ORIENTABLE MANIFOLDS WITH NONZERO DUAL STIEFEL-WHITNEY CLASSES OF LARGEST POSSIBLE GRADING

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ABSTRACT. It is known that, for all n, there exist compact differentiable orientable n-manifolds with dual Stiefel-Whitney class $\overline{w}_{n-\widehat{\alpha}(n)} \neq 0$, and this is best possible, but the proof is nonconstructive. Here $\widehat{\alpha}(n)$ equals the number of 1's in the binary expansion of n if $n \equiv 1 \bmod 4$, and exceeds this by 1 otherwise. We find, for all $n \not\equiv 0 \bmod 4$, examples of real Bott maanifolds with this property.

1. Introduction

The dual Stiefel-Whitney classes $\overline{w}_i(M)$ of a compact differentiable[1] n-manifold M are elements of $H^i(M;\mathbb{Z}_2)$ which have the property that if $\overline{w}_i(M) \neq 0$, then M cannot be embedded in $\mathbb{R}^{n+i}$ nor immersed in $\mathbb{R}^{n+i-1}$. An important early result in algebraic topology was the following theorem of Massey.

Theorem 1.1. ([9]) *Let $\alpha(n)$ denote the number of 1's in the binary expansion of n. If M is an n-manifold and $i > n - \alpha(n)$, then $\overline{w}_i(M) = 0$. Moreover, there exists an n-manifold M with $\overline{w}_{n-\alpha(n)}(M) \neq 0$.*

This theorem was the first step toward Cohen's theorem ([5]) that every n-manifold can be immersed in $\mathbb{R}^{2n-\alpha(n)}$. One attractive feature of this theorem is that the existence part is realized by a well-known manifold: if $n = 2^{e_1} + \cdots + 2^{e_{\alpha(n)}}$ with e_i distinct, then $M = \prod RP^{2^{e_i}}$ is an n-manifold with $\overline{w}_{n-\alpha(n)}(M) \neq 0$.

In [6], the following analogue for orientable manifolds was proved.

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[1]All manifolds considered in this paper will be compact and differentiable, and we will henceforth use the word "manifold" to mean "compact differentiable manifold." All cohomology groups have coefficients in $\mathbb{Z}_2 = \mathbb{Z}/2$.

Theorem 1.2. ([6, Theorem 1.1]) *Let*

$$\widehat{\alpha}(n) = \begin{cases} \alpha(n) & n \equiv 1 \ (4) \\ \alpha(n)+1 & n \not\equiv 1 \ (4). \end{cases}.$$

If M is an orientable n-manifold and $i > n - \widehat{\alpha}(n)$, then $\overline{w}_i(M) = 0$. Moreover, there exists an orientable n-manifold M with $\overline{w}_{n-\widehat{\alpha}(n)}(M) \neq 0$.

This theorem was a reformulation of a result of [12], which drew heavily from results in [10] and [3]. The existence part was nonconstructive, and the goal of this paper is to find explicit orientable n-manifolds with $\overline{w}_{n-\widehat{\alpha}(n)}(M) \neq 0$. Our main theorem, 1.4, accomplishes this with real Bott manifolds for all $n \not\equiv 0$ mod 4. If n is a 2-power ≥ 4, complex projective space $CP^{n/2}$ provides an example of an orientable n-manifold with $\overline{w}_{n-\widehat{\alpha}(n)} \neq 0$. For $n \equiv 0$ mod 4 and not a 2-power, we do not know explicit orientable n-manifolds with $\overline{w}_{n-\widehat{\alpha}(n)} \neq 0$. See Section 3.

Real Bott manifolds have been studied recently by many authors, for example ([4], [7]). The following known result describes the properties relevant to our work.

