DIVISIBILITY BY 2 AND 3 OF CERTAIN STIRLING NUMBERS

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Abstract. The numbers $\widetilde{e}_p(k,n)$ defined as $\min(\nu_p(S(k,j)j!) : j \ge n)$ appear frequently in algebraic topology. Here $S(k,j)$ is the Stirling number of the second kind, and $\nu_p(-)$ the exponent of p. Let $s_p(n) = n-1+\nu_p([n/p]!)$. The author and Sun proved that if L is sufficiently large, then $\widetilde{e}_p((p-1)p^L+n-1,n) \ge s_p(n)$.

In this paper, we determine the set of integers n for which $\widetilde{e}_p((p-1)p^L+n-1,n) = s_p(n)$ when $p=2$ and when $p=3$. The condition is roughly that, in the base-p expansion of n, the sum of two consecutive digits must always be less than p. The result for divisibility of Stirling numbers is, when $p=2$, that for such integers n, $\nu_2(S(2^L+n-1,n)) = [(n-1)/2]$.

We also present evidence for conjectures that, if $n=2^t$ or 2^t+1, then the maximum value over all $k \ge n$ of $\widetilde{e}_2(k,n)$ is $s_2(n)+1$.

1. Introduction

Let $S(k,j)$ denote the Stirling number of the second kind. This satisfies

$$S(k,j)j! = (-1)^j \sum_{i=0}^{j} (-1)^i \binom{j}{i} i^k. \qquad (1.1)$$

Let $\nu_p(-)$ denote the exponent of p. For $k \ge n$, the numbers $\widetilde{e}_p(k,n)$ defined by

$$\widetilde{e}_p(k,n) = \min(\nu_p(S(k,j)j!) : \ j \ge n) \qquad (1.2)$$

are important in algebraic topology. We will discuss these applications in Section 6.

In [7], it was proved that, if L is sufficiently large, then

$$\widetilde{e}_p((p-1)p^L+n-1,n) \ge n-1+\nu_p([n/p]!). \qquad (1.3)$$

Let $s_p(n) = n-1+\nu([n/p]!)$, as this will appear throughout the paper. Our main theorems, 1.7 and 1.9, give the sets of integers n for which equality occurs in (1.3)

Date: July 16, 2008.
Key words and phrases. Stirling number, divisibility, James numbers.
2000 *Mathematics Subject Classification*: 11B73,55Q52.

when $p=2$ and when $p=3$. Before stating these, we make a slight reformulation to eliminate the annoying $(p-1)p^L$.

We define the partial Stirling numbers $a_p(k,j)$ by

$$a_p(k,j)=\sum_{i\not\equiv 0\,(p)}(-1)^i\binom{j}{i}i^k$$

and then

$$e_p(k,n)=\min(\nu_p(a_p(k,j)):\ j\ge n). \tag{1.4}$$

Partial Stirling numbers have been studied in [10] and [9].

The following elementary and well-known proposition explains the advantage of using $a_p(k,j)$ as a replacement for $S(k,j)j!$: it is that $\nu_p(a_p(k,j))$ is periodic in k. In particular, $\nu_p(a_p(n-1,n))=\nu_p(a_p((p-1)p^L+n-1,n))$ for L sufficiently large, whereas $S(n-1,n)n!=0$. Thus when using $a_p(-)$, we need not consider the $(p-1)p^L$. The second part of the proposition says that replacing $S(k,j)j!$ by $a_p(k,j)$ merely extends the numbers $\widetilde{e}_p(k,n)$ for $k\ge n$ in which we are interested periodically to all integers k. An example $(p=3,n=10)$ is given in [4, p.543].

Proposition 1.5. *a. If $t\ge\nu_p(a_p(k,j))$, then*

$$\nu_p(a_p(k+(p-1)p^t,j))=\nu_p(a_p(k,j)).$$

b. If $k\ge n$, then $e_p(k,n)=\widetilde{e}_p(k,n)$.

Proof. a. ([3, 3.12]) For all t, we have

$$a_p(k+(p-1)p^t,j)-a_p(k,j)=\sum_{i\not\equiv 0\,(p)}(-1)^i\binom{j}{i}i^k(i^{(p-1)p^t}-1)\equiv 0\ (p^{t+1}),$$

from which the conclusion about p-exponents is immediate.

b. We have

$$(-1)^jS(k,j)j!-a_p(k,j)\equiv 0\ (p^k) \tag{1.6}$$

since all its terms are multiples of p^k. Since $\widetilde{e}(k,n)\le\nu(S(k,k)k!)<k$, a multiple of p^k cannot affect this value. ■

Our first main result determines the set of values of n for which (1.3) is sharp when $p=2$.

Theorem 1.7. *For $n \geq 1$, $e_2(n-1,n) = s_2(n)$ iff $n = 2^\epsilon(2s+1)$ with $0 \leq \epsilon \leq 2$ and $\binom{3s}{s}$ odd.*

Remark 1.8. Since $\binom{3s}{s}$ is odd iff binary(s) has no consecutive 1's, another characterization of those n for which $e_2(n-1,n) = s_2(n)$ is those satisfying $n \not\equiv 0 \bmod 8$, and the only consecutive 1's in binary(n) are, at most, a pair at the end, followed perhaps by one or two 0's. Alternatively, except at the end, the sum of consecutive bits must be less than 2.

When $p = 3$, the description is similar.

Theorem 1.9. *Let T denote the set of positive integers for which the sum of two consecutive digits in the base-3 expansion is always less than 3. Let $T' = \{n \in T : n \not\equiv 2\,(3)\}$. For integers a and b, let $aT + b = \{an + b : \ n \in T\}$, and similarly for T'. Then $e_3(n-1,n) = s_3(n)$ if and only if*

$$n \in (3T+1) \cup (3T'+2) \cup (9T+3).$$

Remark 1.10. Thus $e_3(n-1,n) = s_3(n)$ iff $n \not\equiv 0, 6$ (9) and the only consecutive digits in the base-3 expansion of n whose sum is ≥ 3 are perhaps $\cdots 21$, $\cdots 12$, or $\cdots 210$, each at the very end.

The following definition will be used throughout the paper.

Definition 1.11. *Let $\overline{n}$ denote the residue of n mod p.*

The value of p will be clear from the context. Similarly $\overline{x}$ denotes the residue of x, etc.

Remark 1.12. As our title suggests, we can interpret our results in terms of divisibility of Stirling numbers. Suppose $p = 2$ or 3 and L is sufficiently large. The main theorem of [7] can be interpreted to say that

$$\nu_p(S((p-1)p^L + n - 1, n)) \geq (p-1)[\tfrac{n}{p}] + \overline{n} - 1. \tag{1.13}$$

Our main result is that equality occurs in (1.13) iff, for $p = 3$, n is as in Theorem 1.9 with $n \not\equiv 2$ (9) or, for $p = 2$, n is as in Theorem 1.7. We also show that, if $p = 3$ and $n = 9x + 2$, then equality occurs in

$$\nu_3(S(2 \cdot 3^L + n - 1, n + 1)) \geq 6x$$

iff $x \in T'$.

In [12, (1.5)], a function $T^p_{k,\alpha}(n,r)$ was introduced, relevant to the proof of (1.3). We recall it in Definition 2.8. Useful in our proofs of 1.7 and 1.9 is the explicit value mod p of $T^p_{k,2}(n,r)$ when $p=2$ and 3. (See 2.10, 3.2, and 3.17.) We obtain this by relating it to $T^p_{k,1}([\frac{n}{p}],[\frac{r}{p}])$ and then evaluating the latter. This extends [12, Thm 1.5] to the case $\alpha=1$. Useful in this proof is Theorem 1.15, which is proved in Section 3 and might be of independent interest.

Definition 1.14. *If n is a positive integer and r is any integer, let*

$$S_1(n,r)=p^{-[\frac{n-1}{p-1}]}\sum_{k\equiv r\,(p)}(-1)^k\binom{n}{k},\ \text{and } S_2(n,r)=p^{-[\frac{n-1}{p-1}]}\sum_{k\equiv pr\,(p^2)}(-1)^k\binom{pn}{k}.$$

These are integers by [13]. They were also studied in [10]. The prime p is implicit.

Theorem 1.15. *Let p be an odd prime.*

a. *For all r, $S_1(n,r)\equiv S_2(n,r)$ mod p.*

b. *Mod p,* $S_1(n,r)\equiv\begin{cases}(-1)^{s-1} & \text{if } n=(p-1)s\\ (-1)^{s-1}(\frac{s+1}{2}+r) & \text{if } n=(p-1)s-1.\end{cases}$

c. *Mod p, $S_1(n+p(p-1),r)\equiv -S_1(n,r)$.*

Of special interest in algebraic topology is

$$\overline{e}_p(n):=\max(e_p(k,n):\ k\in\mathbb{Z}). \tag{1.16}$$

In Section 5, we discuss the relationship between $\overline{e}_2(n)$, $e_2(n-1,n)$, and $s_2(n)$. We describe an approach there toward a proof of the following conjecture.

Conjecture 1.17. *If $n=2^t$, then*

$$\overline{e}_2(n)=e_2(n-1,n)=s_2(n)+1,$$

while if $n=2^t+1$, then

$$\overline{e}_2(n)=e_2(n-1,n)+1=s_2(n)+1.$$

This conjecture suggests that the inequality $e_2(n-1,n)\ge s_2(n)$ fails by 1 to be sharp if $n=2^t$, while if $n=2^t+1$, it is sharp but the maximum value of $e_2(k,n)$ occurs for a value of $k\ne n-1$.

2. Proof of Theorem 1.7

In this section, we prove Theorem 1.7, utilizing results of [12] and some work with binomial coefficients. The starting point is the following result of [12]. In this section, we abbreviate $\nu_2(-)$ as $\nu(-)$.

Theorem 2.1. ([12, 1.2]) *For all $n \ge 0$ and $k \ge 0$,*

$$\nu\left(2^k k! \sum_i \binom{n}{4i+2}\binom{i}{k}\right) \ge \nu([n/2]!).$$

The bulk of the work is in proving the following refinement. The inequality is immediate from 2.1.

Theorem 2.2. *Let n be as in Theorem 1.7, and, if $n > 4$, define n_0 by $n = 2^e + n_0$ with $0 < n_0 < 2^{e-1}$. Then*

$$\nu\left(\binom{n-1}{k} 2^k k! \sum_i \binom{n}{4i+2}\binom{i}{k}\right) \ge \nu([n/2]!) \tag{2.3}$$

for all k, with equality if and only if

$$k = \begin{cases} 0 & 1 \le n \le 4 \\ n_0 - 1 & n \not\equiv 0 \pmod 4,\ n > 4 \\ n_0 - 2 & n \equiv 0 \pmod 4,\ n > 4. \end{cases} \tag{2.4}$$

Proof that Theorem 2.2 implies the "if" part of Theorem 1.7. By (1.3), $e_2(n-1, n) \ge s_2(n)$ for all n. Thus it will suffice to prove that if n is as in Theorem 2.2, then

$$\nu(a_2(n-1, n)) = s_2(n). \tag{2.5}$$

Note that

$$0 = (-1)^n S(n-1, n) n! = -a_2(n-1, n) + \sum \binom{n}{2k}(2k)^{n-1}.$$

Factoring 2^{n-1} out of the sum shows that (2.5) will follow from showing

$$\sum \binom{n}{2k} k^{n-1} = \nu([n/2]!). \tag{2.6}$$

The sum in (2.6) may be restricted to odd values of k, since terms with even k are more 2-divisible than the claimed value. Write $k = 2j + 1$ and apply the Binomial Theorem, obtaining

$$\sum_j \binom{n}{4j+2} \sum_\ell 2^\ell j^\ell \binom{n-1}{\ell} = \sum_j \binom{n}{4j+2} \sum_\ell 2^\ell \binom{n-1}{\ell} \sum_i S(\ell, i) i! \binom{j}{i}. \tag{2.7}$$

Here we have used the standard fact that $j^\ell = \sum S(\ell,i)i!\binom{j}{i}$.

