THE TOPOLOGICAL COMPLEXITY OF THE KLEIN BOTTLE

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Abstract. We use obstruction theory to determine the topological complexity of the Klein bottle. The same result was obtained by Cohen and Vandembroucq several weeks earlier, but our method differs substantially from theirs.

1. Introduction

The topological complexity, $\mathrm{TC}(X)$, of a topological space X, as introduced in [6], is roughly the number of rules required to tell how to move between any two points of X. We use obstruction theory to prove the following new result.

Theorem 1.1. *The topological complexity of the Klein bottle K equals 5.*

A month prior to this work, it was known ([3])[1] that, with N_g denoting the non-orientable compact surface of genus g,

$$\mathrm{TC}(N_g)=\begin{cases}5 & \text{if } g\ge 4\\ 4 \text{ or } 5 & \text{if } g=2 \text{ or } 3\\ 4 & \text{if } g=1.\end{cases}$$

The Klein bottle is, of course, the non-orientable surface N_2 of genus 2. Our result here was obtained several weeks earlier by Cohen and Vandembroucq in [1], and they also settled $\mathrm{TC}(N_3)$. Days prior to their posting, we had posted a version of this paper, but they pointed out a mistake in our argument. We have corrected that mistake here. There is similarity in our approaches in that both approaches use the same result of Costa and Farber ([2]). However, whereas our paper deals directly with the simplicial structure of K, theirs deals with the bar resolution. The calculations

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[1]The paper [3] discussed reduced TC, which is 1 less than the concept with which we deal.

to show the obstruction nonzero are quite different. A superficial difference is that [1] prefers to deal with reduced TC, while we prefer to use the original definition.

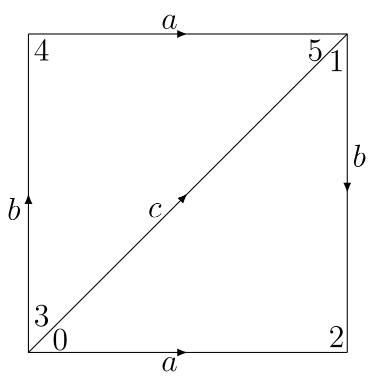
The Klein bottle is homeomorphic to the space of all configurations of various physical systems. For example, it is homeomorphic to the space of all planar 5-gons with side lengths 1, 1, 2, 2, and 3.([8, Table B]) Such a polygon can be considered as linked robot arms. Our result implies that five rules are required to program these arms to move from any configuration to any other.

Our main tool is a result of Costa and Farber ([2]), which we state later as Theorem 4.1, which describes a single obstruction in $H^{2n}(X \times X; G)$, where G is a certain local coefficient system, for an n-dimensional cell complex X to satisfy $\mathrm{TC}(X) \le 2n$. We prove that this class is nonzero for the Klein bottle K, and hence $\mathrm{TC}(K) > 4$.

2. The Δ-complex for $K \times K$

A Δ-complex, as described in [7], is essentially a quotient of a simplicial complex, with certain simplices identified. As noted there, this notion is equivalent to that of semi-simplicial complex introduced in [4]. It is important that vertices be numbered prior to identifications, and that simplices be described by writing vertices in increasing order.

The Δ-complex that we will use for K is given below. It has one vertex v, three edges, $(0,2) = (4,5)$, $(1,2) = (3,4)$, and $(0,1) = (3,5)$, and two 2-cells, $(0,1,2)$ and $(3,4,5)$.



If K and L are simplicial complexes with an ordering of the vertices of each, then the simplices of the simplicial complex $K \times L$ are all $\langle (v_{i_0}, w_{j_0}), \ldots, (v_{i_k}, w_{j_k}) \rangle$ such that $i_0 \le \cdots \le i_k$ and $j_0 \le \cdots \le j_k$ and $\{v_{i_0}, \ldots, v_{i_k}\}$ and $\{w_{j_0}, \ldots, w_{j_k}\}$ are simplices of

