

# THE CONNECTIVE $K$ -THEORY OF THE EILENBERG-MACLANE SPACE $K(\mathbb{Z}_p, 2)$

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ABSTRACT. We compute  $ku^*(K(\mathbb{Z}_p, 2))$  and  $ku_*(K(\mathbb{Z}_p, 2))$ , the connective  $KU$ -cohomology and connective  $KU$ -homology groups of the mod- $p$  Eilenberg-MacLane space  $K(\mathbb{Z}_p, 2)$ , using the Adams spectral sequence. We obtain a striking interaction between  $h_0$ -extensions and exotic extensions. The mod- $p$  connective  $KU$ -cohomology groups, computed elsewhere, are needed in order to establish higher differentials and exotic extensions in the integral groups.

## 1. MAIN RESULTS

In [12] and [5], the authors initiated a partial computation of the connective  $KU$ -homology groups,  $ku_*(K(\mathbb{Z}_2, 2))$ , of the mod-2 Eilenberg-MacLane space  $K(\mathbb{Z}_2, 2)$  in separate studies of Stiefel-Whitney classes of manifolds. We eventually turned to the associated cohomology groups,  $ku^*(K(\mathbb{Z}_2, 2))$ , and were able to give a complete determination, via the Adams spectral sequence (ASS). This generalized nicely to the odd primes, and then we found a duality result ([4]) relating these homology and cohomology groups which enabled us to determine the homology groups  $ku_*(K(\mathbb{Z}_p, 2))$ .

Let  $K_2 = K(\mathbb{Z}_p, 2)$ , with the prime  $p$  implicit. We begin with a description of the  $ku^*$ -module  $ku^*(K_2)$ . Note that  $ku^* = \mathbb{Z}_{(p)}[v]$  with  $|v| = -2(p-1)$ . We find that depiction via ASS charts is the most insightful way to envision the groups. There is a very nice interplay between extensions (multiplication by  $p$ ) seen in Ext ( $h_0$ -extensions) and exotic extensions. We depict the ASS with cohomological (co)degrees increasing from right-to-left. We write  $|x| = d$  if  $x \in ku^d(K_2)$  or the associated  $E_2$ -term.

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In  $ku^*(K_2)$ , there is a trivial submodule whose Poincaré series when  $p = 2$  is described at the end of Section 2. It plays no role and **will be ignored from now on**. As a  $ku^*$ -module,  $ku^*(K_2)$  is generated by certain products of elements of  $E_2^0$ ,  $y_0$ ,  $y_i = y_0^{p^i}$ , with  $|y_i| = 2p^i$ ,  $z_j$  for  $j \geq 0$  with  $|z_j| = 2(p^{j+1} + 1)$ , and  $q$  with  $|q| = 9$  if  $p = 2$ , and  $|q| = 4p - 1$  if  $p$  is odd.

The even-graded part  $ku^{\text{ev}}(K_2)$  is formed from shifted copies of  $ku^*$ -modules  $A_k$  and  $B_k$ , which can be defined inductively as follows.

**Definition 1.1.** *Let  $k_0 = 1$  if  $p$  is odd, and  $k_0 = 2$  if  $p = 2$ . Let  $B_{k_0-1} = 0$ . Let  $A_0 = \langle z_0 \rangle$  for all  $p$ . Inductively*

*$B_k$  is built from  $z_{k-1}^{p-1}B_{k-1}$ ,  $TP_{p^k}[v]z_k$ , and  $y_{k-1}^{p-1}B_{k-1}$ , if  $k \geq k_0$*

*and*

*$A_k$  is built from  $z_{k-1}^{p-1}B_{k-1}$ ,  $TP_{p^k}[v]z_k$ , and  $y_{k-1}^{p-1}A_{k-1}$ , if  $k \geq 1$*

*with extensions determined by*

$$pz_k = vz_{k-1}^p \text{ for } k \geq 2, \text{ and } py_{k-1}^{p-1}z_{k-1} = v^{p^{k-1}(p-1)}z_k. \quad (1.2)$$

Here  $TP_i[v]$  is the truncated polynomial algebra over  $\mathbb{Z}_p$  with generator  $v$  and relation  $v^i$ . When we write something like  $zB$ , we mean that all elements of  $B$  are multiplied by the element  $z$ . Saying “is built from” means that these are successive quotients in a filtration as a  $ku^*$ -module. The extension formulas are only asserted up to multiplication by a unit in  $\mathbb{Z}_p$ , and can both occur on an element. For example, in Figure 1.6, we have, in grading 116 when  $p = 2$ ,  $2y_3z_3z_4 = vy_3z_2^2z_4 + v^8z_4^2$ .