Recall that $S(\ell,i)=0$ if $\ell<i$, and $S(i,i)=1$. Terms in the right hand side of (2.7) with $\ell=i$ yield

$$\sum_i \binom{n-1}{i} 2^i i! \sum_j \binom{n}{4j+2}\binom{j}{i},$$

which we shall call A_n. By Theorem 2.2, if n is as in Theorem 1.7, $\nu(A_n)=\nu([n/2]!)$ since all i-summands have 2-exponent $\ge \nu([n/2]!)$, and exactly one of them has 2-exponent equal to $\nu([n/2]!)$. Terms in (2.7) with $\ell>i$ satisfy

$$\nu(\text{term}) > \nu\left(2^i i! \sum_j \binom{n}{4j+2}\binom{j}{i}\right),$$

the RHS of which is $\ge \nu([n/2]!)$ by 2.1. The claim (2.6), and hence Theorem 1.7, follows. ■

We recall the following definition from [12, 1.5].

Definition 2.8. *Let p be any prime. For $n,\alpha,k\ge 0$ and $r\in\mathbb{Z}$, let*

$$T^p_{k,\alpha}(n,r) := \frac{k!p^k}{[n/p^{\alpha-1}]!}\sum_i (-1)^{p^\alpha i+r}\binom{n}{p^\alpha i+r}\binom{i}{k}.$$

In the remainder of this section, we have $p=2$ and omit writing it as a superscript of T.

By 2.1, Theorem 2.2 is equivalent to the following result, to the proof of which the rest of this section will be devoted.

Theorem 2.9. *If n is as in Theorem 2.2, then $\binom{n-1}{k}T_{k,2}(n,2)$ is odd if and only if k is as in* (2.4)*.*

Central to the proof of 2.9 is the following result, which will be proved at the end of this section. This result applies to all values of n, not just those as in Theorem 2.2. This result is the complete evaluation of $T_{k,2}(n,2)$ mod 2.

Theorem 2.10. *If $4k+2>n$, then $T_{k,2}(n,2)=0$. If $4k+2\le n$, then, mod 2,*

$$T_{k,2}(n,2) \equiv \binom{[n/2]-k-1}{[n/4]}.$$

Proof of Theorem 2.9. The cases $n \le 4$ are easily verified and not considered further.

First we establish that $\binom{n-1}{k} T_{k,2}(n,2)$ is odd for the stated values of k. We have

$$\binom{n-1}{k} = \begin{cases} \binom{2^e+n_0-1}{n_0-1} & \text{if } n_0 \not\equiv 0 \pmod 4 \\ \binom{2^e+n_0-1}{n_0-2} & \text{if } n_0 \equiv 0 \pmod 4, \end{cases}$$

which is clearly odd in both cases. Here and throughout we use the well-known fact that, if $0 \le \epsilon_i, \delta_i \le p-1$, then

$$\binom{\sum \epsilon_i p^i}{\sum \delta_i p^i} \equiv \prod \binom{\epsilon_i}{\delta_i} \pmod p. \tag{2.11}$$

Now we show that $T_{k,2}(n,2)$ is odd when n and k are as 1.7 and (2.4).

Case 1: $n_0 = 8t+4$ with $\binom{3t}{t}$ odd, and $k = 8t+2$. Using 2.10, with all equivalences mod 2,

$$T_{k,2}(n,2) \equiv \binom{2^{e-1}+4t+2-(8t+2)-1}{2^{e-2}+2t+1} \equiv \binom{-4t-1}{2t+1} \equiv \binom{6t+1}{2t+1} \equiv \binom{3t}{t}.$$

Case 2: $n_0 = 4t+\epsilon$, $\epsilon \in \{1,2\}$, $\binom{3t}{t}$ odd, $k = 4t+\epsilon-1$. Then

$$T_{k,2}(n,2) \equiv \binom{2^{e-1}+2t+\epsilon-1-(4t+\epsilon-1)-1}{2^{e-2}+t} \equiv \binom{-2t-1}{t} \equiv \binom{3t}{t}.$$

Case 3: $n_0 = 4t+3$, $\binom{3(2t+1)}{2t+1}$ odd, $k = 4t+2$. Then

$$T_{k,2}(n,2) \equiv \binom{2^{e-1}+2t+1-(4t+2)-1}{2^{e-2}+t} \equiv \binom{-2t-2}{t} \equiv \binom{3t+1}{t} \equiv \binom{2(3t+1)+1}{2t+1}.$$

Now we must show that, if n is as in Theorem 1.7 and k does not have the value specified in (2.4), then $\binom{n-1}{k} T_{k,2}(n,2)$ is even. The notation of Theorem 2.2 is continued. We divide into cases.

Case 1: $k \ge n_0$. Here $\binom{n-1}{k}$ odd implies $k \ge 2^e$, but then $4k+2 > n$ and so by Theorem 2.10, $T_{k,2}(n,2) = 0$. Hence $\binom{n-1}{k} T_{k,2}(n,2)$ is even.

Case 2: $n_0 = 4t+4$, $k = n_0 - 1$. Here $T_{k,2}(n,2) \equiv \binom{-(2t+2)}{t+1} \equiv \binom{3t+2}{t+1}$. If t is even, this is even, and if $t = 2s-1$, this is congruent to $\binom{3s-1}{s}$ which is even, since if $\nu(s) = w$, then $2^w \notin 3s-1$; i.e., the decomposition of $3s-1$ as a sum of distinct 2-powers does not contain 2^w.

Case 3: $n_0 = 4t+\epsilon$, $1 \le \epsilon \le 3$, and $k < n_0 - 1$. Here

$$\binom{n-1}{k} T_{k,2}(n,2) \equiv \binom{4t+\epsilon-1}{k} \binom{2^{e-1}+2t+[\epsilon/2]-k-1}{2^{e-2}+t}.$$

If $k \le 2t+[\epsilon/2]-1$, then the second factor is even due to the $i=e-2$ factor in (2.11). If $k > 2t+[\epsilon/2]-1$, the second factor is congruent to $\binom{-(k+1-2t-[\epsilon/2])}{t} \equiv \binom{k-t-[\epsilon/2]}{t}$. For $\binom{4t+\epsilon-1}{k}\binom{k-t-[\epsilon/2]}{t}$ to be odd would require one of the following:

$$\begin{aligned}
&\epsilon = 1,\ k = 4i,\ \text{and } \tbinom{t}{i}\tbinom{4i-t}{t} \text{ odd}\\
&\epsilon = 2,\ k = 4i+\langle 0,1\rangle,\ \text{and } \tbinom{t}{i}\tbinom{4i-t-\langle 1,0\rangle}{t} \text{ odd.}\\
&\epsilon = 3,\ k = 4i+\langle 0,2\rangle,\ \tbinom{t}{i}\tbinom{4i-t+\langle -1,1\rangle}{t} \text{ odd.}
\end{aligned}$$

But all these products are even if $i<t$ by Lemma 2.14. If $i=t$, since $k<n_0-1$, we obtain a $\binom{3t-1}{t}$ factor, which is even, as in Case 2.

Case 4: $n_0 = 4t+4$ and $k < n_0-2$. Note that t must be even since $n \not\equiv 0$ (8) in 2.2. We have

$$\binom{n-1}{k}T_{k,2}(n,2) \equiv \binom{4t+3}{k}\binom{2^{e-1}+2t+1-k}{2^{e-2}+t+1}.$$

The case $k \le 2t+1$ is handled as in Case 3. If $k > 2t+1$, then, similarly to Case 3, it reduces to $\binom{4t+3}{k}\binom{k-t-1}{t+1}$. If $k=4t$ or $4t+1$, then we obtain $\binom{3t-1}{t+1}$ or $\binom{3t}{t+1}$, which are even since t is even. Now suppose $k=4i+\Delta$ with $0\le\Delta\le 3$ and $i<t$. Since t is even, if Δ is odd, then $\binom{k-t-1}{t+1}$ is even. For $\Delta = 0$ or 2, we obtain $\binom{t}{i}\binom{4i-t\pm 1}{t+1}$. Since t is even, we use $\binom{2A+1}{2B+1}\equiv\binom{2A}{2B}$ to obtain $\binom{t}{i}\binom{4i-t-\langle 0,2\rangle}{t}$, which is even by Lemma 2.14. ■

The following result implies the "only if" part of Theorem 1.7.

Theorem 2.12. *Assume $n \equiv 0$ mod 8 or $n = 2^\epsilon(2s+1)$ with $0 \le \epsilon \le 2$ and $\binom{3s}{s}$ even. Then for all $N \ge n$, we have $\nu_2(a_2(n-1,N)) > s_2(n)$.*

Proof. Combining aspects of 2.2, 2.10, and 4.21, the theorem will follow from showing that for n as in the theorem and $N\ge n$ satisfying $[N/4]=[n/4]$, we have

$$\sum_{4k+2\le N}\binom{n-1}{k}\binom{[N/2]-k-1}{[N/4]} \equiv 0\ (2). \qquad (2.13)$$

Note that if $[N/4] > [n/4]$, then $\frac{[N/2]!}{[n/2]!}$ is even in the 2-primary analogue of the proof of 4.21.

When $n = 8\ell$, it is required to show that $\sum\binom{8\ell-1}{k}\binom{4\ell-k-1}{2\ell}$ and $\sum\binom{8\ell-1}{k}\binom{4\ell-k}{2\ell}$ are both even. The first corresponds to $N=n$ or $n+1$, and the second to $N=n+2$ or

$n+3$. The first is proved by noting easily that the summands for $k=2j$ and $2j+1$ are equal. The second follows from showing that the summands for $k=2j$ and $2j-1$ are equal. This is easy unless $2j=8i$. For this, we need to know that $\binom{2\ell-4i}{\ell}\binom{\ell}{i}$ is always even, and this follows easily from showing that the binary expansions of $\ell-4i$, $\ell-i$, and i cannot be disjoint.

For $n=2^e(2s+1)$ with $\binom{3s}{s}$ even, all summands in (2.13) can be shown to be even when $n=2^e+n_0$ with $0<n_0<2^{e-1}$ and $N=n$ using the proof of Theorem 2.9. For such n and $N>n$, the main case to consider is $n=8a+4$ and $N=n+2$. Then we need $\binom{8a+3}{k}\binom{4a+2-k}{2a+1}\equiv 0 \bmod 2$. For this to be false, k must be odd. But then we have

$$\binom{8a+3}{k}\binom{4a+2-k}{2a+1}\equiv\binom{8a+3}{k-1}\binom{4a+1-(k-1)}{2a+1}\equiv 0$$

by the result for $N=n$ with k replaced by $k-1$.

If $n=2^{e+d}+\cdots+2^e+n_0$ with $d>0$ and $0<n_0<2^{e-1}$, then (2.13) for $n=N$ is proved when k does not have the special value of (2.4) just as in the second part of the proof of 2.9. We illustrate what happens when k does have the special value by considering what happens to Case 1 just after (2.11). The binomial coefficient there becomes

$$\binom{2^{e+d-1}+\cdots+2^{e-1}-4t-1}{2^{e+d-2}+\cdots+2^{e-2}+2t+1},$$

which is 0 mod 2 by consideration of the 2^{e-1} position in (2.11). For $N>n$, the argument is essentially the same as that of the previous paragraph. ■

The following lemma was used above.

