	$2p^{i_0}(c_n+p)+1$	$2p^{i_0'+1}+1$

In particular there are relations

$$\begin{aligned} |w_{n,i}| &\le |u_{n+i}| = 2p^{n+i}+1 \\ \rho_n(i+1/2)+\rho_n(i) &= p^{i+1}, \\ \textit{and} \quad \rho_n(i) &\le p^i < \rho_n(i+1). \end{aligned} \tag{6.5}$$

More explicitly for integers $i \ge 0$,

$$\begin{aligned} \rho_n(i) &= (p^{n+1}-p)(p^{i-(n+1)}+p^{i-2(n+1)}+\cdots+p^{i_0}) \\ &\quad + p^{i_0}+i_1 \\ &= p^i - p^{i-n} + p^{i-n-1} - p^{i-2n-1} + \cdots \\ &\quad + p^{i_0+n+1} - p^{1+i_0} + p^{i_0} + i_1 \\ &= \begin{cases} p^i & \textit{for } 0 \le i \le n \\ p^i - p^{i-n} + p^{i-n-1} + 1 & \textit{for } n+1 \le i \le 2n+1 \\ p^i - p^{i-n} + p^{i-n-1} - p^{i-2n-1} + p^{i-2n-2} + 2 & \\ & \textit{for } 2n+2 \le i \le 3n+2 \\ \vdots & \end{cases} \end{aligned} \tag{6.6}$$

We do not need the values of $f_n(i) \bmod 2(p^n-1)$ in the cases where it is not shown.

Proof of Lemma 6.4. The first five functions are defined explicitly, so filling in the columns for them is straightforward. We also know the values $f_n(i_0)$ for the last four functions listed, so it remains to determine the constants a and b for each of them. The congruences modulo $2(p^n-1)$ are also straightforward.

The constants a and b for $\rho_2(i)$ were given in Proposition 5.8.

Our differentials

$$d^{\rho_n(i)}(y_{n,i}) = v_n^{\rho_n(i)} w_{n,i} \qquad \text{and} \qquad d^{\rho_n(i+1/2)}(y_{n,i+1/2}) = v_n^{\rho_n(i+1/2)} z_{n+i+1}.$$

imply

$$\begin{aligned} & |y_{n,i}| + 1 + 2(p^n-1)\rho_n(i) = |w_{n,i}| \\ \text{and} \quad & |y_{n,i+1/2}| + 1 + 2(p^n-1)\rho_n(i+1/2) = |z_{i+n+1}|. \end{aligned} \tag{6.7}$$

This means that the constants for $\rho_n(i)$ and $\rho_n(i+1/2)$ are determined by those of $|y_{n,i}|$ and $|z_i|$, which are known, and those of $|w_{n,i}|$ and $|y_{n,i+1/2}|$, to which we now turn.

The constants a and b for $|w_{n,i}|$ are given by (6.1).

For $|y_{n,i+1/2}|$, (2.3) implies

$$\begin{aligned} y_{n,i+1/2+n+1} &= y_{n,i+n+1}^{p-1} w_{n,i+n+1} = y_{n,i+n+1}^{p-1} w_{n,i} y_{n,i}^{p-1} z_{i+n+1}^{p^n-1} \\ &= y_{n,i+n+1}^{p-1} y_{n,i+1/2} z_{i+n+1}^{p^n-1}, \\ \text{so} \quad |y_{n,i+1/2+n+1}| &= |y_{n,i+1/2}| + |y_{n,i+n+1}^{p-1} z_{i+n+1}^{p^n-1}| \\ &= |y_{n,i+1/2}| + 2(p-1)p^{i+n+1} + 2(p^n-1)(p^{i+n+1}+1), \end{aligned}$$

which gives the stated values of a and b. □

7. Preliminaries before the proof

Before proving Theorems 1.1 and 1.2, we have the following observation.

Proposition 7.1. Divisibility Criterion. *If in the Adams spectral sequence for $k(n)^*X$, $d^r(\alpha) = v_n^r\beta$, then*

$$|\alpha| + 1 + 2r(p^n-1) = |\beta|,$$

i.e., $|\beta|$ is congruent to $1 + |\alpha|$ modulo $|v_n|$.

Proof of Theorem 1.1. In [Laz01, Corollary 11.8] and [Rob89, Theorem 2.3], the odd primary $k(n)$ is shown to be A_∞. In private communication, Lazarev says that his argument for $k(n)$ works just as well for $p = 2$.

In [Rob87, p. 257], Robinson produces a Universal Coefficient Theorem for A_∞ spectra. In our case this gives the spectral sequence of Theorem 1.1. For spaces of finite type with $K(n)_*(X)$ finitely generated, $k(n)_*(X)$ is the sum of a free module (of finite dimension) over $k(n)_*$ and a sum of torsion modules, $T_k(v_n)$. The above Ext is easy to compute and everything is in Ext^0 and Ext^1. More precisely:

$$\mathrm{Ext}^{0,*}_{k(n)_*}(k(n)_*, k(n)_*) = k(n)^*$$

$$\mathrm{Ext}^{1,*}_{k(n)_*}(T_k(v_n), k(n)_*) = T_k(v_n)$$

$$\text{with generator in } \mathrm{Ext}^{1,|v_n^k|}_{k(n)_*} \text{ and } v_n \in \mathrm{Ext}^{0,-2(p^n-1)}_{k(n)_*}$$

The entire E_2 term is in Ext^0 and Ext^1. This is peculiar to $k(n)$. As a result, the spectral sequence collapses. □

Proof of Theorem 1.2. If we have $d^r(\alpha) = v_n^r\beta$ in the Adams spectral sequence for $k(n)^*(X)$, it means we have (a cohomology) $T_r(v_n)$ with generator in the degree of β. From the UCT, to get this, we must have a (homology) $T_r(v_n)$ with generator in the degree of α. To get this in the Adams spectral sequence for $k(n)_*(X)$, we must have a differential $d_{\rho_n}(\beta') = v_n^r\alpha'$ with the mentioned degrees. Reverse the argument to get the other direction. □

Remark 7.2. There is a way to invert v_n in the Adams spectral sequence which converts it to a full plane spectral sequence, the *localized* Adams spectral sequence, rather than an upper half plane one. Details can be found in [MRS01, §2.3].

Remark 7.3. It seems likely that Theorem 1.2 also follows from the method of synthetic spectra of Piotr Pstrągowski [Pst23], but we prove it with more prosaic methods. We leave the synthetic approach to the interested reader.