K and L, respectively. Note that we may have $v_{i_t} = v_{i_{t+1}}$ or $w_{j_t} = w_{j_{t+1}}$, but not both (for the same t). Now, if K and L are Δ-complexes, i.e., they have some simplices identified, then $K \times L$ has

$$\langle (v_{i_0}, w_{j_0}), \ldots, (v_{i_k}, w_{j_k}) \rangle \sim \langle (v_{i'_0}, w_{j'_0}), \ldots, (v_{i'_k}, w_{j'_k}) \rangle$$

iff $\{v_{i_0}, \ldots, v_{i_k}\} \sim \{v_{i'_0}, \ldots, v_{i'_k}\}$ and $\{w_{j_0} \ldots w_{j_k}\} \sim \{w_{j'_0} \ldots w_{j'_k}\}$, and the positions of the repetitions in $(v_{i_0}, \ldots, v_{i_k})$ and $(v_{i'_0}, \ldots, v_{i'_k})$ are the same, and so are those of $(w_{j_0}, \ldots, w_{j_k})$ and $(w_{j'_0}, \ldots, w_{j'_k})$. This description is equivalent to the one near the end of [5], called $K \Delta L$ there. There is also a discussion in [7, pp.277-278].

Following this description, we now list the simplices of $K \times K$, where K is the above Δ-complex. We write v for the unique vertex when it is being producted with a simplex, but otherwise we list all vertices by their number. We omit commas in ordered pairs; e.g., 24 denotes the vertex of $K \times K$ which is vertex 2 in the first factor and vertex 4 in the second factor. There are 1, 15, 50, 60, and 24 distinct simplices of dimensions 0, 1, 2, 3, and 4, respectively. This is good since $1-15+50-60+24 = 0$, the Euler characteristic of $K \times K$. We number the simplices in each dimension, which will be useful later.

The only 0-simplex is (v, v).

1-simplices:
1: $(0v, 1v) = (3v, 5v)$.
2: $(0v, 2v) = (4v, 5v)$.
3: $(1v, 2v) = (3v, 4v)$.
4: $(v0, v1) = (v3, v5)$.
5: $(v0, v2) = (v4, v5)$.
6: $(v1, v2) = (v3, v4)$.
7: $(00, 11) = (30, 51) = (33, 55) = (03, 15)$.
8: $(00, 12) = (30, 52) = (34, 55) = (04, 15)$.
9: $(01, 12) = (31, 52) = (33, 54) = (03, 14)$.
10: $(00, 21) = (40, 51) = (43, 55) = (03, 25)$.
11: $(00, 22) = (40, 52) = (44, 55) = (04, 25)$.
12: $(01, 22) = (41, 52) = (43, 54) = (03, 24)$.
13: $(10, 21) = (30, 41) = (33, 45) = (13, 25)$.

14: $(10, 22) = (30, 42) = (34, 45) = (14, 25)$.

15: $(11, 22) = (31, 42) = (33, 44) = (13, 24)$.

2-simplices

1: $(0v, 1v, 2v)$.

2: $(3v, 4v, 5v)$.

3: $(v0, v1, v2)$.

4: $(v3, v4, v5)$.

5: $(00, 10, 11) = (30, 50, 51) = (03, 13, 15) = (33, 53, 55)$.

6: $(00, 10, 12) = (30, 50, 52) = (04, 14, 15) = (34, 54, 55)$.

7: $(01, 11, 12) = (31, 51, 52) = (03, 13, 14) = (33, 53, 54)$.

8: $(00, 01, 11) = (30, 31, 51) = (03, 05, 15) = (33, 35, 55)$.

9: $(00, 02, 12) = (30, 32, 52) = (04, 05, 15) = (34, 35, 55)$.

10: $(01, 02, 12) = (31, 32, 52) = (03, 04, 14) = (33, 34, 54)$.

11: $(00, 20, 21) = (40, 50, 51) = (03, 23, 25) = (43, 53, 55)$.

12: $(00, 20, 22) = (40, 50, 52) = (04, 24, 25) = (44, 54, 55)$.

13: $(01, 21, 22) = (41, 51, 52) = (03, 23, 24) = (43, 53, 54)$.

14: $(00, 01, 21) = (40, 41, 51) = (03, 05, 25) = (43, 45, 55)$.

15: $(00, 02, 22) = (40, 42, 52) = (04, 05, 25) = (44, 45, 55)$.

16: $(01, 02, 22) = (41, 42, 52) = (03, 04, 24) = (43, 44, 54)$.

17: $(10, 20, 21) = (30, 40, 41) = (13, 23, 25) = (33, 43, 45)$.

18: $(10, 20, 22) = (30, 40, 42) = (14, 24, 25) = (34, 44, 45)$.

19: $(11, 21, 22) = (31, 41, 42) = (13, 23, 24) = (33, 43, 44)$.

20: $(10, 11, 21) = (30, 31, 41) = (13, 15, 25) = (33, 35, 45)$.