Lemma 2.14. *Let $i<t$, $-2\le\delta\le 1$, and if $\delta=-2$, assume that t is even. Then $\binom{t}{i}\binom{4i-t+\delta}{t}$ is even.*

Proof. Assume that $\binom{t}{i}\binom{4i-t+\delta}{t}$ is odd. Then i, $t-i$, and $4i-2t+\delta$ have disjoint binary expansions. If $\delta=0$ or 1, then letting $\ell=t-i$ and $r=2i-t$, we infer that $\ell+r$, ℓ, and $2r$ are disjoint with ℓ and r positive, which is impossible by Sublemma 2.15.2. If $\delta=-1$ and t is odd, then two of i, $t-i$, and $4i-2t-1$ are odd, and so they cannot be disjoint. Thus we may assume t is even and $\delta=-1$ or -2. Let $\ell=t-i$ and $r=2i-t-1$. Then $\ell+r+1$, ℓ, and $2r$ are disjoint with ℓ and r positive and r odd, which is impossible by Sublemma 2.15.3. ■

Sublemma 2.15. *Let ℓ and r be nonnegative integers.*

(1) *Then ℓ, $2r+1$, and $\ell+r+1$ do not have disjoint binary expansions.*
(2) *If ℓ and r are positive, then ℓ, $2r$, and $\ell+r$ do not have disjoint binary expansions.*
(3) *If ℓ is positive and r is odd, then ℓ, $2r$, and $\ell+r+1$ do not have disjoint binary expansions.*

Proof. (1) Assume that ℓ and r constitute a minimal counterexample. We must have $\ell = 2\ell'$ and $r = 2r'+1$. Then ℓ' and r' yield a smaller counterexample.

(2) Assume that ℓ and r constitute a minimal counterexample. If r is even, then ℓ must be even, and so dividing each by 2 gives a smaller counterexample. If $r = 1$, then ℓ, 2, and $\ell+1$ are disjoint, which is impossible, since the only way for ℓ and $\ell+1$ to be disjoint is if $\ell = 2^e - 1$. If $r = 2r'+1$ with $r' > 0$, and $\ell = 2\ell'$, then ℓ' and r' form a smaller counterexample. If $r = 2r'+1$ and $\ell = 2\ell'+1$, then ℓ', $2r'+1$, and $\ell'+r'+1$ are disjoint, contradicting (1).

(3) Let $r = 2r'+1$. Then ℓ must be even $(= 2\ell')$. Then ℓ', $2r'+1$, and $\ell'+r'+1$ are disjoint, contradicting (1). ■

The following lemma together with Theorem 3.2 implies Theorem 2.10. Its proof uses the following definition, which will be employed throughout the paper.

Definition 2.16. *Let $d_p(-)$ denote the number of 1's in the p-ary expansion.*

Lemma 2.17. *Mod 2,*

$$T_{k,1}(n,r) \equiv \begin{cases} \binom{n-k-1}{[(n-1+\overline{r})/2]} & n > k \\ 0 & n \le k \end{cases} \tag{2.18}$$

Proof. The proof is by induction on k. Let $f_k(n,r)$ denote the RHS of (2.18) mod 2. It is easy to check that $f_0(n,r) = \delta_{d_2(n),1}$, agreeing with $T_{0,1}(n,r)$ as determined in (3.4). Here and throughout $\delta_{i,j}$ is the Kronecker function. From Definition 2.8, mod 2, $T_{k,1}(1,r) \equiv \delta_{k,0}$. This is what causes the dichotomy in (2.18).

By [12, (2.3)], if $k > 0$, then

$$T_{k,1}(n,r) + rT_{k-1,1}(n,r+2) = -T_{k-1,1}(n-1,r+1). \tag{2.19}$$

Noting that f only depends on the mod 2 value of r, the lemma follows from

$$\begin{aligned} f_k(n,0) &= f_{k-1}(n-1,1) \\ f_k(n,1) &= f_{k-1}(n,1)+f_{k-1}(n-1,0), \end{aligned}$$

which are immediate from the definition of f and Pascal's formula. ■

3. Mod p values of T-function

We saw in Theorem 2.9 that knowledge of the mod 2 value of the T-function of [12] played an essential role in proving Theorem 1.7. A similar situation occurs when $p=3$. The principal goal of this section is the determination of $T^3_{k,2}(n,r)$, obtained by combining Theorems 3.2 and 3.17. We also prove Theorem 1.15, which is used in the proof of 3.2, but may be of intrinsic interest.

We begin by recording a well-known proposition which will be used throughout the paper.

Proposition 3.1. *If $n\ge 0$, then $\nu_p(n!)=\frac{1}{p-1}(n-d_p(n))$, and hence $\nu_p(\binom{n}{b})=\frac{1}{p-1}(d_p(b)+d_p(n-b)-d_p(n))$.*

The following result extends [12, Thm 1.5] to include the case $\alpha=1$.

Theorem 3.2. *Let p be any prime. For any $\alpha\ge 1$, we have the congruence, mod p,*

$$T^p_{k,\alpha+1}(n,r)\equiv(-1)^{\overline{r}}\binom{\overline{n}}{\overline{r}}T^p_{k,\alpha}([\tfrac{n}{p}],[\tfrac{r}{p}]).$$

Proof. This was proved for $\alpha\ge 2$ in [12, Thm 1.5]. The only place that the proof of that result does not work when $\alpha=1$ is in the initial step of Case 3 of [12, p.5548]. Required to complete that proof is

$$T^p_{0,2}(pn,pr)\equiv T^p_{0,1}(n,r)\pmod p.$$

This just says, mod p,

$$\tfrac{1}{n!}\sum_{i\equiv pr\,(p^2)}(-1)^i\tbinom{pn}{i}\equiv\tfrac{1}{n!}\sum_{i\equiv r\,(p)}(-1)^i\tbinom{n}{i}. \tag{3.3}$$

When p is odd, this follows immediately from part a of Theorem 1.15, since $\nu_p(n!)\le[(n-1)/(p-1)]$.

We prove (3.3) when $p = 2$ by showing that both sides equal $\delta_{1,d_2(n_0)}$. The RHS equals

$$\frac{1}{2^{n-d_2(n)}u}\cdot 2^{n-1}\equiv 2^{d_2(n)-1}\equiv \delta_{d_2(n),1}\,(\mathrm{mod}\ 2), \tag{3.4}$$

with u odd, while the LHS is $\frac{1}{n!}\sum\limits_{i\equiv 2r\,(4)}\binom{2n}{i}\equiv \frac{1}{2^{n-d_2(n)}}\cdot\begin{cases}2^{2n-2} & n \text{ odd}\\ 2^{n-1}u' & n \text{ even,}\end{cases}$, and this also equals $\delta_{d_2(n),1}$. Here we have used

$$\sum_{i\equiv r\,(4)}\binom{n}{i} = 2^{n-2}+\epsilon_{n,r}2^{[n/2]-1},\ \text{ with } \epsilon_{n,r} = \begin{cases}0 & n-2r\equiv 2 \ (\mathrm{mod}\ 4)\\ 1 & n-2r\equiv -1,0,1 \ (\mathrm{mod}\ 8)\\ -1 & n-2r\equiv 3,4,5 \ (\mathrm{mod}\ 8),\end{cases}$$

which is easily proved by induction on n. ■

Next we discuss Theorem 1.15 and give its proof. First we note that the definitions of S_1 and S_2 in it are similar to [12, (3.4)], but differ regarding the role of the second variable in S_2. We remark that part b of 1.15 was given by Lundell in [10], although he merely said "the proof is a straightforward but somewhat tedious induction." Part a is of particular interest to us.

Proof of Theorem 1.15. Throughout this proof, p denotes an odd prime. We will work with polynomials in the ring $R := \mathbb{F}_p[x]/(x^p - 1)$. In R, let

$$P_n(x) = \sum_{r=0}^{p-1} S_1(n,r)x^r \text{ and } Q_n(x) = \sum_{r=0}^{p-1} S_2(n,r)x^r. \tag{3.5}$$

Also in R, let

$$\psi(x) = \frac{(1-x)^p - (1-x^p)}{p(1-x)} = \frac{(1-x)^{p-1} - (1+\cdots+x^{p-1})}{p}.$$

We will prove later the following result, which immediately implies part a.

Theorem 3.6. *For $1 \le d \le p-1$ and $m \ge 0$, we have in R*

$$P_{(p-1)m+d}(x) = \psi(x)^m(1-x)^d = Q_{(p-1)m+d}(x).$$

Parts b and c follow from Theorem 3.6 and the following result, which we will also prove later. The numbering of the parts is related to the corresponding part of Theorem 1.15.

Lemma 3.7. *We have, in R,*

b.i. $(\psi(x)+1)(1-x)^{p-1}=0$,

b.ii. $(\psi(x)+x^{(p-1)/2})(1-x)^{p-2}=0$, *and*

c. $\psi(x)^p=-1$.

The deduction of 1.15.bc is straightforward. For the first part of b, we have in R

$$\begin{aligned} & (-1)^{s-1}(1+x+\cdots+x^{p-1}) \\ = \;& (-1)^{s-1}(1-x)^{p-1} \\ = \;& \psi(x)^{s-1}(1-x)^{p-1} \\ = \;& P_{(p-1)s}(x) \\ = \;& \sum_{r=0}^{p-1} S_1((p-1)s,r)x^r. \end{aligned}$$

Noting that $(1-x)^{p-2}=1+2x+3x^2+\cdots+(p-1)x^{p-2}$, the second part of 3.6.b follows from the following analysis of coefficients of polynomials in R.

$$\begin{aligned} & S_1((p-1)(s-1)+p-2,r) \\ = \;& [x^r]P_{(p-1)(s-1)+p-2}(x) \\ = \;& [x^r](\psi(x)^{s-1}(1-x)^{p-2}) \\ = \;& [x^r]((-1)^{s-1}x^{(s-1)(p-1)/2}(1-x)^{p-2}) \\ = \;& (-1)^{s-1}[x^{r+(s-1)/2}](1-x)^{p-2} \\ = \;& (-1)^{s-1}(r+(s-1)/2+1). \end{aligned}$$

Note that exponents of x may be considered mod p. The deduction of 1.15.c from 3.7.c is much easier, and omitted. ■

Proof of Theorem 3.6. We first show the theorem is true when $m=0$. The argument for P is similar to, and easier than, the following argument for Q. Let $1\le d\le p-1$. Note that, mod p,

$$Q_d(x^p)=\sum_r S_2(d,r)x^{pr}=\sum_r(-1)^{pr}\binom{pd}{pr}x^{pr}\equiv\sum_{r=0}^{d}(-1)^r\binom{d}{r}x^{pr}=(1-x^p)^d.$$

Thus the same is true when x^p is replaced by x. Note that here we are dealing with polynomials mod p, but not in the ring R used earlier.

Next we prove that for any $n \not\equiv 1 \bmod (p-1)$

$$P_{n+p-1}(x) = \psi(x) P_n(x) \tag{3.8}$$

in R. To see this, first note that if $n \not\equiv 1 \bmod (p-1)$,

$$S_1(n,r) = S_1(n-1,r) - S_1(n-1,r-1). \tag{3.9}$$

Note that the need for $n \not\equiv 1$ is so that $[(n-1)/(p-1)] = [(n-2)/(p-1)]$. Similarly, for $n \not\equiv 1 \bmod (p-1)$

$$S_1(n+p-1,r) = \tfrac{1}{p}\sum_{i=0}^{p}(-1)^i\tbinom{p}{i}S_1(n-1,r-i). \tag{3.10}$$

Since $S_1(n-1,r) = S_1(n-1,r-p)$, this becomes

$$\begin{aligned} S_1(n+p-1,r) &= \sum_{i=1}^{p-1}(-1)^i\tfrac{1}{p}\tbinom{p}{i}S_1(n-1,r-i) \\ &= \sum_{i=1}^{p-2}\alpha_i S_1(n,r-i), \end{aligned} \tag{3.11}$$

where

$$\psi(x) = \sum_{i=1}^{p-2}\alpha_i x^i. \tag{3.12}$$

At the last step, we have used (3.9). The equation (3.11) translates to (3.8).