Before we state the next result, we need

$$H_*K_2 = \Gamma(\iota_2^*) \otimes \Gamma(z_i^* : i > 0) \otimes E(u_i^* : i \geq 0). \tag{7.4}$$

Here we have $y_{n,j}^* = \gamma_{p^j}(\iota_2^*)$ dual to $y_{n,j}$ in cohomology.

In Theorem 9.4(i), we compute the E_2 term for the Adams spectral sequence for $k(n)_*(K_2)$. In particular, $\Gamma(y_{n,1}^*)$ is there.

The following is a refined version of Lemma 5.3. Unfortunately, the proof of the crucial refinement is intertwined with the proof of the computation of the ASS, Theorem 8.1, in the next section. This result seems to best fit this section and it is easy enough to read off what is needed from Theorem 8.1.

Lemma 7.5. *For any prime, the z_i are all permanent cycles in the Adams spectral sequence for $k(n)^*(K_2)$ and there is a non-zero differential $d^r(y_{n,i})$ for some $r \leq p^i$. In the Adams spectral sequence for $k(n)_*(K_2)$, $v_n^r y_{n,i}^*$ is hit by a differential for some $r \leq p^i$.*

Proof. The image of the map

$$BP^*(K(\mathbf{Z}/p,m)) \to H^*K(\mathbf{Z}/p,m)$$

is computed by Tamanoi in [Tam97] (and much earlier in his 1983 masters thesis in Japan) and then again later in [RWY98]. In particular, the answer for $m = 2$ contains the z_i, where $i > 0$. This map factors through $k(n)^*(K_2)$ so we conclude that the z_i cannot support a differential.

Let $b_\ell \in k(n)_{2\ell}\mathbb{C}P^\infty$ be the standard generator and consider the composition

$$\mathbb{C}P^\infty \xrightarrow{p} \mathbb{C}P^\infty \longrightarrow K_2.$$

Define $b(s) = \sum_\ell b_\ell s^\ell$ and $b_{(i)} = b_{p^i}$. Note that $b_{(i)}$ maps to $y_{n,i}^* \in k(n)_*(K_2)$. We follow [RW77, Theorem 3.8(ii)] and use the fact that for $k(n)$, $[p](s) = v_n s^{p^n}$. The composition above takes $b(s)$ to zero, but the first map takes $b(s) \to b(v_n s^{p^n})$. In particular, we see that $v_n^{p^i} b_{(i)}$ maps to zero, giving $v_n^{p^i} y_{n,i}^* = 0 \in k(n)_*(K_2)$.

Since we must have a $d_{\rho_n}(\beta) = v_n^r y_{n,i}^*$ with $r \le p^i$, from Theorem 1.2, we must have a corresponding $d_n^\rho(\alpha') = v_n^r \beta'$ with $|\alpha'| = 2p^i = |y_{n,i}^*|$. We want to show that $y_{n,i} = \alpha'$. We give the proof for odd primes, $p = 2$ requires modifications. To do this, we use induction on i. We can begin the induction with $i = 0$, where $d^1(y_{n,0}) = v_n u_n$, from (2.4). We now assume the result for $y_{n,i-1}$. More than that, we assume Theorem 8.1 to the point where we have computed $d^{\rho_n(i-1)}(y_{n,i-1})$ and obtained $E_{1+\rho_n(i-1)}$. We will be finished if we can show that the only element of $E_{1+\rho_n(i-1)}$ that can have a differential on it in degree $2p^i$ is $y_{n,i}$. The only elements that can have differentials are in the $k(n)^*$-free part of $E_{1+\rho_n(i-1)}$. In Theorem 8.1, we have $\ell = 2(i-1)$, so the $k(n)^*$-free part of $E_{1+\rho_n(i-1)}$ is easy to read off as

$$E(y_{n,i-1/2}) \otimes T_{p^n}(z_{n+i}) \otimes P(y_{n,i}) \otimes W_{n,i-1} \otimes L_n \otimes TZ_{n,i}$$

The elements of L_n cannot be used because they give us $K(n)^*(K_2)$. The lowest degree element of $TZ_{n,i}$ is z_{n+i+1} and its degree is higher than $2p^i$ so we can ignore $TZ_{n,i}$. The degree of z_{n+i} of $T_{p^n}(z_{n+i})$ is also too high. All we have left to eliminate is $E(y_{n,i-1/2}) \otimes W_{n,i-1}$. The element of lowest degree in $W_{n,i-1}$ is $w_{n,i}$. For $i \le n$, the degree of $w_{n,i}$ is $2p^{n+i} + 1$ by (2.2) and too big to consider. When $i > n$, we rely on Lemma 6.4. Here we have the degree of $w_{n,i}$ is given as

$$2(c_1 + p^{n+1} c_n) p^{i-n-1} + 2c_n + |w_{n,i-n-1}|.$$

The term $2p^{n+1} c_n p^{i-n-1} = 2(p^n - 1)p^i$, so the degree is too high to worry about. All that is left is $y_{n,i-1/2}$, but it has odd degree. Although unnecessary at this stage, the degree of $y_{n,i-1/2}$ is also greater than $2p^i$. □

8. The Adams spectral sequence for odd primes

The E_2-term of the odd primary Adams spectral sequence for $k(n)^*(K_2)$ is the subject of Theorem 4.3.

Theorem 8.1. Adams differentials and intermediate terms for $k(n)^*(K_2)$.

(i) In the odd primary Adams spectral sequence for $k(n)^*(K_2)$, the differentials d^r for $r \ge 1$ are

$$d^1(y_{n,0}) = v_n u_n$$

$$d^1(u_s) = \begin{cases} v_n z_{n-s}^{p^s} & \text{for } 0 \le s < n \\ v_n z_{s-n}^{p^n} & \text{for } s > 2n, \end{cases}$$

$$d^{\rho_n(i+1/2)}(y_{n,i+1/2}) = v_n^{\rho_n(i+1/2)} z_{n+i+1} \qquad \text{for } i \ge 0$$

$$\text{and} \qquad d^{\rho_n(i)}(y_{n,i}) = v_n^{\rho_n(i)} w_{n,i} \qquad \text{for } i > 0,$$

where $\rho_n(\ell/2)$ is as in Lemma 6.4.

(ii) For each $\ell \geq 0$,

$$\begin{aligned} E_{1+\rho_n(\ell/2)} &= E_{\rho_n((\ell+1)/2)} \\ &= \bigoplus_{0\leq k\leq \ell} \left(S_{n,k/2}\otimes M_{n,k/2}\right) \\ &\quad \oplus \left(k(n)^* \otimes \left\{ \begin{array}{c} E(y_{n,(\ell+1)/2})\otimes T_{p^n}(z_{n+1+\ell/2}) \\ \text{for } \ell \text{ even} \\ T_p(y_{n,(\ell+1)/2})\otimes E(w_{n,(\ell+1)/2}) \\ \text{for } \ell \text{ odd} \end{array} \right\} \otimes M_{n,(\ell+1)/2} \right) \end{aligned}$$

for $M_{n,\ell/2}$ and $S_{n,\ell/2}$ as in (2.11).