21: $(10, 12, 22) = (30, 32, 42) = (14, 15, 25) = (34, 35, 45)$.

22: $(11, 12, 22) = (31, 32, 42) = (13, 14, 24) = (33, 34, 44)$.

23: $(00, 11, 22)$.

24: $(30, 41, 52)$.

25: $(03, 14, 25)$.

26: $(33, 44, 55)$.

27: $(00, 01, 22) = (40, 41, 52)$.

28: $(30, 31, 52) = (00, 01, 12)$.

29: $(03, 04, 25) = (43, 44, 55)$.

30: $(33, 34, 55) = (03, 04, 15)$.

31: $(00, 10, 22) = (04, 14, 25)$.

32: $(30, 40, 52) = (34, 44, 55)$.

33: $(03, 13, 25) = (00, 10, 21)$.

34: $(33, 43, 55) = (30, 40, 51)$.

35: $(00, 21, 22) = (40, 51, 52)$.

36: $(03, 24, 25) = (43, 54, 55)$.

37: $(30, 51, 52) = (00, 11, 12)$.

38: $(33, 54, 55) = (03, 14, 15)$.

39: $(00, 12, 22) = (04, 15, 25)$.

40: $(30, 42, 52) = (34, 45, 55)$.

41: $(03, 15, 25) = (00, 11, 21)$.

42: $(33, 45, 55) = (30, 41, 51)$.

43: $(01, 12, 22) = (03, 14, 24)$.

44: $(31, 42, 52) = (33, 44, 54)$.

45: $(03, 13, 24) = (01, 11, 22)$.

46: $(33, 43, 54) = (31, 41, 52)$.

47: $(10, 21, 22) = (30, 41, 42)$.

48: $(13, 24, 25) = (33, 44, 45)$.

49: $(10, 11, 22) = (30, 31, 42)$.

50: $(13, 14, 25) = (33, 34, 45)$.

3-simplices

1: $(00, 10, 21, 22)$

2: $(00, 11, 21, 22)$.

3: $(00, 10, 11, 22)$.

4: $(00, 11, 12, 22)$.

5: $(00, 01, 11, 22)$.

6: $(00, 01, 12, 22)$.

7: $(30, 40, 51, 52)$.

8: $(30, 41, 51, 52)$.

9: $(30, 40, 41, 52)$.

10: $(30, 41, 42, 52)$.

11: $(30, 31, 41, 52)$.

12: $(30, 31, 42, 52)$.

13: $(03, 13, 24, 25)$.

14: $(03, 14, 24, 25)$.

15: $(03, 13, 14, 25)$.

16: $(03, 14, 15, 25)$.

17: $(03, 04, 14, 25)$.

18: $(03, 04, 15, 25)$.

19: $(33, 43, 54, 55)$.

20: $(33, 44, 54, 55)$.

21: $(33, 43, 44, 55)$.

22: $(33, 44, 45, 55)$.

23: $(33, 34, 44, 55)$.

24: $(33, 34, 45, 55)$.

25: $(00, 10, 20, 21) = (03, 13, 23, 25)$.

26: $(00, 10, 11, 21) = (03, 13, 15, 25)$.

27: $(00, 01, 11, 21) = (03, 05, 15, 25)$.

28: $(00, 10, 20, 22) = (04, 14, 24, 25)$.

29: $(00, 10, 12, 22) = (04, 14, 15, 25)$.

30: $(00, 02, 12, 22) = (04, 05, 15, 25)$.

31: $(01, 11, 21, 22) = (03, 13, 23, 24)$.

32: $(01, 11, 12, 22) = (03, 13, 14, 24)$.

33: $(01, 02, 12, 22) = (03, 04, 14, 24)$.

34: $(00, 01, 02, 12) = (30, 31, 32, 52)$.

35: $(00, 01, 11, 12) = (30, 31, 51, 52)$.

36: $(00, 10, 11, 12) = (30, 50, 51, 52)$.

37: $(00, 01, 02, 22) = (40, 41, 42, 52)$.

38: $(00, 01, 21, 22) = (40, 41, 51, 52)$.

39: $(00, 20, 21, 22) = (40, 50, 51, 52)$.

40: $(10, 11, 12, 22) = (30, 31, 32, 42)$.

41: $(10, 11, 21, 22) = (30, 31, 41, 42)$.

42: $(10, 20, 21, 22) = (30, 40, 41, 42)$.