A similar argument, sketched below, shows that for any $n \not\equiv 1 \bmod (p-1)$

$$Q_{n+p-1}(x) = \psi(x) Q_n(x) \tag{3.13}$$

in R. The S_2-analogue of (3.9) is true mod p, obtained from

$$S_2(n,r) = S_2(n-1,r) + p^{-[\frac{n-1}{p-1}]}\sum_{i=1}^{p-1}(-1)^i\tbinom{p}{i}\sum_{k\equiv pr-i\ (p^2)}(-1)^k\tbinom{p(n-1)}{k} - S_2(n-1,r-1)$$

by noting that the k-sums are divisible by $p^{[(n-1)/(p-1)]}$ by [13], and so since $\binom{p}{i} \equiv 0$ mod p, then each i-summand is 0 mod p. The S_2-analogue of (3.10), mod p, is obtained similarly, using that $(1-x)^{p^2} \equiv (1-x^p)^p$ mod p. The argument for (3.13) is completed as in (3.11).

Theorem 3.6 with $d \neq 1$ is immediate from (3.8) and (3.13) plus the validity when $m = 0$ established in the first paragraph of this proof. The proof when $d = 1$ requires the following three lemmas.

Lemma 3.14. *If n is odd and $n-2r\equiv 0$ mod p, then $S_1(n,r)=0=S_2(n,r)$.*

Proof. Since $n-r\equiv r$ mod p, both $\binom{n}{r}$ and $\binom{n}{n-r}$ occur in the sum for $S_1(n,r)$, and with opposite sign since n is odd. Hence all terms in the sum occur in cancelling pairs. The same is true of all terms in the sum for $S_2(n,r)$ since $pn-pr\equiv pr$ mod p^2. ■

Lemma 3.15. *If $(1-x)f(x)=0$ in R, then $f(x)=c(1+x+\cdots+x^{p-1})$ for some c.*

Proof. Let $f(x)=c_0+c_1x+\cdots+c_{p-1}x^{p-1}$. The given equation implies $c_0=c_1=\cdots=c_{p-1}$. ■

Lemma 3.16. *For $t\in\mathbb{Z}$, let $R_t\subset R$ denote the span of x^i-x^{t-i} for all i. If $g(x)\in R_t$, then $g(x)\psi(x)\in R_{t-1}$.*

Proof. Since $\psi(x)$ is a linear combination of various x^j+x^{p-1-j}, the lemma follows from the observation that

$$(x^i-x^{t-i})(x^j+x^{-1-j})=x^{i+j}-x^{t-1-i-j}+x^{i-1-j}-x^{t+j-i}.$$

■

Note that if $g(x)\in R_t$, then $[x^{t/2}]g(x)=0$.

Now we prove the case $d=1$ of Theorem 3.6. We have

$$P_{(p-1)m+1}(x)\cdot(1-x)=P_{(p-1)m+2}(x)=(1-x)^2\psi(x)^m.$$

By Lemma 3.15, $\Delta_m(x):=P_{(p-1)m+1}(x)-(1-x)\psi(x)^m$ has all coefficients equal. By Lemma 3.14, if $(p-1)m+1-2r\equiv 0$ mod p, then $[x^r]P_{(p-1)m+1}(x)=0$. Note that here $r=(1-m)/2$, with exponents always considered mod p in R. In the notation of Lemma 3.16, $1-x\in R_1$, and hence by that lemma, $(1-x)\psi(x)^m\in R_{1-m}$. Thus $[x^{(1-m)/2}]((1-x)\psi(x)^m)=0$. Thus $[x^{(1-m)/2}]\Delta_m(x)=0$, and hence $\Delta_m(x)=0$, as desired. ■

Proof of Lemma 3.7. To prove b.i., we prove $\psi(x)+1$ is divisible by $(1-x)$ by showing $\psi(1)\equiv -1$ mod p. Note that $\sum_{i=0}^{p-1}((-1)^i\binom{p-1}{i}-1)=-p$, and hence

$$\psi(1)=\tfrac{1}{p}\sum_{i=0}^{p-1}((-1)^i\tbinom{p-1}{i}-1)=-1.$$

To prove b.ii., we prove $g(x) := \psi(x) + x^{(p-1)/2}$ is divisible by $(1-x)^2$. Since $g(1) = 0$, it remains to show that the derivative satisfies $g'(1) = 0$; i.e., that $\psi'(1) + \frac{p-1}{2} \equiv 0$ mod p. Let $\alpha_i = \frac{1}{p}((-1)^i\binom{p-1}{i} - 1)$. Then $\psi'(1) = \sum_{i=1}^{p-1} i\alpha_i$. Since

$$-(p-1)(1-x)^{p-2} = \tfrac{d}{dx}(1-x)^{p-1} = \textstyle\sum_{i=1}^{p-1}(-1)^i\binom{p-1}{i}ix^{i-1},$$

setting $x = 1$ shows $\sum_{i=1}^{p-1}(-1)^i\binom{p-1}{i}i = 0$ and thus

$$p\psi'(1) = \sum_{i=1}^{p-1} pi\alpha_i = \sum_{i=1}^{p-1}((-1)^i\tbinom{p-1}{i} - 1)i = -\sum_{i=1}^{p-1} i = -\tfrac{p(p-1)}{2},$$

and hence $\psi'(1) + \frac{p-1}{2} = 0$, as desired.

To prove c, we use $x^p = 1$, $(A+B)^p \equiv A^p + B^p$, and $i^p \equiv i$, and obtain, in R,

$$\psi(x)^p = \sum_{i=0}^{p-1} \frac{(-1)^i\binom{p-1}{i} - 1}{p} = 0 - 1.$$

■

Now we give the mod 3 values of $T^3_{k,1}(-,-)$. The mod 3 values of $T^3_{k,2}(-,-)$ can be obtained from this using Theorem 3.2. Throughout the rest of this section and the next, the superscript 3 on T is implicit.

Theorem 3.17. *Let $n = 3m + \delta$ with $0 \le \delta \le 2$.*

- *If $n - k = 2\ell$, then, mod 3, $T_{k,1}(n,r)$ is given by*

		δ = 0	1	2
$r \pmod 3$	0	$\binom{\ell-1}{m-1}$	$\binom{\ell-1}{m}$	$-\binom{\ell-1}{m}$
	1, 2	$-\binom{\ell-1}{m}$	$\binom{\ell-1}{m}$	$-\binom{\ell-1}{m}$

- *If $n - k = 2\ell + 1$, then, mod 3, $T_{k,1}(n,r)$ is given by*

		δ = 0	1	2
$r \pmod 3$	0	0	$\binom{\ell}{m}$	0
	1	$\binom{\ell}{m}$	$-\binom{\ell}{m}$	0
	2	$-\binom{\ell}{m}$	0	0

Proof. By [12, (2.3)], we have

$$T_{k,1}(n,r)+rT_{k-1,1}(n,r+3)=-T_{k-1,1}(n-1,r+2), \tag{3.18}$$

yielding an inductive determination of $T_{k,1}$ starting with $T_{0,1}$. One can verify that the mod 3 formulas of Theorem 3.17 also satisfy (3.18). For example, if $r\equiv 1$ mod 3 and $n-k=2\ell$, then for $\delta=0$, 1, 2, (3.18) becomes, respectively, $-\binom{\ell-1}{m}+\binom{\ell}{m}=\binom{\ell-1}{m-1}$, $\binom{\ell-1}{m}-\binom{\ell}{m}=-\binom{\ell-1}{m-1}$, and $-\binom{\ell-1}{m}+0=-\binom{\ell-1}{m}$.

To initiate the induction we show that, mod 3,

$$T_{0,1}(n,r)\equiv\begin{cases}2 & n=2\cdot 3^e\\ 1 & n=3^{e_1}+3^{e_2},\ 0\le e_1<e_2\\ r & n=3^e,\ e>0\\ r+1 & n=1\\ 0 & \text{otherwise,}\end{cases} \tag{3.19}$$

and observe that the tabulated formulas for $k=0$ also equal (3.19). The latter can be proved by considering separately $n=6t+d$ for $0\le d\le 5$. For example, if $d=3$, then $m=2t+1$, $\delta=0$, and $n-k=2(3t+1)+1$. For $r\equiv 0,1,2$, the tabulated value is, respectively, 0, $\binom{3t+1}{2t+1}$, $-\binom{3t+1}{2t+1}$. Using Proposition 3.1, one shows $\nu_3\left(\binom{3t+1}{2t+1}\right)=d_3(2t+1)-1$. Thus the tabulated value in these cases is 0 mod 3 unless $2t+1$, hence $6t+3$, is a 3-power, and in this case $\binom{3t+1}{2t+1}\equiv 1$ mod 3.

To see (3.19), we note that

$$T_{0,1}(n,r)=\frac{3^{[(n-1)/2]}}{n!}S_1(n,r)$$

with S_1 as in Theorem 1.15, and that, mod 3,

$$\frac{3^{[(n-1)/2]}}{n!}\equiv\begin{cases}1 & n=3^{2e}\text{ or }3^e+3^{e+2k}\\ 2 & n=3^{2e+1},\ 2\cdot 3^e,\ \text{ or }3^e+3^{e+2k-1}\\ 0 & \text{otherwise.}\end{cases}$$

Thus, for example, mod 3, if $e>0$, then, using Theorem 1.15

$$T_{0,1}(3^{2e},r)\equiv S_1(3^{2e},r)=(-1)^{(3^{2e}-1)/2}\Big(\frac{3^{2e}+3}{4}+r\Big)\equiv r,$$

in agreement with (3.19). ■

4. Proof of Theorem 1.9

In this section, we prove Theorem 1.9. We begin with a result, 4.3, which reduces much of the analysis to evaluation of binomial coefficients mod 3.

Definition 4.1. *For* $\epsilon = \pm 1$*, let* $\tau(n,k,\epsilon) := T_{k,1}(n,1) + \epsilon T_{k,1}(n,2)$*, mod* 3.

The following result is immediate from Theorem 3.17.

Proposition 4.2. *Let* $n = 3m + \delta$ *with* $0 \le \delta \le 2$*. If* $n - k = 2\ell$*, then, mod* 3*,* $\tau(n,k,-1) \equiv 0$*, while* $\tau(n,k,1) \equiv (-1)^{\delta}\binom{\ell-1}{m}$*. If* $n - k = 2\ell + 1$*, then, mod* 3*,*

$$\tau(n,k,\epsilon) \equiv \begin{cases} 0 & \text{if } \delta = 2 \text{ or } \epsilon = 1 \text{ and } \delta = 0 \\ -\binom{\ell}{m} & \text{otherwise.} \end{cases}$$

The following result is a special case of Theorem 4.21, which is proved later.

Theorem 4.3. *Define*

$$\phi(n) := \sum \binom{n-1}{k} \tau([\tfrac{n}{3}], k, (-1)^{n-k-1}) \in \mathbb{Z}/3. \qquad (4.4)$$

Then $\nu_3(a_3(n-1,n)) = s_3(n)$ *if and only if* $\phi(n) \ne 0$.

The following definition will be used throughout this section.

Definition 4.5. *An integer* x *is* sparse *if its base-3 expansion has no* 2*'s or adjacent* 1*'s. The pair* (x,i) *is* special *if* x *is sparse and* $i = x - \max\{3^{a_j} : 3^{a_j} \in x\}$.

Some special pairs are $(9,0)$, $(10,1)$, $(30,3)$, and $(91,10)$.

Lemma 4.7 will be used frequently. Its proof uses the following sublemma, which is easily proved.

Sublemma 4.6. *Let* $F_1(x,i) = (3x, 3i)$ *and* $F_2(x,i) = (9x+1, 9i+1)$*. The special pairs are those that can be obtained from* $(1,0)$ *by repeated application of* F_1 *and/or* F_2.

For example $(3^7 + 3^3 + 3, 3^3 + 3) = F_1F_2F_2F_1F_1(1,0)$.