In (ii) note that as ℓ goes to ∞, both the expressions enclosed in braces go to $\mathbf{Z}/p$ and $M_{n,(\ell+1)/2}$ goes to L_n. Hence the last summand goes to $k(n)^*\otimes L_n$, and (ii) implies that the Adams E_∞-term is the $k(n)^*$-module described in Theorem 2.12.

Proof of Theorem 2.12 and Theorem 8.1(ii) assuming Theorem 8.1(i). The Adams E_1-term is

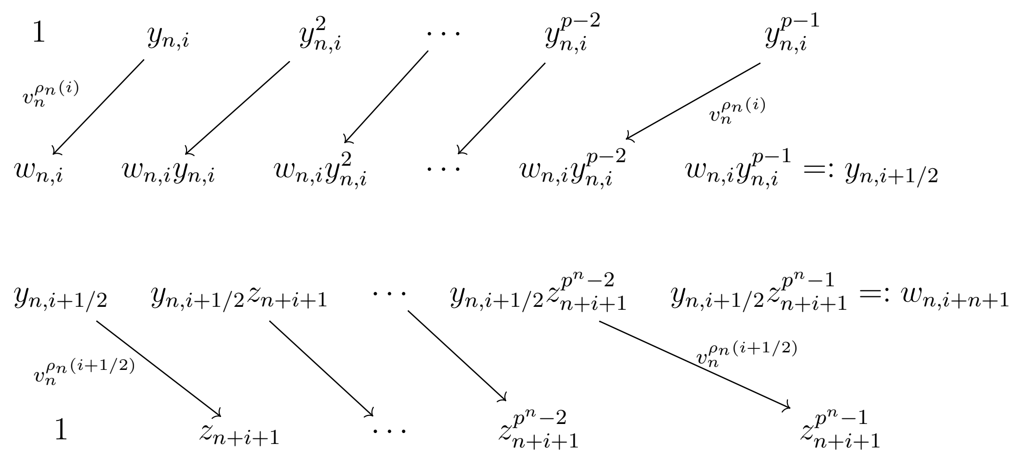
$$k(n)^*\otimes H^*K_2.$$

Our d^1 for an odd prime p was computed in §2.3 and our E_2 in Theorem 4.3. The remaining $k(n)^*$-free part was $k(n)^*\otimes E(y_{n,1/2})\otimes M_{n,0}$, but $M_{n,0} = T_{p^n}(z_{n+1})\otimes M_{n,1/2}$, giving us the answer for the above Adams E_2-term.
Higher differentials involve multiplication by higher powers of v_n, so they cannot affect the torsion submodule.

The remaining Adams differentials, starting with $d^{\rho_n(1/2)}(y_{n,1/2}) = v_n^{\rho_n(1/2)}z_{n+1}$ (where $\rho_n(1/2) = p-1$), have the following effects on the indicated subquotient rings of E_2, $E_{1+\rho_n(i)}$ and $E_{1+\rho_n(i+1/2)}$ for integers $i\geq 0$.

(8.2)
$$\begin{aligned} y_{n,i+1/2} &\mapsto v_n^{\rho_n(i+1/2)}z_{n+i+1} \\ k(n)^*\otimes E(y_{n,i/2})\otimes T_{p^n}(z_{n+i+1}) &\rightsquigarrow (k(n)^*\otimes E(w_{n,n+i+1})) \\ &\qquad \oplus (T_{\rho_n(i+1/2)}(v_n)\otimes z_{n+i+1}T_{p^n-1}(z_{n+i+1})) \\ &= (k(n)^*\otimes E(w_{n,n+i+1}))\oplus S_{n,i+1/2} \\ y_{n,i+1} &\mapsto v_n^{\rho_n(i+1)}w_{n,i+1} \\ k(n)^*\otimes T_p(y_{n,i+1})\otimes E(w_{n,i+1}) &\rightsquigarrow (k(n)^*\otimes E(y_{n,i+3/2})) \\ &\qquad \oplus (T_{\rho_n(i+1)}(v_n)\otimes w_{n,i+1}T_{p-1}(y_{n,i+1})) \\ &= (k(n)^*\otimes E(y_{n,i+3/2}))\oplus S_{n,i+1} \end{aligned}$$

These are illustrated by the following diagrams, in which an arrow $\alpha\to\beta$ labeled by v_n^r for some r means that $d^r\alpha = v_n^r\beta$. Within each of the two diagrams, all arrows should bear the same label, but all but two labels have been omitted to avoid clutter.



(8.3)

Let $S_{n,i}$ and $S_{n,i+1/2}$ be as in (2.11). Then the new torsion modules created by $d^{\rho_n(i)}$ for $i > 0$ and $d^{\rho_n(i+1/2)}$ for $i \geq 0$ are respectively

$$S_{n,i} \otimes M_{n,i} \qquad \text{and} \qquad S_{n,i+1/2} \otimes M_{n,i+1/2}.$$

These give Theorem 8.1(ii).

This means that the Adams E_∞-term has the form indicated in Theorem 2.12. We have to be sure that there are no nontrivial extensions in $k(n)^*$-module structure.

Suppose that for some i, $v_n^{\rho_n(i)}w_{n,i}$ is not zero but instead has higher filtration than expected. This would mean

$$v_n^{\rho_n(i)}w_{n,i} = v_n^{r_{n,i}}x_{n,i} \qquad \text{for some } x_{n,i} \text{ with } r_{n,i} > \rho_n(i).$$

Then we would have

$$v_n^{\rho_n(i)}(w_{n,i} - v_n^{r_{n,i}-\rho_n(i)}x_{n,i}) = 0,$$

and we could define

$$w'_{n,i} := w_{n,i} - v_n^{r_{n,i}-\rho_n(i)}x \qquad \text{with } v_n^{\rho_n(i)}w'_{n,i} = 0$$

in that filtration of the spectral sequence. Of course this could also be in a higher filtration, but this has to end because each degree of Theorem 2.12 is finite. This follows from $K(n)^*(K_2)$ finite over $K(n)^*$ and K_2 being of finite type. In the end, our final element would represent the same element in E_∞ as $w_{n,i}$, so the $k(n)^*$-module structure would still be as stated in Theorem 2.12.