43: $(30, 40, 50, 51) = (33, 43, 53, 55)$.

44: $(30, 40, 41, 51) = (33, 43, 45, 55)$.

45: $(30, 31, 41, 51) = (33, 35, 45, 55)$.

46: $(30, 40, 50, 52) = (34, 44, 54, 55)$.

47: $(30, 40, 42, 52) = (34, 44, 45, 55)$.

48: $(30, 32, 42, 52) = (34, 35, 45, 55)$.

49: $(31, 41, 51, 52) = (33, 43, 53, 54)$.

50: $(31, 41, 42, 52) = (33, 43, 44, 54)$.

51: $(31, 32, 42, 52) = (33, 34, 44, 54)$.

52: $(03, 04, 05, 15) = (33, 34, 35, 55)$.

53: $(03, 04, 14, 15) = (33, 34, 54, 55)$.

54: $(03, 13, 14, 15) = (33, 53, 54, 55)$.

55: $(03, 04, 05, 25) = (43, 44, 45, 55)$.

56: $(03, 04, 24, 25) = (43, 44, 54, 55)$.

57: $(03, 23, 24, 25) = (43, 53, 54, 55)$.

58: $(13, 14, 15, 25) = (33, 34, 35, 45)$.

59: $(13, 14, 24, 25) = (33, 34, 44, 45)$.

60: $(13, 23, 24, 25) = (33, 43, 44, 45)$.

4-simplices

1: $(00, 10, 20, 21, 22)$.

2: $(00, 10, 11, 21, 22)$.

3: $(00, 10, 11, 12, 22)$.

4: $(00, 01, 11, 21, 22)$.

5: $(00, 01, 11, 12, 22)$.

6: $(00, 01, 02, 12, 22)$.

7: $(30, 40, 50, 51, 52)$.

8: $(30, 40, 41, 51, 52)$.

9: $(30, 40, 41, 42, 52)$.

10: $(30, 31, 41, 51, 52)$.

11: $(30, 31, 41, 42, 52)$.

12: $(30, 31, 32, 42, 52)$.

13: $(03, 13, 23, 24, 25)$.

14: $(03, 13, 14, 24, 25)$.
15: $(03, 13, 14, 15, 25)$.
16: $(03, 04, 14, 24, 25)$.
17: $(03, 04, 14, 15, 25)$.
18: $(03, 04, 05, 15, 25)$.
19: $(33, 43, 53, 54, 55)$.
20: $(33, 43, 44, 54, 55)$.
21: $(33, 43, 44, 45, 55)$.
22: $(33, 34, 44, 54, 55)$.
23: $(33, 34, 44, 45, 55)$.
24: $(33, 34, 35, 45, 55)$.

3. $H^4(K \times K)$ with local coefficients

We will need to show that a certain class is nonzero in $H^4(K \times K)$ with coefficients in a certain local coefficient system. In this section, we describe the relations in $H^4(K \times K; G)$ for an arbitrary free abelian local coefficient system G.

For a Δ-complex X with a single vertex x_0, such as the one just described for $K \times K$, a local coefficient system G is an abelian group G together with an action of the group ring $\mathbb{Z}[\pi_1(X; x_0)]$ on G. If C_k denotes the free abelian group generated by the k-cells of X, homomorphisms

$$\delta_{k-1} : \operatorname{Hom}(C_{k-1}, G) \to \operatorname{Hom}(C_k, G)$$

are defined by

$$\delta_{k-1}(\phi)(\langle v_{i_0}, \dots, v_{i_k}\rangle) = \rho_{i_0,i_1} \cdot \phi(\langle v_{i_1}, \dots, v_{i_k}\rangle) + \sum_{j=1}^{k} (-1)^j \phi(\langle v_{i_0}, \dots, \widehat{v_{i_j}}, \dots, v_{i_k}\rangle), \tag{3.1}$$

where $\widehat{v_{i_j}}$ denotes omission of that vertex, and ρ_{i_0,i_1} is the element of $\pi_1(X; x_0)$ corresponding to the edge from v_{i_0} to v_{i_1}. Then

$$H^k(X; G) = \ker(\delta_k)/\operatorname{im}(\delta_{k-1}).$$

This description is given in [4, p.501].