Theorem 1.3. *If $A = (a_{i,j})$ is an n-by-n strictly upper-triangular binary matrix, there is an n-manifold B_n with*

$$H^*(B_n;\mathbb{Z}_2) \approx \mathbb{Z}_2[x_1,\ldots,x_n]/(x_j^2 = \sum_{i=1}^{j-1} a_{i,j}x_ix_j),$$

where $|x_i| = 1$. The manifold is orientable if and only if each row of A contains an even number of 1's.

Proof. The manifold is formed from a Bott tower

$$B_n \to B_{n-1} \to \cdots \to B_1 \to B_0 = \{*\}.$$

Each B_j is the projectification of $\xi_j \oplus \varepsilon$, where ξ_j and ε are line bundles over B_{j-1}, with ε trivial, and $w_1(\xi_j) = \sum_{i=1}^{j-1} a_{i,j}x_i$. The cohomology result is [8, Lemma 2.1]. The orientability claim is well-known. For example, [7, Proposition 2.5]. ■

Our main theorem, 1.4, gives the desired examples when $n \equiv 1$ mod 4, while Corollary 1.5 gives the examples when $n \equiv 2, 3$ mod 4.

Theorem 1.4. *Let $n \equiv 1 \mod 4$. The Bott manifold B_n with relations $x_n^2 = (x_1 + \cdots + x_{n-2})x_n$, $x_i^2 = x_{i-1}x_i$ for $2 \le i \le n-1$, and $x_1^2 = 0$ is orientable with $\overline{w}_{n-\widehat{\alpha}(n)} \neq 0$.*

For example, the Bott manifold B_5 corresponds to the following matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Corollary 1.5. *For $n \equiv 2$ (resp. 3) $\mod 4$, $B_{n-1} \times S^1$ (resp. $B_{n-2} \times S^1 \times S^1$) is an orientable n-dimensional Bott manifold with $\overline{w}_{n-\widehat{\alpha}(n)} \neq 0$. Here B_{n-1} and B_{n-2} are the manifolds of Theorem 1.4.*

Proof. This is immediate from the Whitney product formula, triviality of the tangent bundle of S^1, Theorem 1.4, and that if $m \equiv 1 \bmod 4$, then $m - \widehat{\alpha}(m) = m + 1 - \widehat{\alpha}(m+1) = m + 2 - \widehat{\alpha}(m+2)$. These are Bott manifolds by appending one or two steps to the Bott tower with ξ_j trivial. ■

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2. Proof of Theorem 1.4

The case $n = 1$ has $B_n \approx S^1$ with $n - \widehat{\alpha}(n) = 0$, and $\overline{w}_0(S^1) \neq 0$. From now on, we assume $n > 1$. Recall that $\widehat{\alpha}(n) = \alpha(n)$ since $n \equiv 1 \bmod 4$.

We will prove that there is a class $z \in H^{\alpha(n)}(B_n)$ such that $\chi \operatorname{Sq}^{n-\alpha(n)}(z) \neq 0$. Here χ is the antiautomorphism of the mod 2 Steenrod algebra. This implies Theorem 1.4 by the following standard argument (e.g., [10, §3]). The S-dual of the manifold B_n is the Thom space of its stable normal bundle, and dual to $\chi \operatorname{Sq}^{n-\alpha(n)}$ into the top class of the manifold is $\operatorname{Sq}^{n-\alpha(n)}$ on the Thom class of the normal bundle. But this equals $w_{n-\alpha(n)}$ of the normal bundle, which is $\overline{w}_{n-\alpha(n)}$ of the manifold.

By [11], $\chi \operatorname{Sq}^{n-\alpha(n)}$ equals the sum of all Milnor basis elements of grading $n-\alpha(n)$. Let $n = 2^{p_r} + \cdots + 2^{p_1} + 1$ with $p_r > \cdots > p_1 \ge 2$ and $r = \alpha(n) - 1$. Then $n - \alpha(n) = 2^{p_r} + \cdots + 2^{p_1} - r$. A Milnor basis element $\operatorname{Sq}(t_1, \ldots, t_s)$ has grading $\sum t_i(2^i - 1)$ and vanishes on classes of grading less than $\sum t_i$. We omit the proof of the following lemma, which is very similar to the second half of the proof of [6, Theorem 1.3(c)].