A similar argument works for the relation $v_n^{\rho_n(i+1/2)}z_{n_{i+1}} = 0$. □

Overview of the Proof of Theorem 8.1(i). We can assume by induction that we have $E_{\rho_n(i-1/2)+1}$ and we want to get to $E_{\rho_n(i+1/2)+1}$. There are two parts to the proof. We must establish the two differentials, $\rho_n(i)$ and $\rho_n(i+1/2)$, but at the same time we have to show that there are no other differentials. The logical way this should go is to show that there are no d^r with $\rho_n(i-1/2) < r < \rho_n(i)$, then compute $\rho_n(i)$. After this, show there are no d^r with $\rho_n(i) < r < \rho_n(i+1/2)$, then compute $\rho_n(i+1/2)$. This is not what we do, but it is best to interpret what we do this way. It turns out that computing the differentials and showing there are no other differentials are independent of each other. Furthermore, it is unnecessary to break up the non-existence of the d^r into two parts because the proof is exactly the same for both parts. So, what we do is show the two differentials must exist, no matter if there are other differentials or not. After that, we show there are no extra differentials. The two proofs could go in the opposite order, or be done in the proper sequence in the way that makes the most sense. However, rather than do them in the proper sequence, the reader can interpret the proofs that way.

Proof of Theorem 8.1(i) . Our proof starts with showing the asserted differentials must happen. Then we have to show that there are no additional differentials. This is where the full power of *The Pairing* comes in.

Assume by induction that we have $E_{\rho_n(i-1/2)+1}$. We must have a $d^r(y_{n,i}) = v_n^r q$ where q has odd degree and $r \leq p^i$ by Lemma 7.5. There are few odd degree elements in this range. We will show that if $q = w_{n,i+1}$, we would have $r > p^i$. This eliminates all $q = w_{n,i+j}$, $j > 1$, because their degree is even higher. We want to show

$$|w_{n,i+1}| - 1 - |y_{n,i}| > 2p^i(p^n - 1)$$

Using the formula for $w_{n,i+1}$ of (6.7), the left hand side becomes

$$\begin{aligned} &|y_{n,i+1}| + 1 + 2\rho_n(i+1)(p^n-1) - 1 - |y_{n,i}| \\ &\qquad = 2(p-1)p^i + 2\rho_n(i+1)(p^n-1), \end{aligned}$$

which makes the desired inequality

$$\rho_n(i+1) > \frac{p^{i+n} - p^{i+1}}{p^n - 1} = \frac{p^i(p^n - p)}{p^n - 1}$$

It is enough to have $\rho_n(i+1) > p^i$, because this is larger than the term on the right, but this is in (6.5).

The only remaining elements of odd degree that have degree less than $|y_{n,i}| + 2p^i(p^n-1)$ are $y_{n,i}^s w_{n,i}$ for $s > 0$. However, since we know $w_{n,i}$ meets the *Divisibility Criterion*, we must have s at least $p^n - 1$. Then the index of the differential would be $p^i + \rho_n(i)$, and this is greater than p^i so can't happen. We conclude that we *must* have $d^{\rho_n(i)} y_{n,i} = v_n^{\rho_n(i)} w_{n,i}$ as claimed.

Later, when we show there are no extraneous differentials, that proof actually shows there are no differentials d^r where $r < \rho_n(i)$, so when we do the computation for this differential, there is no interference from other possible differentials, because they do not exist.

Thus, the action of this differential takes place in $k(n)^* \otimes P(y_{n,i}) \otimes E(w_{n,i})$ which can be broken up as $k(n)^* \otimes P(y_{n,i+1}) \otimes T_p(y_{n,i}) \otimes E(w_{n,i})$. The remaining v_n-torsion free part is $k(n)^* \otimes P(y_{n,i+1}) \otimes E(y_{n,i+1/2})$, giving us $E_{\rho_n(i)+1}$.

By Lemmas 5.3 and 7.5, we know that z_{n+i+1} is a permanent cycle and that some $v_n^r z_{n+i+1}$ must be hit by a differential coming from an odd degree element. Remember that we are now working in $E_{\rho_n(i)+1}$. Furthermore, our proof that there are no other differentials than those specified shows that there are no differentials d^r with $\rho_n(i) + 1 < r < \rho_n(i + 1/2)$.

Lemma 8.4. *If $d^r(w_{n,i+j}) = v_n^r z_{n+i+1}$ for some $j > 0$, then $r \leq \rho_n(i - 1/2)$.*

Proof. It is enough to study the $j = 1$ case. If this differential is too short, then it is even shorter for $j > 1$. We would have

$$|w_{n,i+1}| + 1 + 2r(p^n - 1) = |z_{n+i+1}|.$$

Replace $|w_{n,i+1}|$ using (6.7)

$$|y_{n,i+1}| + 1 + 2\rho_n(i + 1)(p^n - 1) + 1 + 2r(p^n - 1) = |z_{n+i+1}|.$$

Plugging in the numbers for $y_{n,i+1}$ and z_{n+i+1} and rearranging, we get

$$2\rho_n(i + 1)(p^n - 1) + 2r(p^n - 1) = 2p^{n+i+1} - 2p^{i+1} = 2p^{i+1}(p^n - 1).$$

So, $r = p^{i+1} - \rho_n(i + 1)$.

We need to show this is $\leq \rho_n(i - 1/2)$. We use the formulas from Lemma 6.4. We need to show that $p^{i+1} \leq \rho_n(i + 1) + \rho_n(i - 1/2)$. This is easy for small i, so using the formulas and induction, we need to show

$$p^{i+n+2} \leq \rho_n(i + n + 2) + \rho_n(i + n + 1/2).$$

The right hand side is

$$p^{i+2}(p^n - 1) + 1 + \rho_n(i + 1) + p^{n+i}(p - 1) - 1 + \rho_n(i - 1/2).$$

Expanding and using induction, this is greater than or equal to

$$p^{n+i+2} - p^{i+2} + p^{n+i+1} - p^{n+i} + p^{i+1}.$$

For this to be greater than or equal to p^{n+i+2}, we need

$$p^{n+i+1} + p^{i+1} \geq p^{n+i} + p^{i+2}.$$

This is obvious for $n > 1$, and we get an equality when $n = 1$. □

Lemma 8.4 rules out all $w_{n,i+j}$ with $j>0$ as the source of a differential hitting a v_n-multiple of z_{n+i+1} because we assume that $E_{1+\rho_n(i)}$ has already been computed.