We have

$$\pi_1(K; v) = \langle a, b, c\rangle/(c = ab^{-1} = ba)$$

and

$\pi_1(K \times K; (v, v)) = \langle a, b, c, a', b', c' \rangle / (c = ab^{-1} = ba,\ c' = a'b'^{-1} = b'a'),$

where the primes correspond to the second factor. We will prove the following key result, in which $\gamma_1, \dots \gamma_{24}$ denote the generators corresponding to the 4-cells of $K \times K$ listed at the end of the previous section, and G is any free abelian local coefficient system.

Theorem 3.2. *Let*

$$\varepsilon_j = \begin{cases} -1 & j \equiv 2 \mod 3 \\ 1 & j \equiv 0, 1 \mod 3, \end{cases}$$

and let $\psi \in \mathrm{Hom}(C_4, G)$. *Then* $[\psi] = 0 \in H^4(K \times K; G)$ *if and only if* $\sum_{j=1}^{24} \varepsilon_j \psi(\gamma_j)$ *is 0 in the quotient of* G *modulo the action of* $b - 1$, $b' - 1$, $c + 1$, *and* $c' + 1$.

We find the image of $\delta : \mathrm{Hom}(C_3, G) \to \mathrm{Hom}(C_4, G)$ by row-reducing the 60-by-24 matrix M whose entries $m_{i,j}$ satisfy

$$\delta(\phi)(\gamma_j) = \sum_{i=1}^{60} m_{i,j} \phi(\beta_i), \quad j = 1, \dots, 24. \tag{3.3}$$

Here β_i denotes the generator of C_3 corresponding to the ith 3-cell.

We will show that the row-reduced form of this matrix has 27 nonzero rows, with its only nonzero elements in rows $i = 1, \dots, 23$ being 1 in position (i, i) and $-\varepsilon_i$ in position $(i, 24)$, while in rows 24 through 27 the only nonzero element is in column 24, and equals $b - 1$, $b' - 1$, $c + 1$, and $c' + 1$, respectively.

A row $(r_1, \dots, r_{24})$, with $r_j = \psi(\gamma_j)$ for $\psi \in \mathrm{Hom}(C_4, G)$, is equivalent, modulo the first 23 rows just described, to a row with 0's in the first 23 positions, and $\sum_{j=1}^{24} \varepsilon_j r_j$ in the 24th column. The last four rows of the reduced matrix yield the claim of Theorem 3.2.

Now we prove the claim about the row-reduction of M. We list the matrix M below. Each row has two nonzero entries, as each 3-cell is a face of two 4-cells, while each column has five nonzero entries, as each 4-cell is bounded by five 3-cells. For example, $m_{42,1} = c$ because $(10, 20, 21, 22)$ is obtained from $(00, 10, 20, 21, 22)$ by omission of the initial vertex, and $\rho_{00,10} = c$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	-1	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	-1	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	-1	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	-1	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	-1	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	-1	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	-1	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	-1	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	0
24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
25	1	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0
26	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
27	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0
28	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b'	0	0	0	0	0	0	0	0
29	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	b'	0	0	0	0	0	0	0
30	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	b'	0	0	0	0	0	0
31	0	0	0	c'	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
32	0	0	0	0	c'	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	c'	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
34	0	0	0	0	0	1	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	-1	0	0	b	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	1	0	0	0	b	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
39	-1	0	0	0	0	0	b	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
40	0	0	c	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
41	0	c	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
42	c	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
43	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0
44	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
45	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	-1
46	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b'	0	0
47	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	b'	0
48	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	b'
49	0	0	0	0	0	0	0	0	0	c'	0	0	0	0	0	0	0	0	1	0	0	0	0	0
50	0	0	0	0	0	0	0	0	0	0	c'	0	0	0	0	0	0	0	0	1	0	0	0	0
51	0	0	0	0	0	0	0	0	0	0	0	c'	0	0	0	0	0	0	0	0	0	1	0	0
52	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	-1
53	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0
54	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	-1	0	0	0	0	0
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	b	0	0	0
56	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	b	0	0	0	0
57	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	0	0	0	b	0	0	0	0	0
58	0	0	0	0	0	0	0	0	0	0	0	0	0	0	c	0	0	0	0	0	0	0	0	1
59	0	0	0	0	0	0	0	0	0	0	0	0	0	c	0	0	0	0	0	0	0	0	1	0
60	0	0	0	0	0	0	0	0	0	0	0	0	c	0	0	0	0	0	0	0	1	0	0	0

That the row reduced form of this matrix is as claimed above is easily verified. It can be done by hand in less than 30 minutes.

4. Our specific obstruction class

In [2], the following theorem is proved.