Lemma 2.1. *A tuple* $(t_1,\dots,t_s)$ *satisfies* $\sum t_i(2^i-1)=2^{p_r}+\cdots+2^{p_1}-r$ *and* $\sum t_i\le r$ *if and only if*

$$t_i=\begin{cases}1 & i=p_j \text{ for some } j\\ 0 & \text{otherwise.}\end{cases}$$

Let β denote the Milnor basis element $\mathrm{Sq}(t_1,\dots,t_{p_r})$ for the t_i's of Lemma 2.1. Then Lemma 2.1 and the preceding paragraph imply that $\chi\,\mathrm{Sq}^{n-\alpha(n)}(z)=\beta(z)$ on any class z with $|z|\le r$.

For typographical reasons, we let $P_i=2^{p_i}$, so $n=P_r+\cdots+P_1+P_0$ with $P_0=1$. A basis for $H^{\alpha(n)}(B_n)$ consists of all $x_{i_1}\cdots x_{i_{\alpha(n)}}$ with $1\le i_1<\cdots<i_{\alpha(n)}\le n$. Our class z is chosen to be the one with

$$i_j=\begin{cases}P_j+\cdots+P_1 & 1\le j\le r\\ n & j=\alpha(n)=r+1.\end{cases}\tag{2.2}$$

Recall that $\mathcal{P}^0_t$ denotes the Milnor basis element with only nonzero entry a 1 in position t, and that $\mathcal{P}^0_t(x)=x^{2^t}$ if $|x|=1$. Using this and the Cartan formula, $\chi\,\mathrm{Sq}^{n-\alpha(n)}(z)$ is the sum over all permutations of $\{P_0,\dots,P_r\}$ of products of $x_{i_1},\dots,x_{i_{\alpha(n)}}$ raised to these 2-power exponents. Here, i_j are as defined in (2.2). See (2.11) for a restatement involving notation defined in the interim.

The relations in $H^*(B_n)$ on classes x_i for $i<n$ imply that

$$x_i^e=\begin{cases}x_{i-e+1}\cdots x_i & e\le i\\ 0 & e>i.\end{cases}\tag{2.3}$$

Also, $x_n^{2^p}=(x_1+\cdots+x_{n-2})^{2^p-1}x_n$, as is easily proved by induction on p. We will use these facts repeatedly.

As a warmup, we consider the case $\alpha(n)=2$, so $n=P+1$ with $P=2^p$, $P\ge4$. Then $z=x_Px_n$, and

$$\chi\,\mathrm{Sq}^{n-\alpha(n)}(z)=x_P^Px_n+x_Px_n^P=x_1\cdots x_n+(x_1+\cdots+x_{n-2})^{P-1}x_{n-1}x_n.$$

Since $x_1\cdots x_n$ is the nonzero element of $H^n(B_n)$, the case $\alpha(n)=2$ of Theorem 1.4 follows from the following key result.

Theorem 2.4. *Let $m = 2^p - 1$ with $p > 1$. Let Q_m denote the ring $\mathbb{Z}_2[x_1, \ldots, x_m]$ with relations $x_i^2 = x_{i-1}x_i$ for $2 \le i \le m$, and $x_1^2 = 0$. Then*

$$(x_1 + \cdots + x_m)^m = 0.$$

We precede the proof of Theorem 2.4 with two useful results.