Now the only odd degree elements left in degrees less than $|z_{n+i+1}|$ are the $y^s_{n,i+1}y_{n,i+1/2}$. We know $y_{n,i+1/2}$ would work with differential $\rho_n(i+1/2)$ because of the *Divisibility Criterion*, Proposition 7.1 and (6.7). The *Divisibility Criterion* requires s to be a multiple of p^n-1. The lowest non-zero s is $s=p^n-1$ and this would give a differential of length $\rho_n(i+1/2)-p^{i+1}$, but $p^{i+1}>\rho_n(i+1/2)$ so this cannot happen. We *must* have $d^{\rho_n(i+1/2)}(y_{n,i+1/2})=v_n^{\rho_n(i+1/2)}z_{n+i+1}$.

The part of $E_{\rho_n(i)+1}$ that the action of $d^{\rho_n(i+1/2)}$ takes place in is $P(v_n)\otimes E(y_{n,i+1/2})\otimes T_{p^n}(z_{n+i+1})$ and results in the $P(v_n)$-free part being $E(w_{n,i+(n+1)})$, giving us $E_{\rho_n(i+1/2)+1}$.

Having computed these differentials, we can use *The Pairing* of Theorem 1.2 to get the dual differentials for the Adams spectral sequence for $k(n)_*(K_2)$ in Theorem 9.4. We first show $d_{\rho_n(i)}(w^*_{n+i})=v_n^{\rho_n(i)}y^*_i$. We know, from Lemma 7.5 that some differential must hit some $v_n^r y^*_i$ with $r\le p^i$. From *The Pairing*, we know that some element, q, in the degree of y^*_i must have $d_{\rho_n(i)}(m)=v_n^{\rho_n(i)}q$. However, in $E_{\rho_n(i-1/2)}$, we see that y^*_i is the only element there is in that degree and w^*_{n+i} is the only odd degree generator in the correct degree. The only option is the expected result. Again, the pairing gives us a $d_{\rho_n(i+1/2)}$ in degrees corresponding to z^*_{n+i+1} and $w^*_{n+i+1/2}$. There are no other options, so $d_{\rho_n(i+1/2)}$ is as advertised.

Having computed these two must-have differentials, it gives us the description of $E_{\rho_n(i+1/2)+1}$ of Theorem 8.1(ii).

We are not finished. *We must show that there are no extraneous d^rs.* We assume, by induction, that we have $E_{\rho_n(i-1/2)+1}$. The first step is to show there are no differentials with $\rho_n(i-1/2)+1<r<\rho_n(i)$. Any such differential would take place on $E_{\rho_n(i-1/2)+1}$. There are no differentials on the z's because they are permanent cycles. The element $y_{n,i}$ is reserved for our special differentials as is $w_{n,i}$. The only possible differentials are on the $w_{n,i+j}$ with $1\le j\le n$. If we show there are no such differentials, we get our $\rho_n(i)$. Then we have to consider d^r with $\rho_n(i)+1<r<\rho_n(i+1/2)$. These would take place on $E_{\rho_n(i)+1}$. Here we cannot use $y_{n,i+1/2}$ or z_{n+1+i} because they are reserved for the special differentials already found to be necessary. The remaining options are $y_{n,i+1}$ and $w_{n,i+j}$ with $1\le j\le n$. But this is the same as we had before with the exception of $y_{n,i+1}$. This is easy to eliminate because the lowest odd degree element is $w_{n,i+1}$. From our computation of the special differentials, we know this would require (from (6.7)) $\rho_n(i+1)>\rho_n(i+1/2)$, so the differential would be too long. We can now concentrate on showing there are no extra differentials d^r with $\rho_n(i-1/2)<r\le\rho_n(i+1/2)$ that start on a $w_{n,i+j}$, $1\le j\le n$.

Let r be the smallest r in the range $\rho_n(i-1/2) < r \leq \rho_n(i+1/2)$ with $d^r(w_{n,i+j}) = v_n^r\beta \neq 0$, for the $w_{n,i+j}$ of smallest degree. We know from Theorem 1.2, *The Pairing*, that there is a β', with $|\beta'| = |\beta|$, in the homology Adams spectral sequence with $d_r(\beta') = v_n^r\alpha' \neq 0$. If β' is decomposable, then there must be an element, β'', with lower degree than β' with $d_r(\beta'') = v_n^r\alpha'' \neq 0$. For example, if $\beta' = ab$, then $d_r(\beta') = d_r(a)b \pm ad_r(b)$ and either $d_r(a)$ or $d_r(b)$ is non-zero. In either case, we get our β'' with degree less than $|\beta'|$. Again, by *The Pairing*, there is an α in the cohomology Adams spectral sequence with $|\alpha| = |\alpha''| < |w_{n,i+j}|$ with $d^r(\alpha) \neq 0$. This contradicts our choice of $w_{n,i+j}$. We conclude that if there is such an r, β' is indecomposable. Theorem 1.2, *The Pairing*, is pretty vague about what the corresponding elements are. All it really gives us are degrees.

Since we started with the odd degree $w_{n,i+j}$, we are looking for an even degree target element. However, we know where all the even degree indecomposables are in the homology spectral sequence, dual to $E_{\rho_n(i-1/2)+1}$. These elements are the y_s^*, $s \geq i$, and the $\gamma_{p^k}(z^*_{n+i+s})$, with $k < n$ and $s > 0$. . We have similar looking elements in $E_{\rho_n(i-1/2)+1}$, namely, $y_{n,s}$, $s \geq i$, and $z^{p^k}_{n+i+s}$, $s > 0$, $k < n$. They are not known to be "dual" in any sense, but they are in the right degrees. All we will use about these cohomology elements is their degree. If we can show that there are no differentials that hit elements in these degrees, we are done. We overlooked some elements in our original proof, but a very persistent referee forced us to find them. This led to a complete reworking of the proof, a dramatic improvement.

We have three main ways to show a differential cannot exist. (1) We can use the *Divisibility Criterion*, (2) we can show that a prospective d^r has $r > \rho_n(i+1/2)$, or (3) we can show that $r \leq \rho_n(i-1/2)$.

First we have to check to see if there is some s with $d^r(w_{n,i+j}) = v_n^r y_{n,s}$, again, we repeat, only using the degree of $y_{n,s}$. (If we could actually use $y_{n,s}$ this would be easy because we know it is a source and cannot be a target.) For this, we must have

$$|w_{n,i+j}| + 1 + 2r(p^n - 1) = |y_{n,s}|$$

but we can replace the first term using (6.7)

$$|y_{n,i+j}| + 1 + 2\rho_n(i+j)(p^n - 1) + 1 + 2r(p^n - 1) = |y_{n,s}|$$

so

$$|y_{n,s}| - |y_{n,i+j}| - 2 = 2p^s - 2p^{i+j} - 2$$

is both positive and divisible by $2(p^n - 1)$. This cannot be zero mod $2(p^n - 1)$ so the *Divisibility Criterion* tells us we cannot have this differential.