Figure 1.6 should enable the reader to envision  $A_k$  and  $B_k$  for  $p = 2$  and  $k \leq 5$ , and, by extrapolating, for all  $k$ . Elements connected by dashed lines are in  $A_5$  but not in  $B_5$ . The long red lines, sometimes slightly curved, are the exotic extensions. The portion in gradings  $\leq 102$ , not including the top  $v$ -tower or the extensions to it, is  $y_4A_4$  (or  $y_4B_4$  if the dashed part is omitted). The portion in gradings  $\geq 106$ , not including the  $v$ -tower on  $z_5$  or the  $h_0$ -extensions from it, is  $z_4B_4$ . The portion in the lower right corner from gradings  $\leq 84$  is  $y_3y_4A_3$ , and  $y_2y_3y_4A_2$  is in gradings  $\leq 74$ . In Figure 1.7, we present a schematic of  $A_3$  and  $B_3$  at the odd primes. Again the dashed portion is in  $A_3$ , but not  $B_3$ , and the triangle in the lower right portion is  $y_1^{p-1}y_2^{p-1}A_1$ .

A generating set as a  $\mathbb{Z}_p[v]$ -module for  $B_k$  is

$$\left\{ z_j \prod_{i=j}^{k-1} \{z_i^{p-1}, y_i^{p-1}\} : k_0 \leq j \leq k \right\}, \quad (1.3)$$

while  $A_k$  has additional generators

$$\begin{cases} z_1 y_1 \cdots y_{k-1} & p = 2 \\ z_0 y_0^{p-1} \cdots y_{k-1}^{p-1} & \text{all } p. \end{cases}$$

The notation here means a product over all choices of one of the two elements in each factor. For example,  $\prod_{i=1}^2 \{z_i^{p-1}, y_i^{p-1}\} = z_1^{p-1} z_2^{p-1} + z_1^{p-1} y_2^{p-1} + y_1^{p-1} z_2^{p-1} + y_1^{p-1} y_2^{p-1}$ . An empty product is defined to equal 1.

The following theorem explains how the portion of  $ku^*(K_2)$  in even gradings is a direct sum of shifted versions of  $A_k$  and  $B_k$ .

**Theorem 1.4.** *Let  $M_p[S]$  denote the set of monomials in the elements of a set  $S$  raised to powers  $< p$ . Let*

$$\mathcal{M}_k = (M_p[z_k, y_k] - \{z_k^{p-1}, y_k^{p-1}\}) \cdot M_p[z_i, y_i : i > k]. \quad (1.5)$$

*Let  $\mathcal{M}_k^A$  be the set of monomials in  $\mathcal{M}_k$  with no  $z$ -factors, and  $\mathcal{M}_k^B = \mathcal{M}_k - \mathcal{M}_k^A$ . Then*

$$ku^{ev}(K_2) = \bigoplus_{k \geq 1} \left( \bigoplus_{M \in \mathcal{M}_k^A} M \cdot A_k \oplus \bigoplus_{M \in \mathcal{M}_k^B} M \cdot B_k \right)$$

*plus a trivial  $ku^*$ -module.*

Note that the monomial 1 is in  $\mathcal{M}_k^A$ , so  $A_k$  appears by itself, but  $B_k$  does not.