Lemma 4.7. *Mod* 3*,*

(1) *If* $x - i$ *is even, then* $\binom{x}{i}\binom{(3x-9i)/2}{x} \equiv 0$*;*

(2) *If* $x - i$ *is odd, then* $\binom{x}{i}\binom{(3x-9i-1)/2}{x} \equiv \begin{cases} 1 & \text{if } (x,i) \text{ special} \\ 0 & \text{otherwise;} \end{cases}$

(3) *If* $x-i$ *is odd, then* $\binom{x}{i}\binom{(3x-9i-3)/2}{x} \equiv \begin{cases} 1 & \text{if } (x,i) \text{ special and } x \equiv 0\,(3) \\ 0 & \text{otherwise.} \end{cases}$

Proof. We make frequent use of (2.11).

(1) If $\binom{x}{i} \not\equiv 0$, then $\nu_3(i) \ge \nu_3(x)$, but then the second factor is $\equiv 0$ for a similar reason.

(2) Say (x,i) satisfies C if $\binom{x}{i}\binom{(3x-9i-1)/2}{x} \not\equiv 0$. Note that $(1,0)$ satisfies C. We will show that (x,i) satisfies C iff either $(x,i) = (3x', 3i')$ and (x', i') satisfies C or $(x,i) = (9x''+1, 9i''+1)$ and (x'', i'') satisfies C. The result then follows from the sublemma and the observation that the binomial coefficients maintain a value of 1 mod 3.

If $x = 3x'$, then $\binom{x}{i} \not\equiv 0$ implies $i = 3i'$. Then

$$\binom{(3x-9i-1)/2}{x} \equiv \binom{(9x'-27i'-1)/2}{3x'} \equiv \binom{\frac{1}{2}(9x'-27i'-3)+1}{3x'} \equiv \binom{(3x'-9i'-1)/2}{x'}.$$

If $x = 3x'+1$, then $0 \not\equiv \binom{\frac{1}{2}(9x'-9i)+1}{3x'+1}$ implies $x' = 3x''$. The product becomes $\binom{9x''+1}{i}\binom{(3x''-i)/2}{x''}$. For this to be nonzero, i cannot be $9i''$ by consideration of the second factor, similarly to case (1). If $i = 9i''+1$, the product becomes $\binom{x''}{i''}\binom{(3x''-9i''-1)/2}{x''}$, as claimed. If $x = 3x'+2$, a nonzero second factor would require the impossible condition $(9x'-9i+5)/2 \equiv 2$.

(3) To get nonzero, we must have $x = 3x'$ then $i = 3i'$. The product then becomes $\binom{x'}{i'}\binom{(3x'-9i'-1)/2}{x'}$, which is analyzed using case (2). ■

Next we prove a theorem which, with 4.3, implies one part of the "if" part of Theorem 1.9.

Theorem 4.8. *With* T *as in Theorem 1.9, if* $n \in (3T+1)$ *then* $\phi(n) \neq 0$.

Proof. Define $f_1(x) = \phi(3x+1)$. The lengthy proof breaks up into four cases, which are easily seen to imply the result, that

$$f_1(x) \neq 0 \text{ if } x \in T. \tag{4.9}$$

(1) If x is sparse, then $f_1(x) \neq 0$.
(2) For all x, $f_1(3x) = f_1(x)$.
(3) If x is not sparse and $x \not\equiv 2$ mod 3, or if x is sparse and $x \equiv 1$ mod 3, then $f_1(3x+1) = \pm f_1(x)$.

(4) If $x\equiv 0$ mod 3, then $f_1(3x+2)=f_1(x)$.

Moreover, this inductive proof of (4.9) will establish at each step that

$$\text{if } \tbinom{3x}{k}\tau(x,k,(-1)^{x-k})\neq 0, \text{ then } 3x-k\equiv 0\ (2)$$
$$\text{unless } (3x,k) \text{ is special.} \tag{4.10}$$

Case 1: Let x be sparse and

$$3x=\sum_{j=1}^{t}3^{a_j}$$

with $a_j-a_{j-1}\geq 2$ for $2\leq j\leq t$. Then

$$f_1(x)=\sum\tbinom{3x}{3i}\tau(x,3i,(-1)^{x-i}).$$

We will show that

$$\tbinom{3x}{3i}\tau(x,3i,(-1)^{x-i})=\begin{cases}-1 & 3i=3x-3^{a_t}\\ (-1)^j & 3i=3x-3^{a_t}-3^{a_j},\ j\geq 1\\ 0 & \text{otherwise.}\end{cases} \tag{4.11}$$

This will imply Case 1.

In the first case of (4.11), (x,i) is special. If $x=3x'$, then $i=3i'$ with (x',i') special, and we have

$$\tau(x,3i,-1)=-\tbinom{(3x'-9i'-1)/2}{x'}\equiv -1$$

by Lemma 4.7.(2). If $x=3x'+1$, then $i=3i'+1$ with (x',i') special. Also, since x is sparse, we must have $x'=3x''$ and then $i'=3i''$. Thus

$$\tau(x,3i,-1)=-\tbinom{(x-3i-1)/2}{x'}=-\tbinom{(3x''-9i''-1)/2}{x''}\equiv -1$$

by Lemma 4.7.(2).

For the second case of (4.11), let $3i=3x-3^{a_t}-3^{a_j}$. This time $x-3i=2\ell$ with

$$\ell=\sum_{s=a_{t-1}}^{a_t-2}3^s+\cdots+\sum_{s=a_{j+1}}^{a_{j+2}-2}3^s+\sum_{s=a_{j-1}}^{a_j-2}3^s+\cdots+\sum_{s=a_1}^{a_2-2}3^s+\sum_{s=a_1}^{a_{j+1}-2}3^s+2\cdot 3^{a_1-1}.$$

Then $\ell-1$ is obtained from this by replacing $2\cdot 3^{a_1-1}$ with $3^{a_1-1}+2\sum_{s=0}^{a_1-2}3^s$. Hence

$$\tau(x,3i,(-1)^{x-i})=(-1)^{\overline{x}}\tbinom{\ell-1}{[x/3]}\equiv 2^j\equiv(-1)^j.$$

Here we have used that for $\overline{x}=0,1$, we have $[\frac{x}{3}]=\sum_{j=\overline{x}+1}^{t}3^{a_j}$.

We complete the argument for Case 1 by proving the third part of (4.11). The binomial coefficient $\binom{3x}{3i}$ is 0 unless $3i = 3x - 3^{a_{j_1}} - \cdots - 3^{a_{j_r}}$ with $j_1 < \cdots < j_r$. We must have $j_r = t$ or else $x - 3i$ would be negative. Hence $r > 2$. If $r = 2w + 1 > 1$ is odd, then

$$\tau(x, 3i, (-1)^{x-i}) = -\binom{\ell}{[x/3]}$$

with

$$2\ell+1 = x-3i = \sum_{j\notin\{j_1,\dots,j_r\}} (3^{a_{j+1}-1}-3^{a_j})+\sum_{h=1}^{w}(3^{a_{j_{2h+1}}-1}+3^{a_{j_{2h}}-1})+3^{a_{j_1}-1},$$

and hence

$$\ell = \sum_{j\notin\{j_1,\dots,j_r\}} \sum_{i=a_j}^{a_{j+1}-2} 3^i + \sum_{h=1}^{w}\left(3^{a_{j_{2h}}-1} + \sum_{i=a_{j_{2h}}-1}^{a_{j_{2h+1}}-2} 3^i\right) + \sum_{i=0}^{a_{j_1}-2} 3^i.$$

Using (2.11), we see that $\binom{\ell}{[x/3]} \equiv 0$ by consideration of position $a_{j_2} - 2$. A similar argument works when r is even.

Case 2: We are comparing

$$f_1(x) = \sum \binom{3x}{3i}\tau(x, 3i, (-1)^{3x-3i})$$

with

$$f_1(3x) = \sum \binom{9x}{9i}\tau(3x, 9i, (-1)^{9x-9i}),$$

mod 3. Clearly the binomial coefficients agree. Let $x = 3y + \delta$ with $0 \le \delta \le 2$.

If $x - 3i = 2\ell$, let $Q = (x - 3i)/2$. We have

$$\tau(x, 3i, 1) = (-1)^\delta \binom{Q-1}{y} \equiv \binom{3Q-1}{3y+\delta} = \tau(3x, 9i, 1).$$

If $x - 3i = 2\ell + 1$, let $Q = (x - 3i - 1)/2$. If $\delta \ne 2$, we have

$$\tau(x, 3i, -1) = -\binom{Q}{y} \equiv -\binom{3Q+1}{3y+\delta} = \tau(3x, 9i, -1),$$

while if $\delta = 2$, we have $\tau(x, 3i, -1) = 0$ by 4.2, and $\binom{3Q+1}{3y+\delta} = 0$.

Case 3: Let $x = 3y + \delta$ with $\delta \in \{0, 1\}$. Except for the single special term when x is sparse, we have $f_1(x) = \sum \binom{3x}{3i}\tau(x, 3i, 1)$, and will show that

$$f_1(3x + 1) = \sum \binom{9x+3}{9i+3}\tau(3x + 1, 9i + 3, 1). \qquad (4.12)$$

If $x-3i = 2\ell$, then $\tau(x, 3i, 1) = (-1)^\delta \binom{\ell-1}{y}$ and $\tau(3x+1, 9i+3, 1) = -\binom{3\ell-2}{3y+\delta} \equiv -\binom{\ell-1}{y}$ since $\delta \ne 2$. Thus $f_1(3x + 1) = (-1)^{\delta+1} f_1(x)$. To see that (4.12) contains all possible nonzero terms, note that terms $\binom{9x+3}{9i}\tau(3x+1,9i,(-1)^{x-i-1})$ contribute 0 to $f_1(3x+1)$ since the τ-part is $-\binom{(3x-9i)/2}{x}\equiv 0$ or $-\binom{(3x-9i-1)/2}{x}\equiv 0$, since (x,i) is not special.

If x is sparse, the special term (x,i) contributes -1 to $f_1(x)$. If also $x\equiv 1$ mod 3, then the corresponding term in (4.12) is $\tau(3x+1,9i+3,-1)$ with $x-i$ odd, equaling $-\binom{(3x-9i-3)/2}{x}\equiv -1$ by 4.7.(3). That the terms added to each are equal is consistent with $f_1(3x+1)=(-1)^{\delta+1}f_1(x)$.

Case 4: Let $x=3y$. Ignoring temporarily the special term when x is sparse, we have $f_1(x)=\sum\binom{3x}{3i}\tau(x,3i,1)$ and will show that $f_1(3x+2)=\sum\binom{9x+6}{9i+6}\tau(3x+2,9i+6,1)$. If $x-3i=2\ell$, then

$$\tau(x,3i,1)\equiv\binom{\ell-1}{y}\equiv\binom{3\ell-3}{3y}\equiv\tau(3x+2,9i+6,1).$$

If the $9i+6$ in the sum for $f_1(3x+2)$ is replaced by $9i$ or $9i+3$, then the associated τ is 0, for different reasons in the two cases.

We illustrate what happens to a special term (x,i) when x is sparse, using the case $x=30$ and $i=3$. It is perfectly typical. This term contributes -1 to $f_1(x)$. We will show that it also contributes -1 to $f_1(3x+2)$, using $9i+3$ rather than $9i+6$, which is what contributed in all the other cases. The reader can check that for terms with $k=9i+\langle 0,3,6\rangle$, the τ-terms are, respectively

$$\tau(92,27,-1)=0,\ \tau(92,30,1)\equiv\binom{30}{30}\equiv 1,\ \tau(92,33,-1)=0.$$

The binomial coefficient accompanying the case $i=30$ is $\binom{9\cdot 30+6}{9\cdot 3+3}\equiv 2$. ■

Next we prove a theorem, similar to 4.8, which, with 4.3, implies another part of the "if" part of Theorem 1.9.