The elements $z^{p^k}_{n+i+s}$ below are in degrees that correspond to the degrees of the remaining even degree generators in the homology version. We have to show, using only their degrees, that there is **no** differential

$$d^r(w_{n,i+j}) = v_n^r z_{n+i+s}^{p^k} \quad \text{with}$$

(8.5)
$$0 < j \le n,\ 0 < s, 0 \le k < n,\ 0 < i$$
$$\text{and } \rho_n(i-1/2) < r \le \rho_n(i+1/2).$$

We have

$$|w_{n,i+j}| + 1 + 2r(p^n-1) = |z_{n+i+s}^{p^k}|$$

We replace $|w_{n,i+j}|$ with $|y_{n,i+j}| + 1 + 2\rho_n(i+j)(p^n-1)$ from (6.7) so we have

$$|y_{n,i+j}| + 1 + 2\rho_n(i+j)(p^n-1) + 1 + 2r(p^n-1) = |z_{n+i+s}^{p^k}|.$$

Turning this into numbers and rearranging,

$$\begin{aligned} 2r(p^n-1) &= |z_{n+i+s}^{p^k}| - 2 - |y_{n,i+j}| - 2\rho_n(i+j)(p^n-1) \\ &= 2p^{n+i+k+s} + 2p^k - 2 - 2p^{i+j} - 2\rho_n(i+j)(p^n-1) \end{aligned}$$

First we ask, when is this too big, that is, when is

$$2\rho_n(i+1/2)(p^n-1) < 2p^{n+i+k+s} + 2p^k - 2 - 2p^{i+j} - 2\rho_n(i+j)(p^n-1)$$

Rearranging, when is

$$2\rho_n(i+1/2)(p^n-1) + 2 + 2p^{i+j} + 2\rho_n(i+j)(p^n-1) < 2p^{n+i+k+s} + 2p^k$$

From (6.5) we know $\rho_n(i+j) \le p^{i+j}$, so when is

$$2\rho_n(i+1/2)(p^n-1) + 2 + 2p^{i+j} + 2p^{i+j}(p^n-1) < 2p^{n+i+k+s} + 2p^k.$$

The large terms on left and right are the $2p^{n+i+j}$ and $2p^{n+i+k+s}$. So this inequality holds when $n+i+j < n+i+k+s$, $j < k+s$, or $s > j-k$.

What we have left is $s \le j-k$, or $s = j-k-t$ with $t \ge 0$. We can't have t too big because $j \le n$. Using the above approach, it is easy to see that the differential is too short if $j > k+s+1$, or $s < j-k-1$. Unfortunately, that misses a couple of cases, namely $s = j-k-\epsilon$ when ϵ is 0 or 1. Those cases are more delicate, but since they are also too short, we do them all at once.

When our differential is too short, we have

$$2\rho_n(i-1/2)(p^n-1) \ge 2p^{n+i+k+s} + 2p^k - 2 - 2p^{i+j} - 2\rho_n(i+j)(p^n-1)$$

Substitute $s = j-k-t$ to get

$$2\rho_n(i-1/2)(p^n-1) \ge 2p^{n+i+j-t} + 2p^k - 2 - 2p^{i+j} - 2\rho_n(i+j)(p^n-1)$$

When $i+j < n+1$ we can compute all the numbers and show this is true easily, so we will assume $i+j \ge n+1$. Rearrange and use Lemmas 6.3 and 6.4 to get

$$2\rho_n(i-1/2)(p^n-1) + 2 + 2p^{i+j}$$

$$+2\Big(p(p^n-1)p^{i+j-n-1}+\rho_n(i+j-n-1)+1\Big)(p^n-1)$$
$$\geq 2p^{n+i+j-t}+2p^k$$

Replace $\rho_n(i-1/2)$ with $p^i-\rho_n(i-1)$ and multiply everything out and rearrange to get the left hand side as

$$2p^{n+i}+2\rho_n(i-1)+2+2p^{n+i+j}+2p^{i+j-n}+2\rho_n(i+j-n-1)p^n+2p^n$$

and the right hand side

$$2p^i+2\rho_n(i-1)p^n+2p^{i+j}+2\rho_n(i+j-n-1)+2p^{n+i+j-t}+2p^k.$$

Because we know $\rho_n(i-1)\leq p^{i-1}$, $k<n$, and $\rho_n(i+j-n-1)\leq p^{i+j-n-1}$, the largest term on the left is the $2p^{n+i+j}$ and the inequality holds when $t>0$. When $t=0$ we can cancel the big terms on both sides and we have the left side is

$$2p^{n+i}+2\rho_n(i-1)+2+2p^{i+j-n}+2\rho_n(i+j-n-1)p^n+2p^n$$

and the right

$$2p^i+2\rho_n(i-1)p^n+2p^{i+j}+2\rho_n(i+j-n-1)+2p^k.$$

The largest term on the left is now $2p^{n+i}$. On the right, the largest is $2p^{i+j}$ when $j=n$. When this happens those two terms cancel and we are looking at

$$2\rho_n(i-1)+2+2p^i+2\rho_n(i-1)p^n+2p^n\geq 2p^i+2\rho_n(i-1)p^n+2\rho_n(i-1)+2p^k$$

where lots of things cancel to give us

$$2+2p^n\geq 2p^k$$

which is true because $k<n$.

This concludes all of the cases we needed to check. There are no more differentials than those already produced.

Of course if there are no more differentials in the Adams spectral sequence for $k(n)^*(K_2)$, then *The Pairing* says there are no more for $k(n)_*(K_2)$. □

9. From cohomology to homology

We now turn from $k(n)^*(K_2)$ to $k(n)_*(K_2)$. We have already given H_*K_2 in (7.4). We need the Q_n homology of H_*K_2, but it is just dual to the Q_n homology of H^*K_2 described in Section 4. It gives us the E_2 term of the Adams spectral sequence for $k(n)_*(K_2)$, which we spell out in Theorem 9.4(i).

We give the E_2 term of the Adams spectral sequence for $k(n)_*(K_2)$ and describe all the differentials in Theorem 9.4 , and give the final result as a $k(n)_*$-module in Theorem 9.5 . The proofs are dual to the proofs for $k(n)^*(K_2)$.