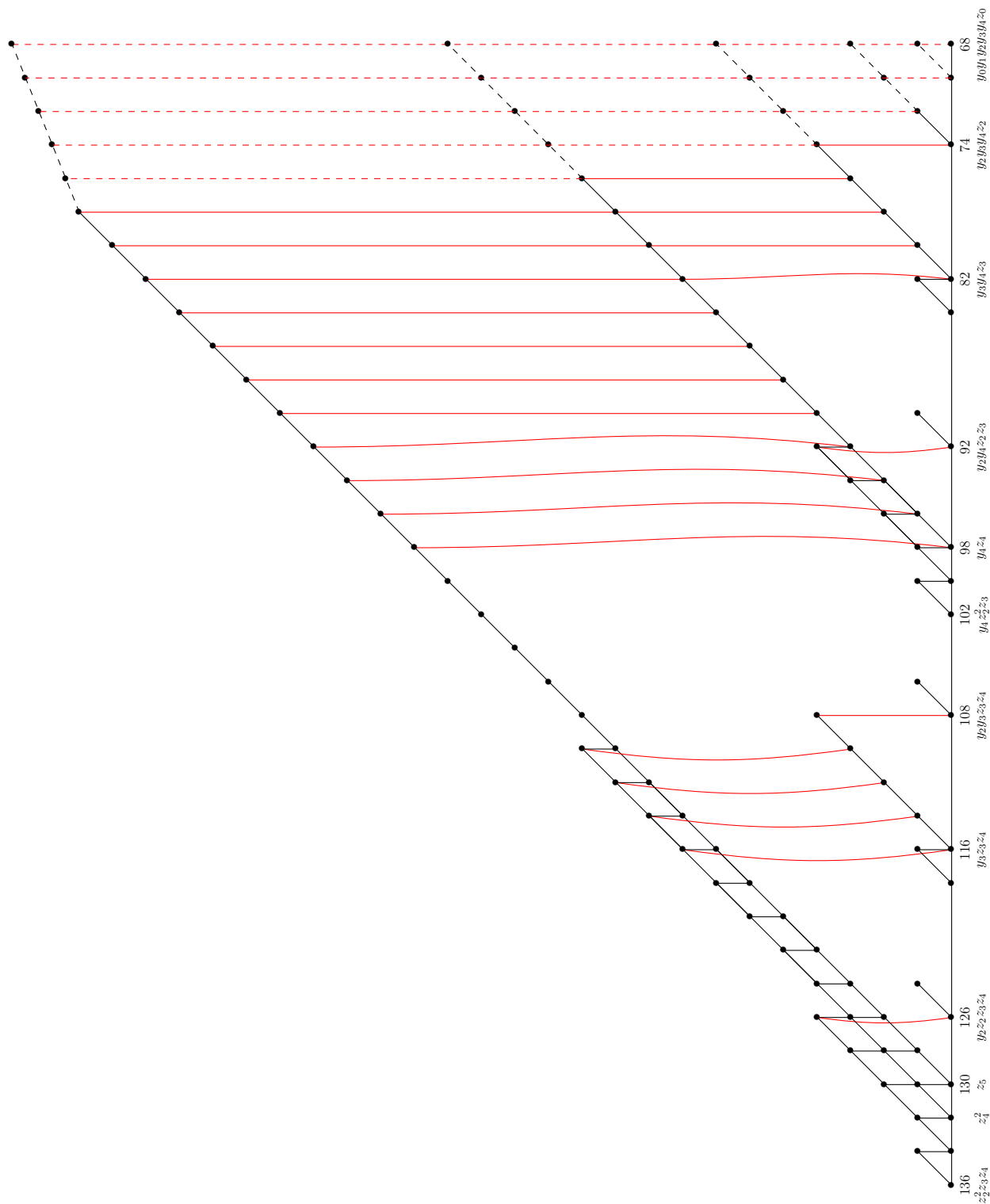
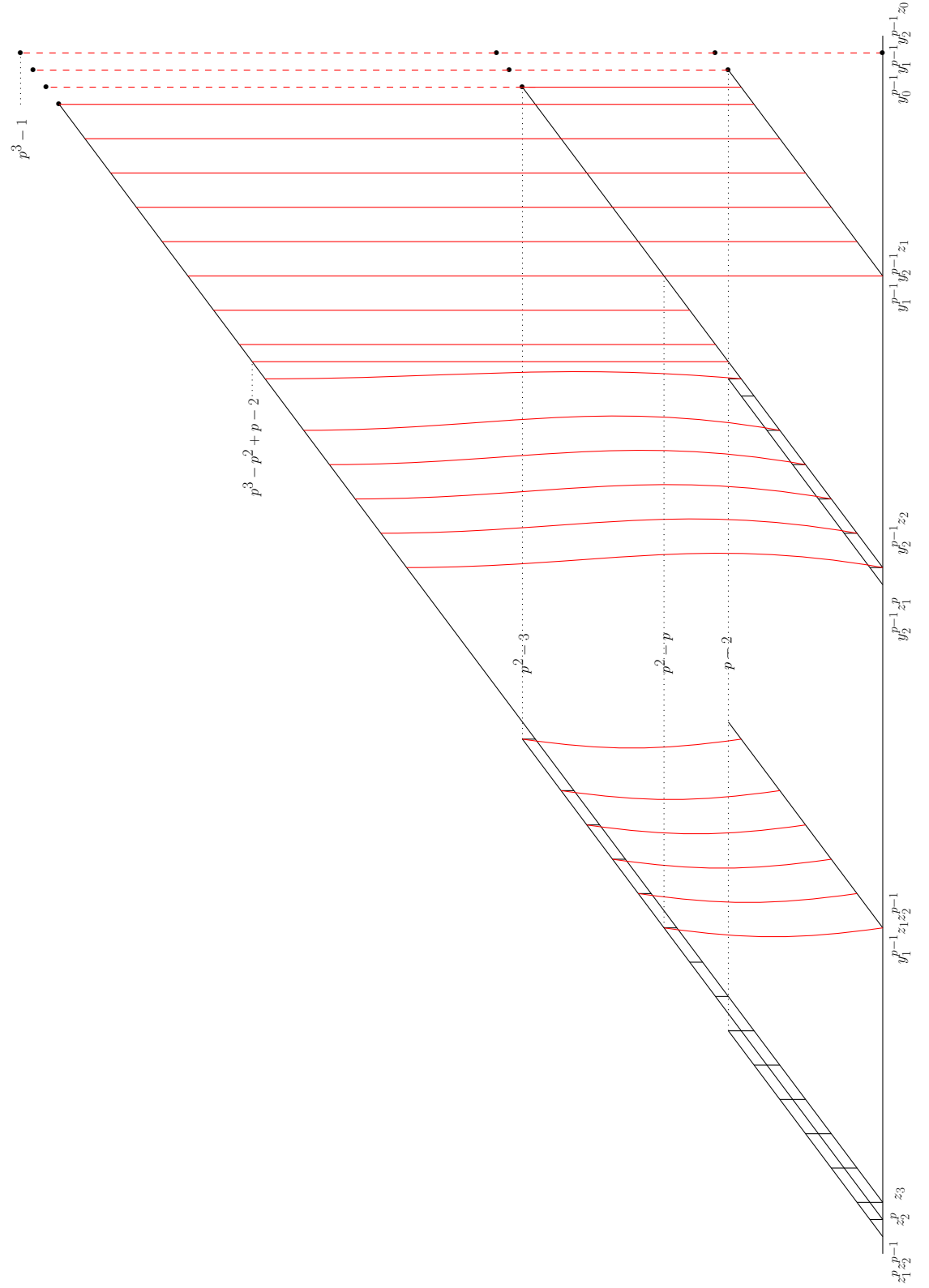
Figure 1.6.  $B_5$  and  $A_5$  when  $p = 2$ .