Theorem 4.13. *With T as in Theorem 1.9, if $n\in(9T+3)$ then $\phi(n)\neq 0$.*

Proof. We define $f_3(x)=\phi(9x+3)$ and write $2\in x$ to mean that a 2 occurs somewhere in the 3-ary expansion of x. We organize the proof into four cases, which imply the result.

(1) If $2\notin x$, then $f_3(x)\neq 0$.
(2) For all x, $f_3(3x)=f_3(x)$.
(3) For all x, $f_3(9x+2)=f_3(x)$.
(4) If x is not sparse and $x\not\equiv 2$ mod 3, then $f_3(3x+1)=(-1)^{\overline{x}+1}f_3(x)$.

Case 1: Let $9x = \sum_{i=1}^{t} 3^{a_i}$ with $a_i > a_{i-1}$ and $a_1 \geq 2$. Let i_0 be the largest $i \geq 1$ such that $a_{i+1} - a_i = 1$. Note that x is sparse iff no such i exists; let $i_0 = 1$ in this situation. For any j, let $p(j)$ denote the number of $i \leq j$ for which $a_{i-1} < a_i - 1$ or $i = 1$. We will sketch a proof that, mod 3,

$$\binom{9x+2}{k}\tau(3x+1,k,(-1)^{x-k}) \equiv \begin{cases} 1\cdot(-1)^{p(j)+1} & k = 9x+2-3^{a_t}-3^{a_j},\ i_0 \leq j < t \\ 2\cdot(-1) & k = 9x+1-3^{a_t},\ n \text{ sparse} \\ 0 & \text{otherwise.} \end{cases} \tag{4.14}$$

We have written the values in a form which separates the binomial coefficient factor from the τ factor. The binomial coefficient factor follows from (2.11). One readily verifies from (4.14) that the nonzero terms in (4.4) written in increasing k-order alternate between 1 and -1 until the last one which repeats its predecessor. Thus the sum is nonzero.

The hard part in all of these is discovering the formula; then the verifications are straightforward, and extremely similar to those of the preceding proof. We give one, that shows where $(-1)^{p(j)+1}$ comes from.

If $k = 9x+2-3^{a_t}-3^{a_j} = 2+3^{a_1}+\cdots+3^{a_{j-1}}+3^{a_{j+1}}+\cdots+3^{a_{t-1}}$, then $3x+1-k = 2\ell+1$ with

$$\ell = \sum_{\substack{i=2\\ i\neq j+1}}^{t}\ \sum_{s=a_{i-1}}^{a_i-2} 3^s + \sum_{s=0}^{a_{j+1}-2} 3^s + \sum_{s=0}^{a_1-2} 3^s.$$

We desire $\tau(3x+1,k,1) = -\binom{\ell}{x}$ with $x = \sum_{i=1}^{t} 3^{a_i-2}$. Note that ℓ has a 3^{a_i-2}-summand for each $i \neq j+1$ for which $a_{i-1} \neq a_i - 1$, and another for each $i \leq j+1$. Thus the 3-ary expansion of ℓ will have 0 in position $a_i - 2$, causing $\tau = 0$, if $i > j+1$ and $a_i = a_{i-1}+1$. That explains the choice of i_0. If $j \geq i_0$, then $\binom{\ell}{x}$ from (2.11) has a factor $\binom{2}{1}$ in positions i enumerated by $p(j)$.

Case 2: If x is sparse, the result follows from the proof of Case 1, and so we assume x is not sparse. Then we are comparing

$$f_3(x) = \sum \binom{9x+2}{9i+2}\tau(3x+1, 9i+2, (-1)^{x-i}), \tag{4.15}$$

mod 3, with

$$f_3(3x) = \sum \binom{27x+2}{27i+2}\tau(9x+1, 27i+2, (-1)^{x-i}). \tag{4.16}$$

The binomial coefficients are clearly equal, mod 3. One can show that, for the other possible contributors to (4.16), $\tau(9x+1, 27i+1, (-1)^{x-i+1}) = 0 = \tau(9x+1, 27i, (-1)^{x-i})$. If $x-i$ is odd, the τ-terms in (4.15) and (4.16) are 0, while if $x-i$ is even and $Q = \frac{x-3i}{2}$, then

$$\tau(3x+1, 9i+2, 1) \equiv -\binom{3Q-1}{x} \equiv -\binom{9Q-1}{3x} \equiv \tau(9x+1, 27i+2, 1).$$

Case 3: If x is not sparse, we are comparing

$$f_3(x) = \sum \binom{9x+2}{9i+2}\tau(3x+1, 9i+2, (-1)^{x-i})$$

with

$$f_3(9x+2) = \sum \binom{9(9x+2)+2}{9(9i+2)+2}\tau(3(9x+2)+1, 9(9i+2)+2, (-1)^{x-i}). \tag{4.17}$$

We will show below that no other terms can contribute to (4.17). Given this, then the binomial coefficients clearly agree, mod 3.

When $x-i$ is odd, the terms in both sums are 0, since they are of the form $\tau(3m+1, 3m+1-2\ell, -1)$.

Suppose $x-i$ is even. Let $Q = \frac{x-3i}{2}$. The first τ is $-\binom{3Q-1}{x}$, while the second is the negative of $\binom{27Q-7}{9x+2} \equiv \binom{3Q-1}{x}$, as desired.

As a possible additional term in (4.17), if $k = 9(9i+2)+2$ is replaced with $k = 9(9i+\alpha)+\beta$ with $0 \le \alpha, \beta \le 2$, which are the only ways to obtain a nonzero binomial coefficient, then we show that the relevant τ is 0. Still assuming $x-i$ even, if $\alpha+\beta$ is odd, then we obtain $\tau(3m+1, 3m+1-2\ell, -1) = 0$, while if $\beta = 0$ and $\alpha \ne 1$, then we obtain $\tau = \binom{3y}{9x+2} \equiv 0$ for some y. Finally, if $\beta = 2$ and $\alpha = 0$,

$$\tau = \binom{9(3x-9i)/2+2}{9x+2} \equiv \binom{(3x-9i)/2}{x}.$$

Since, in order to have $\binom{9(9x+2)+2}{9(9i+2)+2} \not\equiv 0$, we must have $\nu_3(i) \ge \nu_3(x)$, we conclude $\binom{(3x-9i)/2}{x} \equiv 0$ mod 3. The case $x-i$ odd is handled similarly.

If $9x = 3^{a_1} + \cdots + 3^{a_t}$ is sparse and $9i = 9x - 3^{a_t}$, there is an additional term, $\binom{9x+2}{9i+1}\tau(3x+1, 9i+1, 1) \equiv 1$, in the sum for $f_3(x)$. The additional term in $f_3(9x+2)$ is

$$\binom{9(9x+2)+2}{9(9i+1)+2}\tau(3(9x+2)+1, 9(9i+1)+2, 1) \equiv \binom{\ell}{m},$$

with $m = 9x+2 = 2+3^{a_1}+\cdots+3^{a_t}$, and $2\ell+1 = 3(9x+2)+1-9(9i+1)-2$, so that

$$\ell = \sum_{j=2}^{t}\sum_{s=a_{j-1}+2}^{a_j} 3^s + \sum_{s=2}^{a_1} 3^s + 2,$$

and so the additional term in $f(9x+2)$ is 1.

Case 4: We first show

$$\begin{aligned}(-1)^{\overline{x}+1} f_3(x) &= (-1)^{\overline{x}+1}\sum \tbinom{9x+2}{9i+2}\tau(3x+1, 9i+2, (-1)^{x-i})\\ &= \sum \tbinom{27x+11}{27i+11}\tau(9x+4, 27i+11, (-1)^{x-i})\\ &= f_3(3x+1)\end{aligned}$$

for $x \equiv 0, 1$ (3). Both τ's are 0 if $x-i$ is odd, while if $x-i$ is even and $x \equiv 0, 1$ (3), then

$$(-1)^{\overline{x}+1}\tau(3x+1, 9i+2, 1) \equiv (-1)^{\overline{x}}\tbinom{3Q+2}{x} \equiv -\tbinom{9Q+5}{3x+1} = \tau(9x+4, 27i+11, 1),$$

where $Q = (x-3i-2)/2$.

We must also show that $\binom{27x+11}{k}\tau(9x+4, k, (-1)^{x+1-k}) \equiv 0$ for $k \not\equiv 11$ (27). When $k \equiv 2$ (27), the result follows from Lemma 4.7. When $k \equiv 0, 9$ (27), τ is of the form $\binom{3A}{3x+1} \equiv 0$. ■

The "if" part of Theorem 1.9 when $n = 3T'+2$ divides into two parts, Theorems 4.18 and 4.22, noting that $3T'+2 = (9T+2) \cup (9T'+5)$.

Theorem 4.18. *If T is as in 1.9 and $n \in (9T'+5)$, then $\phi(n) \neq 0$.*

Proof. Let $f_5(x) = \phi(9x+5)$. We will prove that if $x \in T'$ then

$$f_5(x) = (-1)^{\overline{x}} f_3(x). \tag{4.19}$$

With Theorem 4.13, this implies the result.

Case 1: Assume x not sparse and recall $x \not\equiv 2$ (3). We show that, mod 3,

$$\tbinom{9x+4}{k}\tau(3x+1, k, (-1)^{9x+4-k}) \equiv (-1)^{\overline{x}}\tbinom{9x+2}{k-2}\tau(3x+1, k-2, (-1)^{9x+4-k}). \tag{4.20}$$

Since $f_5(x)$ is the sum over k of the LHS, and $(-1)^{\overline{x}} f_3(x)$ the sum over k of the RHS, (4.19) will follow when x is not sparse.

We first deal with cases when the RHS of (4.20) is nonzero. By the proof of 4.13, this can only happen when $k-2=9i+2$, $\binom{x}{i}\not\equiv 0$ mod 3, and $x-i$ is even. Mod 3, we have $\binom{9x+4}{9i+4}\equiv\binom{9x+2}{9i+2}$ by (2.11). The two τ's in (4.20) are, with $Q:=\frac{3x-9i}{2}$, $-\binom{Q-2}{x}$ and $-\binom{Q-1}{x}$, respectively. Since $Q\equiv 0$ mod 3, these are equal if $x\equiv 0$ and negatives if $x\equiv 1$.

We conclude the proof of (4.20) by showing that other values of k cause $\binom{9x+4}{k}\tau(3x+1,k,(-1)^{x-k})\equiv 0$. If $k\not\equiv 0,1,3,4$ mod 9, then $\binom{9x+4}{k}\equiv 0$. If $k=9i+1$ or $9i+3$ and $x-i$ even, or if $k=9i$ or $9i+4$ and $x-i$ odd, then $\tau=0$ by 4.2. If $k=9i$ and $x-i$ is even, then $\tau\equiv\binom{3x-9i}{x}\equiv 0$. For $k=9i+1$ or $9i+3$ and $x-i$ odd, the result follows from Lemma 4.7.

Case 2: Assume x is sparse. Let $9x=\sum_{j=1}^{t}3^{a_j}$ with $a_j-a_{j-1}\ge 2$. We call $k=9i+d$, $d\in\{0,1,3,4\}$, special if $(9x,9i)$ is special. The analysis of Case 1 shows that the f_5-sum over non-special values of k equals $(-1)^{\overline{x}}$ times the f_3-sum over non-special values of k.

We saw in (4.14) that the only special value of k giving a nonzero summand for $f_3(x)$ is $k=9i+1$ (with $9i=9x-3^{a_t}$) and this summand is 1. We will show that if $x\equiv 1$ (3), then the only special value of k giving a nonzero summand for $f_5(x)$ is $k=9i+1$, and it gives -1, while if $x\equiv 0$ (3), both $k=9i+1$ and $k=9i+3$ give summands of -1 for $f_5(x)$. This will imply the claim.