Theorem 4.1. *Let X be an n-dimensional Δ-complex with a single vertex x_0, and let $\pi = \pi_1(X, x_0)$. Let $I \subset \mathbb{Z}[\pi]$ denote the augmentation ideal. The action of $\pi \times \pi$ on I by $(g,h) \cdot \alpha = g\alpha h^{-1}$ extends to an action of $\mathbb{Z}[\pi \times \pi]$, defining a local coefficient system I over $X \times X$. If $C_1(X \times X)$ denotes the free abelian group on the set of edges of $X \times X$, then the homomorphism $f : C_1(X \times X) \to I$ defined by $(e_1, e_2) \mapsto [e_1][e_2]^{-1} - 1$ defines an element $\nu \in H^1(X \times X; I)$. Then* $\mathrm{TC}(X) \leq 2n$ *iff $\nu^{2n} = 0 \in H^{2n}(X \times X; I^{\otimes 2n})$, where $\pi \times \pi$ acts diagonally on $I^{\otimes 2n}$.*

The discussion of the function f of this theorem in [2] refers to [9, Ch.6:Thm 3.3], and we use the proof of that result for our interpretation of the function.

When $X = K$ is the Klein bottle, we have $\pi = \pi_1(K) = \{a^m b^n\}$ with the multiplication of these elements determined by the relation $ab^{-1} = ba$. Also relevant for us is the element $c = ab^{-1} = ba$. The following lemma will be useful.

Lemma 4.2. *In $\pi_1(K)$, for any integers m and n,*

$$\begin{aligned} ba^m b^n &= a^m b^{n+(-1)^m} \\ ab^{-1}a^m b^n &= a^{m+1} b^{n+(-1)^{m+1}} \\ a^m b^n a^{-1} b^{-1} &= a^{m-1} b^{-n-1}. \end{aligned}$$

Proof. These follow from $ba^m = a^m b^{(-1)^m}$, $b^{-1}a^m = a^m b^{(-1)^{m+1}}$, and $ab^n = b^{-n}a$, each of which is easily proved by induction. ■

The ideal I for us is the free abelian group with basis $\{\alpha_{m,n} = a^m b^n - 1 : (m,n) \in \mathbb{Z} \times \mathbb{Z} - \{(0,0)\}\}$. Using the numbering of the 1-cells of $K \times K$ given in Section 2, we obtain that the function f is given as in the following table.

$$\begin{array}{c|cccccccc|}
i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
f(e_i) & \alpha_{1,-1} & \alpha_{1,0} & \alpha_{0,1} & \alpha_{-1,-1} & \alpha_{-1,0} & \alpha_{0,-1} & 0 & \alpha_{0,1} \\
\hline \hline
i & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \\
f(e_i) & \alpha_{1,-2} & \alpha_{0,-1} & 0 & \alpha_{1,-1} & \alpha_{-1,-2} & \alpha_{-1,-1} & 0 & \\
\hline
\end{array} \tag{4.3}$$

For example, $f(e_1) = \alpha_{1,-1}$ because $c = ab^{-1}$, while $f(e_{12}) = \alpha_{1,-1}$ because the edge from 0 to 2 is a, while that from 1 to 2 is b.

In [4, p.500], it is noted that the Alexander-Whitney formula for cup products in simplicial complexes,

$$(f^p \cup g^{n-p})\langle v_0, \ldots, v_n\rangle = (-1)^{p(n-p)} f(v_0, \ldots, v_p) \otimes g(v_p, \ldots, v_n),$$

applies also to Δ-complexes. We apply this to $f^4(\gamma_j)$, where $f^4 = f \cup f \cup f \cup f$ with f the function on 1-cells defined above, and γ_j is any of the 24 4-cells listed in Section 2. For example,

$$\begin{aligned}
f^4(\gamma_1) &= f(00,10) \otimes f(10,20) \otimes f(20,21) \otimes f(21,22) \\
&= f(e_1) \otimes f(e_3) \otimes f(e_4) \otimes f(e_6) \\
&= \alpha_{1,-1} \otimes \alpha_{0,1} \otimes \alpha_{-1,-1} \otimes \alpha_{0,-1}.
\end{aligned} \tag{4.4}$$

We note that for all of our 4-cells, consecutive vertices are constant in one factor, and so only $f(e_i)$ for $i \le 6$ will be relevant for f^4. In (4.5), we list $abcd$ such that $f^4(\gamma_j) = f(e_a) \otimes f(e_b) \otimes f(e_c) \otimes f(e_d)$, for each of the 24 4-cells γ_j.