Using the notation of (3.6), we have elements

$$
(9.1)\qquad
\begin{aligned}
v_n &\in G_2^{2(p^n-1),1} = E_2^{1,2(p^n-1)+1},\\
y^*_{n,j} &\in G_2^{2p^j,0} = E_2^{0,2p^j},\\
w^*_{n,i} &\in G_2^{2p^{n+i}+1,0} = E_2^{0,2p^{n+i}+1}\\
\text{and}\qquad z^*_{j,s} = \gamma_{p^s}(z^*_j) &\in G_2^{2p^s(p^j+1),0} = E_2^{0,2p^s(p^j+1)}.
\end{aligned}
$$

The dual analog of (2.6) is

$$
(9.2)\qquad
\begin{aligned}
EE^*_n &:= E(u^*_s : 0 \le s < n) \otimes E(u^*_{2n+s} : s > 0),\\
W^*_{n,i} &:= E(w^*_{n,i+s} : 1 \le s \le n),\\
L^*_n &:= \bigotimes_{0<s<n} \Gamma_{p^{n-s}}(z^*_s),\\
TZ^*_{n,i} &:= \Gamma_{p^n}(z^*_{n+s} : s > i), \qquad \text{and}\\
PZ_n &:= \Gamma(\gamma_{e_n(s)}(z^*_s) : s > 0) \qquad \text{with } e_n(s) \text{ as in (2.6).}
\end{aligned}
$$

We also need dual analogs of (2.9) and (2.11).

$$
(9.3)\qquad
\begin{aligned}
D^*_1 &:= \Gamma_p(y^*_{n,0}) \otimes E(w^*_{n,0}) \otimes EE^*_n \otimes PZ^*_n,\\
S^*_{n,0} &:= D^*_1,\\
M^*_{n,i} &:= \Gamma(y^*_{n,i+1}) \otimes W^*_{n,i} \otimes L^*_n \otimes TZ^*_{n,i} \qquad \text{for } i \ge 0,\\
M^*_{n,i+1/2} &:= \Gamma(y^*_{n,i+1}) \otimes W^*_{n,i} \otimes L^*_n \otimes TZ^*_{n,i+1} \qquad \text{for } i \ge 0,\\
S^*_{n,i} &:= T_{\rho_n(i)}(v_n) \otimes y^*_{n,i}\overline{\Gamma_{p-1}(y^*_{n,i})},\\
&= T_{\rho_n(i)}(v_n) \otimes \left\{\gamma_s(y^*_{n,i}) : 1 \le s \le p-1\right\} \qquad \text{for } i > 0,\\
\text{and}\qquad S^*_{n,i+1/2} &:= T_{\rho_n(i+1/2)}(v_n) \otimes z^*_{n+i+1}\overline{\Gamma_{p^n-1}(z^*_{n+i+1})}\\
&= T_{\rho_n(i+1/2)}(v_n) \otimes \left\{\gamma_s(z^*_{n+i+1}) : 1 \le s \le p^n-1\right\} \qquad \text{for } i \ge 0.
\end{aligned}
$$

Using this notation, we can now state the dual of Theorems 4.3 and 8.1.

Theorem 9.4. The Adams E_2-term, differentials and intermediate terms for $k(n)_*(K_2)$.

(i) The E_2 term of the odd primary Adams spectral sequence for $k(n)_*(K_2)$ is

$$P(v_n) \otimes L^*_n \otimes \Gamma(y^*_{n,1}) \otimes E(y^*_{n,1/2}) \otimes W^*_{n,0} \otimes TZ^*_{n,0}$$

plus a computable family of filtration-0 $\mathbf{Z}/p$'s annihilated by v_n coming from the dual of the $E(Q_n)$-free part of H_*K_2, specified in Theorem 2.12.

(ii) In the odd primary Adams spectral sequence for $k(n)_*(K_2)$, the differentials d_r for $r \geq 1$ are

$$\begin{aligned} d_1(u_n^*) &= v_n y_{n,0}^*, && \\ d_1(\gamma_{p^s}(z_{n-s}^*)) &= v_n u_s^* && \text{for } 0 \leq s < n, \\ d_1(\gamma_{p^n}(z_{s-n}^*)) &= v_n u_s^* && \text{for } s > 2n, \\ d_{\rho_n(i+1/2)}(z_{n+i+1}^*) &= v_n^{\rho_n(i+1/2)} y_{n,i+1/2}^* && \text{for } i \geq 0, \\ \text{and} \qquad d_{\rho_n(i)}(w_{n,i}^*) &= v_n^{\rho_n(i)} y_{n,i}^* && \text{for } i > 0. \end{aligned}$$

(iii) For each $\ell \geq 0$,

$$\begin{aligned} E_{1+\rho_n(\ell/2)} &= E_{\rho_n((\ell+1)/2)} \\ &= \bigoplus_{0 \leq k \leq \ell} \left(S_{n,k/2}^* \otimes M_{n,k/2}^*\right) \\ &\quad \oplus \left(k(n)^* \otimes \left\{ \begin{array}{c} E(y_{n,(\ell+1)/2}^*) \otimes \Gamma_{p^n}(z_{n+1+\ell/2}^*) \\ \text{for } \ell \text{ even} \\ \Gamma_p(y_{n,(\ell+1)/2}^*) \otimes E(w_{n,(\ell+1)/2}^*) \\ \text{for } \ell \text{ odd} \end{array} \right\} \otimes M_{n,(\ell+1)/2}^* \right) \end{aligned}$$

for $M_{n,\ell/2}^*$ and $S_{n,\ell/2}^*$ as in (9.3).

The dual of Theorem 2.12 is

Theorem 9.5. *For an odd prime p, $k(n)_*(K_2)$ has the following three summands as a $k(n)_*$-module:*

(i) The $k(n)_*$ free summand, *$k(n)^* \otimes L_n^*$, for L_n as in (9.2).*

(ii) The higher torsion summand,

$$\bigoplus_{\ell > 0} \left(M_{n,\ell/2}^* \otimes S_{n,\ell/2}^*\right),$$

for $M_{n,\ell/2}^*$ and $S_{n,\ell/2}^*$ as in (9.3).

(iii) The elementary torsion summand, *$S_{n,0}^* \otimes M_{n,0}^*$ as in (9.3).*

These follow from Theorems 8.1 and 2.12 using the duality of Theorem 1.2.

10. Modifications for $p = 2$

All we do in this section is to lay out the results for $k(n)^*(K_2)$ for $p = 2$. We skip the homology version and proofs. We do this with a twinge of guilt. The very first case done was the $p = 2$, $n = 1$ case, and there, the generally useful *Divisibility*

Criterion is worthless. Consequently, there are lots of little *ad hoc* arguments that must be done in that case.