Recall $9i=9x-3^{a_t}$, and hence $x-i$ is odd. If $k=9i+\langle 0,4\rangle$, then the τ-factor is $\tau(3x+1,9i+\langle 0,4\rangle,-1)=0$. If $k=9i+\langle 1,3\rangle$, the relevant term in $f_5(x)$ is

$$\binom{9x+4}{9i+\langle 1,3\rangle}\tau(3x+1,9i+\langle 1,3\rangle,1)=-\binom{\ell}{x},$$

where

$$\ell=\sum_{i=2}^{t}\sum_{s=a_{i-1}}^{a_i-2}3^s+\sum_{s=0}^{a_1-2}3^s+\langle 0,-1\rangle.$$

Using (2.11), $\binom{\ell}{x}\equiv 1$ in the $(9i+1)$-case, while in the $(9i+3)$-case

$$\binom{\ell}{x}\equiv\binom{(\sum_{s=0}^{a_1-2}3^s)-1}{3^{a_1-2}},$$

which is 0 if $x\equiv 1$ (3), since then $a_1=2$, but is 1 if $x\equiv 0$ (3) since then $a_1\ge 3$. ■

When $n \in (9T+2)$, the equality of $e_3(n-1,n)$ and $s_3(n)$ in Theorem 1.9 comes not from $\nu_3(a_3(n-1,n))$, as it has in the other cases, but rather from $\nu_3(a_3(n-1,n+1))$. To see this, we first extend Theorem 4.3, as follows.

Theorem 4.21. *If $N \ge n$, then $\nu_3(a_3(n-1,N)) = s_3(n)$ iff $[N/9] = [n/9]$ and*

$$\sum \binom{n-1}{k}\tau([\tfrac{N}{3}], k, (-1)^{n-k-1}) \not\equiv 0\,(3).$$

Proof. This is very similar to the proof, centered around (2.7), that Theorem 2.2 implies Theorem 1.7. We have

$$0 = (-1)^N S(n-1,N)N! = a_3(n-1,N) + 3^{n-1}\sum(-1)^k\binom{N}{3k}k^{n-1}.$$

Thus $\nu_3(a_3(n-1,N)) = s_3(n)$ iff $B \not\equiv 0$ (3), where, mod 3,

$$\begin{aligned}
B &:= \tfrac{1}{[n/3]!}\sum(-1)^k\binom{N}{3k}k^{n-1}\\
&\equiv \sum_{d=1}^{2}\tfrac{1}{[n/3]!}\sum_{k\equiv d\,(3)}(-1)^k\binom{N}{3k}k^{n-1}\\
&\equiv \tfrac{1}{[n/3]!}\sum_{d=1}^{2}(-1)^d\sum_j(-1)^j\binom{N}{9j+3d}\sum_\ell 3^\ell j^\ell\binom{n-1}{\ell}d^{n-1-\ell}\\
&\equiv \tfrac{1}{[n/3]!}\sum_{d=1}^{2}(-1)^d\sum_j(-1)^j\binom{N}{9j+3d}\sum_\ell 3^\ell\binom{n-1}{\ell}d^{n-1-\ell}\sum_i S(\ell,i)i!\binom{j}{i}\\
&\equiv \tfrac{1}{[n/3]!}\sum_{d=1}^{2}(-1)^d\sum_j(-1)^j\binom{N}{9j+3d}\sum_i 3^i\binom{n-1}{i}d^{n-1-i}i!\binom{j}{i}\\
&\equiv \tfrac{[N/3]!}{[n/3]!}\sum_i\binom{n-1}{i}(T_{i,2}(N,3)+(-1)^{n-1-i}T_{i,2}(N,6))\\
&\equiv \tfrac{[N/3]!}{[n/3]!}\sum_i\binom{n-1}{i}(T_{i,1}([\tfrac{N}{3}],1)+(-1)^{n-1-i}T_{i,1}([\tfrac{N}{3}],2))\\
&= \tfrac{[N/3]!}{[n/3]!}\sum_i\binom{n-1}{i}\tau([\tfrac{N}{3}], i, (-1)^{n-i-1}).
\end{aligned}$$

■

The "if" part of 1.9 when $n \in (9T+2)$ now follows from Theorem 4.21 and the following result.

Theorem 4.22. *If T is as in 1.9 and $n \in (9T+2)$, then*

$$\sum\binom{n-1}{k}\tau([\tfrac{n+1}{3}], k, (-1)^{n-k-1}) \not\equiv 0\,(3).$$

Proof. We prove that for such n

$$\sum \binom{n-1}{k}\tau([\tfrac{n+1}{3}],k,(-1)^{n-k-1}) \equiv \sum \binom{n}{k}\tau([\tfrac{n+1}{3}],k,(-1)^{n-k}) \tag{4.23}$$

and then apply Theorem 4.13. Note that the RHS is $\phi(n+1)$.

If $n = 9x+2$ with x not sparse, then the proof of 4.13 shows that the nonzero terms of the RHS of (4.23) occur for $k = 9i+2$ with $\binom{x}{i} \not\equiv 0$ (3) and $x-i$ even. Now (4.23) in this case follows from

$$\binom{9x+1}{9i+1}\tau(3x+1,9i+1,1) \equiv -\binom{x}{i}\binom{(3x-9i-2)/2}{x} \equiv \binom{9x+2}{9i+2}\tau(3x+1,9i+2,1). \tag{4.24}$$

One must also verify that no other values of k contribute to the LHS of (4.23); this is done by the usual methods.

If $n = 9x+2$ with x sparse, (4.24) holds unless (x,i) is special. For such i, the contribution to the RHS of (4.23) using $k = 9i+1$ is $2\cdot 2 \equiv 1$. The LHS of (4.23) obtains contributions of $1\cdot 2$ from both $k = 9i$ and $k = 9i+1$. Indeed both τ's equal $-\binom{(3x-9i-1)/2}{x} \equiv -1$ by 4.7. ■

The "if" part of Theorem 1.9 is an immediate consequence of Theorems 4.8, 4.13, 4.18, and 4.22. We complete the proof of Theorem 1.9 by proving the following result.

Proposition 4.25. *If n is not one of the integers described in Theorem 1.9, then for all integers $N \ge n$ satisfying $[N/9] = [n/9]$, we have*

$$\sum \binom{n-1}{k}\tau([\tfrac{N}{3}],k,(-1)^{n-k-1}) \equiv 0 \ (3).$$

Proof. We break into cases depending on n mod 9, and argue by induction on n with the integers ordered so that $9x+3$ immediately precedes $9x+2$.

Case 1: $n \equiv 0$ (9). Let $n = 9a$. If $[\frac{N}{3}] = 3a$ or $3a+2$, then $\tau([\frac{N}{3}],k,(-1)^{a-1-k}) = 0$ by 4.2. Now suppose $[\frac{N}{3}] = 3a+1$. We show that for each nonzero term in

$$\sum_k \binom{9a-1}{k}\tau(3a+1,k,(-1)^{a-k-1})$$

with $a-k$ odd, the $(k+1)$-term is the negative of the k-term. Thus the sum is 0.

Both τ's equal $-\binom{(3a-k-1)/2}{a}$. Since $\binom{9a-1}{k} + \binom{9a-1}{k+1} = \binom{9a}{k+1}$, the binomial coefficients are negatives of one another unless $k+1 = 9t$ with $\binom{a}{t} \not\equiv 0$ (3). Then $\nu(t) \ge \nu(a)$ and so $\binom{(3a-k-1)/2}{a} = \binom{(3a-9t)/2}{a} \equiv 0$ (3), so the τ's were 0.

Case 2: $n \equiv 6, 7, 8\ (9)$. In these cases, $[N/9] = [n/9]$ implies $[N/3] = [n/3]$ and so we need not consider $N > n$. By 4.2, $\tau(3x+2, k, (-1)^{x+1-k}) = 0$, which implies $\phi(9x+6) = 0 = \phi(9x+8)$. We have

$$\phi(9x+7) = \sum \tbinom{9x+6}{k} \tau(3x+2, k, (-1)^{x-k}).$$

This is 0 if $x-k$ is odd, while if $x-k$ is even, a summand is $\binom{9x+6}{k}\binom{(3x-k)/2}{x}$, which is 0 unless $k \equiv 0\ (3)$ and hence $x \equiv 0\ (3)$. In the latter case, with $x = 3x'$ and f_1 as in the proof of 4.8, we have $\phi(n) = f_1(9x'+2)$, which, by Case 4 of the proof of 4.8, equals $f_1(3x')$, and this is 0 by induction unless $x' \in T$.

Case 3: $n = 9x+5$. If $x \equiv 0, 1\ (3)$, then $\phi(9x+5) = \pm\phi(9x+3)$ was proved in Case 1 of the proof of 4.18. The induction hypothesis thus implies the result for $N = n$ in these cases. If $x = 3y+2$, then

$$\phi(n) = \sum \tbinom{27y+22}{k} \tau(9y+7, k, (-1)^{y-k}).$$

The k-term is 0 if $y-k$ is odd, while if $y-k$ is even, $\tau = -\binom{(9y+6-k)/2}{3y+2}$. This is 0 unless $k \equiv 2 \bmod 3$, but then $\binom{27y+22}{k} \equiv 0\ (3)$. The k-term for $N = n+1$ is nonzero iff the k-term in $\phi(n)$ is nonzero; this is true because $\tau(3z+2, k, (-1)^{z-k}) = \pm\tau(3z+1, k, (-1)^{z-k})$. Thus the sum for $N = n+1$ is 0 if $x \notin T'$.

Case 4: $n = 9x+2$. Since, for $\epsilon = 0$ or 2, $\tau(3x+\epsilon, k, (-1)^{x-k+1}) = 0$, we deduce that $\sum \binom{n-1}{k}\tau([\frac{N}{3}], k, (-1)^{n-k-1}) = 0$ for $N = n$ and $N = n+4$. For $N = n+1$, this is just the LHS of (4.23). By (4.23), it equals $\phi(n+1)$, which is 0 for $x \notin T$ by the induction hypothesis.

Case 5: $n = 9x+3$. Let $f_3(x) = \phi(9x+3)$. Let x be minimal such that $x \notin T$ and $f_3(x)$ has a nonzero summand. By the proof of 4.13, x is not 0 mod 3, 2 mod 9, 1 mod 9, or 4 mod 9.

If $x \equiv 5$, 7, or 8 mod 9, then $f_3(x)$ has no nonzero summands. For example, if $x = 9t+7$, the summands are $\binom{81t+65}{k}\tau(27t+22, k, (-1)^{t-k-1})$. This is 0 if $t-k$ is even, while if $t-k$ is odd, the τ-factor is $\binom{(27t+21-k)/2}{9t+7}$. For this to be nonzero, we must have $k \equiv 5$ or 7 mod 9, but these make the first factor 0. Other cases are handled similarly.

One can show that for $\epsilon = 0$, 1, 2,

$$\tau(3x+2, 9i+\epsilon, (-1)^{x-i-\epsilon}) = \pm\tau(3x+1, 9i+\epsilon, (-1)^{x-i-\epsilon}) \in \mathbb{Z}/3.$$

This implies that when we use $N = n + 3$, nonzero terms will be obtained iff they were obtained for n.

Case 6: $n \equiv 1, 4 \bmod 9$. Let $f_1(x) = \phi(3x+1)$. By the proof of Theorem 4.8, there can be no smallest $x \equiv 0, 1 \bmod 3$ which is not in T and has $f_1(x) \neq 0$. When using $N = n+2$ or, if $n \equiv 1\ (9)$, $N = n+5$, then the k-summands, $\binom{9x}{k}\tau(3x+1, k, (-1)^{x-k})$, $\binom{9x+3}{k}\tau(3x+2, k, (-1)^{x+1-k})$, and $\binom{9x}{k}\tau(3x+2, k, (-1)^{x-k})$, are easily seen to be 0. ■

5. Discussion of Conjecture 1.17

In this section we discuss the relationship between $\overline{e}_2(n)$, $e_2(n-1, n)$, and $s_2(n)$. In particular, we discuss an approach to Conjecture 1.17, which suggests that the inequality $e_2(n-1, n) \geq s_2(n)$ fails by 1 to be sharp if $n = 2^t$, while if $n = 2^t + 1$, it is sharp but the maximum value of $e_2(k, n)$ occurs for a value of $k \neq n - 1$. The prime $p = 2$ is implicit in this section; in particular, $\nu(-) = \nu_2(-)$ and $a(-, -) = a_2(-, -)$.