Lemma 2.5. *Let Q_m be as in Theorem 2.4.*

- *A degree-m monomial $x_1^{e_1} \cdots x_m^{e_m}$ equals 0 in Q_m iff for some d, $e_1 + \cdots + e_d > d$.*
- *If d is maximal with respect to this property, then $e_{d+1} = 0$. This is called a* gap *at x^{d+1}.*

Proof. We define excess numbers $\Delta_i = e_1 + \cdots + e_i - i$. The excess sequence $(\Delta_1, \ldots, \Delta_{m-1}, \Delta_m = 0)$ satisfies $\Delta_i \ge \Delta_{i-1} - 1$, as their difference is e_i. The effect of a relation is to increase Δ_{i-1} by 1 if $\Delta_i > \Delta_{i-1}$. The only way to create a positive Δ is if $\Delta_i = 0$ and $\Delta_{i+1} > 0$. Thus if the excess sequence starts with no positive Δ's, it can never have any. But such a sequence can be reduced to the 0-sequence since the negative entry with largest subscript can always be increased by 1. This shows that if a monomial has no d as in the lemma, then the monomial equals $x_1 \cdots x_m \ne 0$ in Q_m.

On the other hand, if there is a d as in the lemma, then the excess sequence has a positive entry. Let Δ_i be the first positive entry. Then we can increase Δ_{i-1} until it equals Δ_i. We can continue this until making $\Delta_1 > 0$, which says the monomial equals 0.

The second part of the lemma is clear, since having $e_{d+1} > 0$ would contradict maximality of d. ■

Corollary 2.6. *A degree-m monomial equals 0 in Q_m iff it can be decomposed as $A \cdot B$, where A has degree D but involves just x_i for $i < D$, while B is equivalent to $x_{D+1} \cdots x_m$ in Q_m.*

Proof. Here D is the $e_1 + \cdots + e_d$ of the lemma for maximal d, and $A = x_1^{e_1} \cdots x_d^{e_d}$. Then $B = x_{D+1}^{e_{D+1}} \cdots x_m^{e_m}$ has degree $m - D$. Maximality of d implies that the excess numbers of B are $\le -D$, and its last excess number equals $-D$. Consideration of the last excess number less than $-D$ shows that the excess sequence of B can be reduced to $(-D,\ldots,-D)$, implying the claim about B. ■

We illustrate with the monomial $x_2^3x_3x_6^2$ in Q_6. The maximal d of the lemma is 3, while D of the corollary is 4. We have $A=x_2^3x_3$, while $B=x_5^0x_6^2$ with initial excess sequence $(-5,-4)$.

Proof of Theorem 2.4. The polynomial $(x_1+\cdots+x_m)^m$ can be expanded as

$$\prod_{i=0}^{p-1}(x_{2^i}^{2^i}+\cdots+x_m^{2^i}), \tag{2.7}$$

using (2.3). The total number of monomials when this is expanded is $\prod_{i=0}^{p-1}(m+1-2^i)$, which is even since $p>1$. We will enumerate the number of these monomials which equal 0 in Q_m.

Those having a gap at x^D with $D=E_1+\cdots E_s$ for distinct 2-powers E_i have their A of Corollary 2.6 equal to any monomial in the expansion of

$$\prod_{i=1}^{s}(x_{E_i}^{E_i}+\cdots+x_{D-1}^{E_i}).$$

The number of such monomials is even, using the factor for $E_i=1$ if D is odd, and any factor if D is even. For any fixed D, each monomial A will occur multiplied by the same set of monomials B. Thus the number of monomials in the original expansion having a gap at x^D is even, and so the number of monomials with some gap is even, and this is the number which equal 0 in Q_m. Since the total number of monomials was even, we deduce that the number which are nonzero is even. Since every nonzero monomial equals $x_1\cdots x_m$ in Q_m, we deduce that the polynomial equals 0.

■

Definition 2.8. *We introduce some notation in $H^*(B_n)$.*

- $T_i=P_i+\cdots+P_1$ *for* $i\ge 1$ *and* $T_0=0$. *Here* $P_j=2^{p_j}$ *as in (2.2).*
- $S=x_1+\cdots+x_{n-2}$;
- $[i,j]=x_i\cdots x_j$ *and* $(i,j]=x_{i+1}\cdots x_j$;
- *If W is a homogeneous polynomial of degree d, then L_t (in L_tW) denotes any product of $n-d$ classes x_i (not necessarily distinct) with $i\le t$.*

Also, recall that $n = P_r + \cdots + P_1 + 1$.