j	$abcd$	j	$abcd$	j	$abcd$	j	$abcd$
1	1346	7	3246	13	1365	19	3265
2	1436	8	3426	14	1635	20	3625
3	1463	9	3462	15	1653	21	3652
4	4136	10	4326	16	6135	22	6325
5	4163	11	4362	17	6153	23	6352
6	4613	12	4632	18	6513	24	6532

(4.5)

Our class then is the cohomology class of the element $f^4 \in \mathrm{Hom}(C_4, I^{\otimes 4})$ defined using (4.5) and (4.3). For example, $f^4(\gamma_1)$ is as in (4.4).

5. Proof that our class is nonzero

We will apply Theorem 3.2 to $\psi = f^4$, where $f^4(\gamma_j)$ is obtained from (4.5) and (4.3). For example, the term $\varepsilon_2\psi(\gamma_2)$ in 3.2 is $-\alpha_{1,-1} \otimes \alpha_{-1,-1} \otimes \alpha_{0,1} \otimes \alpha_{0,-1}$.

Although the classes $\alpha_{m,n}$ are a convenient shorthand and a convenient basis for I, we prefer to work in $\mathbb{Z}[\pi]$, using the classes $a^m b^n$ as the basis. Using Lemma 4.2, the four relations in 3.2 say that $a^{m_1}b^{n_1} \otimes a^{m_2}b^{n_2} \otimes a^{m_3}b^{n_3} \otimes a^{m_4}b^{n_4}$ is equivalent to each of the following.

type 1 : $a^{m_1}b^{n_1+(-1)^{m_1}} \otimes a^{m_2}b^{n_2+(-1)^{m_2}} \otimes a^{m_3}b^{n_3+(-1)^{m_3}} \otimes a^{m_4}b^{n_4+(-1)^{m_4}}$

type 2 : $a^{m_1}b^{n_1-1} \otimes a^{m_2}b^{n_2-1} \otimes a^{m_3}b^{n_3-1} \otimes a^{m_4}b^{n_4-1}$

type 3 : $a^{m_1+1}b^{n_1+(-1)^{m_1+1}} \otimes a^{m_2+1}b^{n_2+(-1)^{m_2+1}} \otimes a^{m_3+1}b^{n_3+(-1)^{m_3+1}} \otimes a^{m_4+1}b^{n_4+(-1)^{m_4+1}}$

type 4 : $a^{m_1-1}b^{-n_1-1} \otimes a^{m_2-1}b^{-n_2-1} \otimes a^{m_3-1}b^{-n_3-1} \otimes a^{m_4-1}b^{-n_4-1}$.

For typographical reasons, we shall henceforth list just the exponents, in the order $(m_1, m_2, m_3, m_4, n_1, n_2, n_3, n_4)$. Iterating type 2 relations yields

$$(m_1, m_2, m_3, m_4, n_1, n_2, n_3, n_4) \sim (m_1, m_2, m_3, m_4, n_1{+}k, n_2{+}k, n_3{+}k, n_4{+}k)$$

for any integer k. Then iterating type 4 relations expands this to

$$(m_1, m_2, m_3, m_4, n_1, n_2, n_3, n_4) \sim (m_1{+}\ell, m_2{+}\ell, m_3{+}\ell, m_4{+}\ell, n_1{+}k, n_2{+}k, n_3{+}k, n_4{+}k)$$

for any integers k and ℓ. The effect of (iterated) types 1 and 3 is to allow also a "$+2h$" accompanying each $n_i + k$ for which $m_i \not\equiv m_1 \mod 2$. (Same h for each such i.) Thus, with R denoting the relations of Theorem 3.2, we obtain that $\mathbb{Z}[\pi]^{\otimes 4}/R$ has a basis in 1-1 correspondence with

$$B = \{\langle (d_2, d_3, d_4), S_1, \varepsilon, S_2 \rangle\},$$

where $d_i = m_i - m_1$, $S_1 = \langle n_i - n_1 \in \mathbb{Z} : d_i \equiv 0 \mod 2\rangle$, $\varepsilon = (n_{s_0} - n_1) \mod 2$, where $s_0 = \min\{i : d_i \equiv 1 \mod 2\}$, and $S_2 = \langle n_i - n_{s_0} \in \mathbb{Z} : i > s_0,\ d_i \equiv 1 \mod 2\rangle$. Note that some of S_1, ε, and S_2 can be empty, but each element of B will contain six numbers. Here are a few examples.