For $p=2$, $H^*K_2 = P(\iota_2)\otimes_{i\geq 0} P(u_i)$, with $u_i = Q_i\iota_2$. We let $u_i^2 = z_{i+1} = Q_{i+1}Q_0\iota_2$ ($z_0 = 0$). In an attempt to be as similar as possible with notation, we have

$$\begin{aligned} & |u_i| = 2\times 2^i + 1 \\ \text{and} \quad & |z_i| = 2(2^i+1) \qquad \text{for } i>0 \\ & |Q_n| = 2\times 2^n - 1 \end{aligned}$$

$$y_{n,j} = \iota_2^{2^j} \qquad \text{for } j\geq 0 \text{ in degree } 2\times 2^j$$

with

$$Q_n u_i = \begin{cases} (u_{n-i-1})^{2^{i+1}} = z_{n-i}^{2^i} & \text{for } 0\leq i<n \\ 0 & \text{for } n\leq i\leq 2n \\ (u_{i-n-1})^{2^{n+1}} = z_{i-n}^{2^n} & \text{for } i>2n \end{cases}$$

$$Q_n y_{n,0} = u_n$$

The formulas used for odd primes mostly work here

$$w_{n,i} := \begin{cases} u_n & \text{for } i=0 \\ u_{n+i} + u_{n-i}z_i^{2^n-2^{n-i}} & \text{for } 0<i\leq n \\ u_{2n+1} + y_{n,1/2}z_{n+1}^{2^n-1} & \text{for } i=n+1 \\ y_{n,i-n-1/2}z_i^{2^n-1} & \text{for } i>n+1 \end{cases}$$

$$y_{n,j+1/2} := y_{n,j}w_{n,j}.$$

The only difference between the above and (2.2) is the definition of $w_{n,n+1}$, which here includes u_{2n+1}. For odd primes we have

$$d^1 u_{n+1} = v_n z_{n+1}^{p^n}, \qquad \text{and} \qquad d^{p-1}y_{n,1/2} = v_n^{p-1}z_{n+1}$$

making the heuristic expression

$$w_{n,n+1} := y_{n,1/2}z_{n+1}^{p^n-1} - v_n^{p-2}u_{n+1}$$

a cycle. We do not see the second term for $p>2$ because it has higher filtration, but for $p=2$ both terms have filtration 0.

We can compute the Q_n-homology of H^*K_2 with a diagram like that of (4.1).

Theorem 10.1. *We have elements* $v_n \in G^2_{-2(2^n-1),1}$, $y_{n,1}\in G^2_{4,0}$, $w_{n,i}\in G^2_{2^{n+i+1}+1,0}$, *and* $z_j \in G^2_{2^{j+1}+2,0}$. *The* E_2 *term of the* $p=2$ *Adams spectral sequence for* $k(n)^*(K_2)$ *is*

$$P(v_n)\otimes \bigotimes_{0<i<n} T_{2^i}z_{n-i}\otimes P(y_{n,1})$$
$$\otimes T_{2^{n+1}}(w_{n,i+1} : 0\leq i\leq n)\otimes T_{2^n}(z_{2n+1+i+1} : i>0)$$

plus a computable family of filtration-0 $\mathbf{Z}/2$'s annihilated by v_n coming from the $E(Q_n)$-free part of H^*K_2.

For convenience we reset $z_{n+i+1} = w_{n,i}^2$ for $0 < i \leq n+1$.

Proposition 10.2. *For $p = 2$, the differentials in the $p = 2$ Adams spectral sequence for $k(n)^*(K_2)$ are:*

(i) For $0 < j \leq n+1$, $\rho_n(j) = 2^j = \rho_n(j+1/2)$. Although $y_{n,j+1/2} = y_{n,j}w_{n,j}$, for $j \leq n+1$, this is not a generator.

$$d^{2^j}(y_{n,j}) = v_n^{2^j} w_{n,j}$$

$$\text{and} \qquad d^{2^j}(y_{n,j}w_{n,j}) = v_n^{2^j} w_{n,j}^2 = v_n^{2^j} z_{n+j+1}.$$

(ii) For $j > n+1$,

$$d^{\rho_n(j)}(y_{n,j}) = v_n^{\rho_n(j)} w_{n,j}$$

$$\text{and} \qquad d^{\rho_n(j+1/2)}(y_{n,j+1/2}) = v_n^{\rho_n(j+1/2)} z_{n+j+1}.$$

(iii) For $0 < j \leq n+1$, $\rho_n(j) = 2^j = \rho_n(j+1/2)$. Ignoring the permanent free terms and the previously created v_n-torsion,

$$E_{2^j+1} = k(n)^* \otimes P(y_{n,j+1}) \otimes \bigotimes_{j \leq i \leq n} T_{2^{n+1}}(w_{n,i+1})$$
$$\otimes \bigotimes_{0 \leq i < j} E(w_{2n+2+i}) \otimes \bigotimes_{0 < s} T_{2^n}(z_{2n+2+s}).$$

(iv) For $n+1 < j$,

$$E_{\rho_n(j)+1} = k(n)^* \otimes P(y_{n,j+1}) \otimes E(y_{n,j+1/2})$$
$$\otimes \bigotimes_{0 < i \leq n} E(w_{n,j+i}) \otimes \bigotimes_{0 \leq s} T_{2^n}(z_{n+j+s+1})$$

We could rewrite $T_{2^{n+1}}(w_{n,i+1})$ as $E(w_{n,i+1}) \otimes T_{2^n}(z_{n+i+2})$ for $0 \leq i \leq n$. If we did that, we could write proposition 10.2 without the exceptional cases. Since our interest is in the $k(n)^*$-module structure and not so much in the multiplicative structure, we do this for our final result.

Theorem 10.3. *The 2-primary $k(n)^*(K_2)$ as a $k(n)^*$-module is the sum of the following three summands:*

$$P(v_n) \otimes \bigotimes_{0 < i < n} T_{2^{n-i}}(z_{i+1})$$

$$\bigoplus_{j > 0} \left(T_{\rho_n(j)}(v_n) \otimes P(y_{n,j+1}) \otimes \overline{E}(w_{n,j}) \otimes \bigotimes_{0 < i \leq n} E(w_{n,j+i}) \otimes \bigotimes_{s \geq 0} T_{2^n}(z_{n+j+s+1}) \right)$$

$$\bigoplus_{j>0}\left(T_{\rho_n(j+1/2)}(v_n)\otimes P(y_{n,j+1})\otimes\bigotimes_{0<i\le n}E(w_{n,j+i})\right.$$
$$\left.\otimes\overline{TP}_{2^n}(z_{n+j+1})\otimes\bigotimes_{s>0}T_{2^n}(z_{n+j+s+1})\right)$$

plus a computable family of $\mathbf{Z}/2$'s annihilated by v_n coming from the $E(Q_n)$-free part of H^*K_2.

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