Theorem 2.9. *The following elements of* $H^n(B_n)$ *equal* 0.

a. $L_{T_j - P_i - 1} \cdot (T_j - P_i, n] \cdot S^{P_j - 1}$, $1 \le i < j$;
b. $[1, T_{j-1}] \cdot [T_j, n] \cdot S^{P_j - 1}$, $j \ge 1$.

Proof. [a.] Let $D_j = T_j - P_i$. Write

$$S^{P_j - 1} = \prod_{k=0}^{p_j - 1} (x_1^{2^k} + \cdots + x_{n-2}^{2^k}). \tag{2.10}$$

For any k, and ℓ satisfying $D_j \le \ell \le n - 2$,

$$(x_{D_j+1} \cdots x_n) \cdot x_\ell^{2^k} = x_{D_j+1-2^k} \cdots x_n,$$

independent of ℓ. Since the number of values of ℓ is even,

$$(x_{D_j+1} \cdots x_n) \cdot (x_{D_j}^{2^k} + \cdots + x_{n-2}^{2^k}) = 0.$$

We are left with

$$L_{D_j - 1} \cdot (x_{D_j+1} \cdots x_n) \cdot \prod_{k=0}^{p_j - 1} (x_1^{2^k} + \cdots + x_{D_j - 1}^{2^k}).$$

This is 0 because it has a gap at x_{D_j}. The part complementary to the middle factor has degree D_j, which is greater than its largest subscript.

[b.] The argument begins by reducing S to $(x_1 + \cdots + x_{T_j - 1})$, similarly to part (a). ⟦Write $S^{P_j - 1}$ as in (2.10). The even number of terms $x_\ell^{2^k}$ in each factor with $T_j \le \ell \le n - 2$ can be ignored in the presence of $x_{T_j} \cdots x_n$.⟧ Now we have

$$(x_1 \cdots x_{T_{j-1}}) \cdot (x_{T_j} \cdots x_n) \cdot (x_1 + \cdots + x_{T_j - 1})^{P_j - 1}.$$

Monomials in $(x_1 + \cdots + x_{T_j - 1})^{P_j - 1}$ containing any factor x_i with $i \le T_{j-1}$ yield 0 upon multiplication by $x_1 \cdots x_{T_{j-1}}$. So now we have

$$(x_1 \cdots x_{T_{j-1}}) \cdot (x_{T_j} \cdots x_n) \cdot (x_{T_{j-1}+1} + \cdots + x_{T_j - 1})^{P_j - 1}.$$

Because $x_{T_{j-1}+1}^2 = 0$ in the presence of $x_1 \cdots x_{T_{j-1}}$, we see that $x_{T_{j-1}+1}$ is acting here like x_1 does in Q_m. The uniformity of the other relations implies that we can effectively reduce the subscripts in $(x_{T_{j-1}+1} + \cdots + x_{P_j + T_{j-1} - 1})^{P_j - 1}$ by T_{j-1}, so that it behaves like $(x_1 + \cdots + x_{P_j - 1})^{P_j - 1}$, which equals 0 by Theorem 2.4.

■

Now we prove Theorem 1.4. There are 2^r bijections

$$\{1,\ldots,r+1\}\xrightarrow{\sigma}\{0,\ldots,r\}$$

satisfying $\sigma(i)\le i$ for all i. $[\![\sigma(1)\in\{0,1\}$, $\sigma(i)\in\{0,\ldots,i\}-\{\sigma(1),\ldots,\sigma(i-1)\}$, and then $\sigma(r+1)$ is uniquely determined.$]\!]$ Then

$$\chi\,\mathrm{Sq}^{n-\alpha(n)}(x_{T_1}\cdots x_{T_r}x_n)=\sum_\sigma\biggl(\Bigl(\prod_{i=1}^r x_{T_i}^{P_{\sigma(i)}}\Bigr)x_n^{P_{\sigma(r+1)}}\biggr). \tag{2.11}$$