$$\begin{aligned} ab^{-1}\otimes b\otimes a^{-1}b^{-1}\otimes b^{-1} &\leftrightarrow \langle(-1,-2,-1),0,0,-2\rangle\\ a^0b^0\otimes b\otimes a^{-1}b^{-1}\otimes b^{-1} &\leftrightarrow \langle(0,-1,0),(1,-1),1,\emptyset\rangle\\ a^0b^0\otimes b\otimes a^0b^0\otimes b^{-1} &\leftrightarrow \langle(0,0,0),(1,0,-1),\emptyset,\emptyset\rangle\\ a^7b^4\otimes a^6b^3\otimes a^4b^4\otimes b^5 &\leftrightarrow \langle(-1,-3,-7),\emptyset,1,(1,2)\rangle. \end{aligned}$$

For the last of these, the monomials equivalent to it are $a^mb^n\otimes a^{m-1}b^{n+o}\otimes a^{m-2}b^{n+o+1}\otimes a^{m-7}b^{n+o+2}$ with o odd. It is essential to note that we have accounted for all relations in $\mathbb{Z}[\pi]^{\otimes 4}$ and hence in $I^{\otimes 4}$.

Our element $\sum^{24}\varepsilon_j f^4(\gamma_j)$ is in $I^{\otimes 4}\subset\mathbb{Z}[\pi]^{\otimes 4}$. It expands as a sum of $16\cdot 24$ classes of the form $a^{m_1}b^{n_1}\otimes a^{m_2}b^{n_2}\otimes a^{m_3}b^{n_3}\otimes a^{m_4}b^{n_4}$, each with coefficient ±1. For example, (4.4) has the 16 terms in the expansion of

$$(a^1b^{-1}-a^0b^0)\otimes(b-a^0b^0)\otimes(a^{-1}b^{-1}-a^0b^0)\otimes(b^{-1}-a^0b^0).$$

When the relations are imposed, elements of the basis B will often appear more than once among the 384 terms in our class, sometimes with signs that cancel, and sometimes with signs that combine. Here are two examples.

The basis element $\langle(0,0,0),(0,1,0),\emptyset,\emptyset\rangle$ can only occur in our sum with $m_1=m_2=m_3=m_4=0$ since there are not four distinct $i\le 6$ with $f(e_i)=\alpha_{1,n}$ or four with $f(e_i)=\alpha_{-1,n}$. It must have $(n_1,n_2,n_3,n_4)=(0,0,1,0)$, since $b^{-1}\otimes b^{-1}\otimes a^0b^0\otimes b^{-1}$ does not appear in our expansion. The term $a^0b^0\otimes a^0b^0\otimes b\otimes a^0b^0$ appears in the expansions of $-f^4(\gamma_2)$, $f^4(\gamma_4)$, $f^4(\gamma_{12})$, $-f^4(\gamma_{14})$, $f^4(\gamma_{16})$, and $f^4(\gamma_{24})$. These are the $f^4(\gamma_j)$'s with a 3 in the third position in (4.5). Each of these has an additional factor $(-1)^3$ because of the three a^0b^0's. Thus the coefficient of this term of B in our sum is -2.

The basis element $\langle(-1,-1,-1),\emptyset,0,(-1,-2)\rangle$ can occur in our class only with $m_1=1$ and $m_2=m_3=m_4=0$ since there are not three distinct $i\le 6$ with $f^4(e_i)=\alpha_{-1,n}$. It must have $(n_2,n_3,n_4)=(1,0,-1)$ and $n_1=\pm1$. Since $\alpha_{1,1}$ does not occur as $f^4(e_i)$ for $i\le 6$, this can only occur as $a^1b^{-1}\otimes b\otimes a^0b^0\otimes b^{-1}$ in $f^4(\gamma_1)$, where it gets a minus sign because of the a^0b^0. Thus its coefficient is -1. This is consistent with the argument in [1], which says that the obstruction is actually a mod 2 obstruction.

This analysis already is enough to show that our class has independent terms which are nonzero mod R, and hence our obstruction is nonzero, and our theorem is proved. There are many more such terms, and a `Maple` program was run to do all this combining and listed the many basis terms that appear in our $\sum \varepsilon_j f^4(\gamma_j)$ with nonzero coefficient after the relations are imposed, but this list is not necessary for our conclusion.

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