Although our focus will be on the two families of n with which Conjecture 1.17 deals, we are also interested, more generally, in the extent to which equality is obtained in each of the inequalities of

$$s_2(n) \leq e_2(n-1, n) \leq \overline{e}_2(n). \tag{5.1}$$

In Table 1, we list the three items related in (5.1) for $2 \leq n \leq 38$, and also the smallest positive k for which $e_2(k, n) = \overline{e}_2(n)$. We denote this as $k_{\max}$, since it is the simplest k-value giving the maximum value of $e_2(k, n)$. Note that in this range $k_{\max}$ always equals $n - 1$ plus possibly a number which is rather highly 2-divisible.

We return to more specific information leading to Conjecture 1.17. To obtain the value of $\overline{e}_2(n)$, we focus on large values of $e_2(k, n)$. For $n = 2^t$ and $2^t + 1$, this is done in the following conjecture, which implies Conjecture 1.17. Note that $s_2(2^t) = 2^t + 2^{t-1} - 2$, and $s_2(2^t + 1) = 2^t + 2^{t-1} - 1$. We employ the usual convention $\nu(0) = \infty$.

Conjecture 5.2. *If $t \geq 3$, then*

$$e_2(k, 2^t) \begin{cases} = \min(\nu(k+1-2^t) + 2^t - t,\ 2^t + 2^{t-1} - 1) & \text{if } k \equiv -1 \pmod{2^{t-1}} \\ < 2^t + 2^{t-1} - 1 & \text{if } k \not\equiv -1 \pmod{2^{t-1}}; \end{cases}$$

TABLE 1. Comparison for (5.1) when $p = 2$

n	$s_2(n)$	$e_2(n, n-1)$	$\bar{e}_2(n)$	$k_{\max}$
2	1	1	1	1
3	2	2	2	2
4	4	4	4	3
5	5	5	6	$4 + 2^3$
6	6	6	8	$5 + 2^3$
7	7	8	8	6
8	10	11	11	7
9	11	11	12	$8 + 2^6$
10	12	12	14	$9 + 2^6$
11	13	13	15	$10 + 2^6$
12	15	15	15	11
13	16	18	18	12
14	17	21	21	13
15	18	22	22	14
16	22	23	23	15
17	23	23	24	$16 + 2^{11}$
18	24	24	26	$17 + 2^{11}$
19	25	25	28	$18 + 2^{11}$
20	27	27	28	$19 + 2^{11}$
21	28	28	28	20
22	29	29	30	$21 + 2^{10}$
23	30	31	31	22
24	33	34	34	23
25	34	36	38	$24 + 2^{16}$
26	35	37	40	$25 + 2^{16}5$
27	36	38	40	$26 + 2^{16}$
28	38	40	40	27
29	39	42	44	$28 + 2^{18}$
30	40	43	45	$29 + 2^{18}$
31	41	46	46	30
32	46	47	47	31
33	47	47	48	$32 + 2^{20}$
34	48	48	50	$33 + 2^{20}$
35	49	49	52	$34 + 2^{20}$
36	51	51	53	$35 + 2^{20}$
37	52	52	54	$36 + 2^{20}3$
38	53	53	56	$37 + 2^{20}7$

$$e_2(k,2^t{+}1)\begin{cases} = \min(\nu(k-2^t-2^{2^{t-1}+t-1})+2^t-t,\ 2^t+2^{t-1}) & \textit{if } k\equiv 0 \pmod{2^{t-1}} \\ < 2^t+2^{t-1} & \textit{if } k\not\equiv 0 \pmod{2^{t-1}}. \end{cases}$$

Note from this that conjecturally the smallest positive value of k for which $e_2(k,n)$ achieves its maximum value is $n-1$ when $n=2^t$ but is $n-1+2^{2^{t-1}+t-1}$ when $n=2^t+1$. The reason for this is explained in the next result, involving a comparison of the smallest $\nu(a(k,j))$ values.

Conjecture 5.3. *There exist odd 2-adic integers u, whose precise value varies from case to case, such that*

(1) *if* $k\equiv -1 \pmod{2^{t-1}}$, *then*

$$\begin{aligned} \nu(a(k,2^t+1)) &= \nu(k+1-2^t-2^{2^{t-1}+t-1}u)+2^t-t \\ \nu(a(k,2^t+2)) &= \nu(k+1-2^t-2^{2^{t-1}+t-2}u)+2^t-t+1 \\ \nu(a(k,2^t+3)) &= \nu(k+1-2^t-2^{2^{t-1}+t-2}u)+2^t-t+1; \end{aligned}$$

(2) *if* $k\equiv 0 \pmod{2^{t-1}}$, *then*

$$\begin{aligned} \nu(a(k,2^t+1)) &= \nu(k-2^t-2^{2^{t-1}+t-1}u)+2^t-t \\ \nu(a(k,2^t+2)) &= \nu(k-2^t-2^{2^{t-1}+t}u)+2^t-t+1 \\ \nu(a(k,2^t+3)) &= \nu(k-2^t-2^{2^{t-1}+t-2}u)+2^t-t+2. \end{aligned}$$

For other values of $j\ge 2^t$ (resp. 2^t+1), $\nu(a(k,j))$ is at least as large as all the values appearing on the RHS above.

Note that, for fixed j, $\nu(a(k,j))$ is an unbounded function of k; it is the interplay among several values of j which causes the boundedness of $e_2(k,n)$ for fixed n.

We show now that Conjecture 5.3 implies the "= min"-part of Conjecture 5.2. In part (1), the smallest $\nu(a(k,j))$ for $j\ge 2^t$ is

$$\begin{cases} \nu(k+1-2^t)+2^t-t & \text{if } \nu(k+1-2^t)\le 2^{t-1}+t-2, \text{ using } j=2^t+1 \\ 2^t+2^{t-1}-1 & \text{if } \nu(k+1-2^t)= 2^{t-1}+t-1, \text{ using } j=2^t+2 \\ 2^t+2^{t-1}-1 & \text{if } \nu(k+1-2^t)> 2^{t-1}+t-1, \text{ using either.} \end{cases}$$

In part (2), the smallest $\nu(a(k,j))$ for $j\ge 2^t+1$ is

$$\begin{cases} \nu(k-2^t)+2^t-t & \text{if } \nu(k-2^t)\le 2^{t-1}+t-2, \text{ using } j=2^t+1 \\ 2^t+2^{t-1} & \text{if } \nu(k-2^t)= 2^{t-1}+t-1, \text{ using } j=2^t+2 \\ 2^t+2^{t-1}-1 & \text{if } \nu(k-2^t)\ge 2^{t-1}+t, \text{ using } j=2^t+1. \end{cases}$$

Conjecture 5.3 can be thought of as an application of Hensel's Lemma, following Clarke ([2]). We are finding the first few terms of the unique zero of the 2-adic function $f(x) = \nu(a(x, 2^t + \epsilon))$ for x in a restricted congruence class.

6. Relationships with algebraic topology

In this section, we sketch how the numbers studied in this paper are related to topics in algebraic topology, namely James numbers and v_1-periodic homotopy groups.

Let $W_{n,k}$ denote the complex Stiefel manifold consisting of k-tuples of orthonormal vectors in $\mathbb{C}^n$, and $W_{n,k} \to S^{2n-1}$ the map which selects the first vector. In work related to vector fields on spheres, James ([8]) defined $U(n, k)$ to be the order of the cokernel of

$$\pi_{2n-1}(W_{n,k}) \to \pi_{2n-1}(S^{2n-1}) \approx \mathbb{Z},$$

now called James numbers. A bibliography of many papers in algebraic topology devoted to studying these numbers can be found in [4]. It is proved in [11] that

$$\nu_p(U(n, k)) \geq \nu_p((n-1)!) - \widetilde{e}_p(n-1, n-k).$$

Our work implies the following sharp result for certain James numbers.

Theorem 6.1. *If $p = 2$ or 3, n is as in Theorems 1.7 or 1.9, and L is sufficiently large, then*

$$\nu_p(U((p-1)p^L + n, (p-1)p^L)) = p^L - (p-1)[\tfrac{n}{p}] - \nu_p(n) - \overline{n}.$$

Proof. We present the argument when $p = 3$. By [4, 4.3] and 1.9, we have

$$\nu_3(U(2 \cdot 3^L + n, 2 \cdot 3^L)) = \nu_3((2 \cdot 3^L + n - 1)!) - (n - 1 + \nu_3([n/3]!)).$$

Using Proposition 3.1, this equals

$$\tfrac{1}{2}(2 \cdot 3^L - n - 1 - d_3(n-1) - [\tfrac{n}{3}] + d_3([\tfrac{n}{3}])).$$

If $\overline{n} \neq 0$ and $n = 3m + \overline{n}$, this equals $3^L - 2m - \overline{n}$, while if $n = 3m$, we use $d_3(k-1) = d_3(k) - 1 + 2\nu_3(k)$ to obtain $3^L - 2m - \nu_3(3m)$. ■

The p-primary v_1-periodic homotopy groups of a topological space X, denoted $v_1^{-1}\pi_*(X)_{(p)}$ and defined in [5], are a first approximation to the p-primary actual homotopy groups $\pi_*(X)_{(p)}$. Each group $v_1^{-1}\pi_i(X)_{(p)}$ is a direct summand of some homotopy group $\pi_j(X)$. It was proved in [4] that for the special unitary group $SU(n)$, we have, if p or n is odd,

$$v_1^{-1}\pi_{2k}(SU(n))_{(p)} \approx \mathbb{Z}/p^{e_p(k,n)},$$

and $v_1^{-1}\pi_{2k-1}(SU(n))_{(p)}$ has the same order. The situation when $p = 2$ and n is even is slightly more complicated; it was discussed in [1] and [6]. In this case, there is a summand $\mathbb{Z}/2^{e_2(k,n))}$ or $\mathbb{Z}/2^{e_2(k,n)-1}$ in $v_1^{-1}\pi_{2k}(SU(n))_{(2)}$. From Theorems 1.9 and 1.7 we immediately obtain

Corollary 6.2. *If n is as in Theorem 1.9 and $k \equiv n-1$ mod $2 \cdot 3^{s_3(n)}$, then*

$$v_1^{-1}\pi_{2k}(SU(n))_{(3)} \approx \mathbb{Z}/3^{s_3(n)}.$$

If n is as in Theorem 1.7 and is odd, and $k \equiv n-1$ mod $2^{s_2(n)-1}$, then

$$v_1^{-1}\pi_{2k}(SU(n))_{(2)} \approx \mathbb{Z}/2^{s_2(n)}.$$

We are especially interested in knowing the largest value of $e_p(k,n)$ as k varies over all integers, as this gives a lower bound for $\exp_p(SU(n))$, the largest p-exponent of any homotopy group of the space. It was shown in [7] that this is $\geq s_p(n)$ if p or n is odd. Our work here immediately implies Corollary 6.3 since $v_1^{-1}\pi_{2n-2}(SU(n))_{(p)}$ has p-exponent greater than $s_p(n)$ in these cases.

Corollary 6.3. *If $p = 3$ and n is not as in 1.9 or $p = 2$ and n is odd and not as in 1.7, then* $\exp_p(SU(n)) > s_p(n)$.

Table 1 illustrates how we expect that $k = n-1$ will give almost the largest group $v_1^{-1}\pi_{2k}(SU(n))_{(p)}$, but may miss by a small amount. There is much more that might be done along these lines.

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