We divide the 2^r terms of (2.11) into four types, determined by $\sigma^{-1}(r-1)$ and $\sigma^{-1}(r)$. The first type is when $\sigma(r)=r-1$ and $\sigma(r+1)=r$. Then this summand of (2.11) equals $\cdots x_{T_{r-1}}^{P_i}x_{T_r}^{P_{r-1}}x_n^{P_r}$, with $i<r-1$. In the notation of Theorem 2.9, this equals $L_{T_{r-1}}\cdot(T_r-P_{r-1},n]\cdot S^{P_r-1}$, which equals 0 by Theorem 2.9[a], since $T_{r-1}\le T_r-P_{r-1}-1$.

The second type is very similar: $\sigma(r)=r$ and $\sigma(r+1)=r-1$. If $\sigma(r-1)=i$ with $i<r-1$, then the summand of (2.11) can be written as $L_{T_{r-2}}\cdot(T_{r-1}-P_i,n]\cdot S^{P_{r-1}-1}$, which equals 0 by Theorem 2.9[a]. Here we have used that $x_{T_{r-1}}^{P_i}x_{T_r}^{P_r}=(T_{r-1}-P_i,n-1]$.

The third type has $\sigma(r-1)=r-1$ and $\sigma(r+1)=r$. If $\sigma(r)=i\ne0$, then we have $L_{T_{r-1}}\cdot(T_r-P_i,n]\cdot S^{P_r-1}$, so again we get 0 by part [a]. If $\sigma(r)=0$, then $\sigma(i)=i$ for $1\le i\le r-1$, and our expression is of the form $[1,T_{r-1}]\cdot[n-1,n]\cdot S^{P_r-1}$ since $T_r=n-1$. This equals 0 by part [b] of Theorem 2.9.

The fourth type has $\sigma(r-1)=r-1$ and $\sigma(r)=r$. Let $\sigma(i)=i$ for $k\le i\le r$ and, if $k>1$, $\sigma(k-1)<k-1$. Then $\sigma(r+1)=k-1$. If $\sigma(k-1)=j>0$, then our expression equals $L_{T_{k-2}}\cdot(T_{k-1}-P_j,n]\cdot S^{P_{k-1}-1}$, which equals 0 by 2.9[a]. If $\sigma(k-1)=0$, then the expression equals $[1,T_{k-2}]\cdot[T_{k-1},n]\cdot S^{P_{k-1}-1}$, which is 0 by part [b]. If $k=1$, then $\sigma(r+1)=0$ and the expression equals $x_1\cdots x_n$, which is the only nonzero summand of (2.11).

3. Other manifolds

In this section, we discuss our attempts to find orientable n-manifolds with $n\equiv 0$ mod 4 and not a 2-power which have $\overline{w}_{n-\widehat{\alpha}(n)}\ne0$.

First we discuss the real Bott manifold B_{12} with relations as in Theorem 1.4. One way of showing $\overline{w}_9(B_{12})=0$ is by the method of Section 2. We verified that $\chi\,\mathrm{Sq}^9(x_ix_{11}x_{12})=0$ for $1\le i\le 10$, which are the only classes needing consideration.

Another way is computer dependent. By [7],

$$w_i(B_{12}) = \sigma_i(x_1, \dots, x_{10}, x_1 + \cdots + x_{10}),$$

where σ_i denotes the elementary symmetric polynomial. We use `Maple` to compute this for $i = 2, 3, 5$, and 9, and then to reduce each to a sum of i-fold products of distinct x_j's using the relations $x_j^2 = x_{j-1}x_j$. Then we use that $\overline{w}_9 = w_9 + w_2^2 w_5 + w_3^3$ for orientable manifolds, and use the relations again to obtain $\overline{w}_9 = 0$.

We tried several other 12-by-12 matrices A, such as the one whose only nonzero entries are $a_{i,i+1} = a_{i,i+2} = 1$ for $1 \le i \le 10$, and always obtained $\overline{w}_9 = 0$.

We also looked at generalized Dold manifolds, but the values of n for which we found examples of the desired type formed a small subset of those in Theorem 1.4 and Corollary 1.5.

Definition 3.1. *The generalized Dold manifold $P(n; m_1, \dots, m_r)$ is defined to be*

$$S^n \times_T (CP^{m_1} \times \cdots \times CP^{m_r}),$$

where the involution T is antipodal on S^n and conjugation on the complex projective spaces.

Dold manifolds have been studied by many people, but our source for generalized Dold manifolds is [13], where the following result is proved.

Theorem 3.2. ([13, Corollary 5.4]) *Let c and d_i satisfy $|c| = 1$ and $|d_i| = 2$. Then*

$$H^*(P(n; m_1, \dots, m_r); \mathbb{Z}_2) = \mathbb{Z}_2[c, d_1, \dots, d_r]/(c^{n+1}, d_1^{m_1+1}, \dots, d_r^{m_r+1}).$$

The total Stiefel-Whitney class is

$$w(P(n; m_1, \dots, m_r)) = (1 + c)^{n+1-r} \prod (1 + c + d_i)^{m_i+1}.$$

Our first example, an ordinary Dold manifold, was already noted in [10]. We omit our proof.

Theorem 3.3. *For $n = 2^e + 1$, $P(2^e - 3; 2)$ is an orientable n-manifold with $\overline{w}_{n-\widehat{\alpha}(n)} \neq 0$.*

Theorem 3.4. *If $n = 3 + 2^e + 2^{f+1}$ or $3 + 2^e + 2^{f+1} + 2^{g+1}$ with $2 \le e \le f < g$, then $P(2^e - 1; 2, 2^f)$ and $P(2^e - 1; 2, 2^f, 2^g)$, respectively, are orientable n-manifolds with $\overline{w}_{n-\widehat{\alpha}(n)} \neq 0$.*

Proof. We sketch the proof in the first case. The second case is proved similarly.

We have

$$w=(1+c)^{2^e-2}(1+c+d_1)^3(1+c+d_2)^{2^f+1}.$$

The manifold is orientable. We will show that the coefficient of $c^{2^e-2}d_1^1 d_2^{2^f-1}$, a class of grading $2^e-2+2^{f+1}=n-\widehat{\alpha}(n)$ in $\overline{w}$, is odd.

The dual class $\overline{w}$ can be written as

$$(1+c)^{2^L-2^{e+1}+2^e+2}(1+c+d_1)^{2^L-4}(1+c+d_1)(1+c+d_2)^{2^L-2^{f+1}}(1+c+d_2)^{2^f-1},$$

for L sufficiently large. The $d_2^{2^f-1}$ must come from the last factor, and d_1^1 from the third. The d_1 and d_2 can now be omitted from the other factors, whose product becomes $(1+c)^p$ with $p=3\cdot 2^L-2^{f+1}-2^{e+1}+2^e-2$, and $\binom{p}{2^e-2}$ is odd.

■

This procedure will not work for $n=3+2^e+2^{f+1}+2^{g+1}+2^{h+1}$ because the factor $(1+c)^{2^e-4}$ in w leads to a $(1+c)^{2^e+4}$ in $\overline{w}$, which when combined with $(1+c+d_1)^{2^L-4}$ yields $(1+c)^{2^L+2^e}$, which equals 1 on RP^{2^e-1}. A `Maple` check has led us to feel that there are no other values of n for which there is an n-dimensional orientable generalized Dold manifold with $\overline{w}_{n-\widehat{\alpha}(n)}\neq 0$.

In [1] and [2], dual Stiefel-Whitney classes of some non-orientable manifolds are obtained.

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