Figure 1.7. Schematic of  $A_3$  and  $B_3$  for odd  $p$ .

Now we describe the portion of  $ku^*(K_2)$  in odd gradings. Let  $P[S]$  denote the polynomial algebra on a set  $S$ , and  $TP_i[S] = P[S]/(s^i : s \in S)$ , the truncated polynomial algebra. Let  $\Lambda_j = TP_p[z_i : 1 \leq i \leq j]$ . Note that if  $p = 2$ ,  $\Lambda_j$  is an exterior algebra. For  $i \leq j$ , let

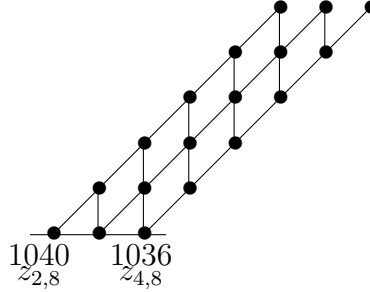
$$z_{i,j} = z_i(z_i \cdots z_{j-1})^{p-1}. \quad (1.8)$$

If  $j = i$ , then  $z_{i,j} = z_i$ .

**Definition 1.9.** For  $\ell > k \geq 1$ , let  $S_{k,\ell} = TP_{k+1}[v]\langle z_{k_0,\ell}, \dots, z_{\ell-k-1+k_0,\ell} \rangle$  with  $pz_{i,\ell} = vz_{i-1,\ell}$  and  $pz_{k_0,\ell} = 0$ .

For example,  $S_{5,8}$  with  $p = 2$  is depicted in Figure 1.10.

**Figure 1.10.**  $S_{5,8}$  if  $p = 2$



The following result describes the portion of  $ku^*(K_2)$  in odd gradings. The exponent of  $p$  in an integer  $i$  is denoted simply by  $\nu(i)$ ; the prime  $p$  is implicit. The element  $q$  here has grading 9 or  $4p - 1$ , as mentioned earlier.

**Theorem 1.11.** *There is an isomorphism of  $ku^*$ -modules*

$$ku^{odd}(K_2) \approx \bigoplus_{i \geq 1} \bigoplus_{\ell \geq \nu(i)+2} qy_1^{i-1} S_{\nu(i)+1,\ell} \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$

The non-visual, formulaic form of our result is as follows.

**Theorem 1.12.** *The  $ku^*$ -module  $ku^*(K_2)$  is isomorphic to a trivial  $ku^*$ -module plus*

$$P[y_1^{p-1}]y_0^{p-1}z_0 \oplus \bigoplus_{t \geq 1} TP_{p^t}[v] \otimes P[y_t]z_t \quad (1.13)$$

$$\oplus \bigoplus_{t \geq k_0} TP_{p^t-t}[v] \otimes P[y_t]z_t \Lambda_t \quad (1.14)$$

$$\oplus \bigoplus_{i \geq 1} TP_{\nu(i)+2}[v]qy_1^{i-1} \bigoplus_{\ell \geq 0} z_{k_0+\ell, \ell+\nu(i)+2} \Lambda_{\ell+\nu(i)+2}. \quad (1.15)$$

Multiplication by  $p$  in (1.13) and (1.14) is determined by (1.2) and in (1.15) as in Definition 1.9.

Our initial interest in this project was  $ku_*(K_2)$  ([12],[5]), but we first achieved success in computing  $ku^*(K_2)$ . In [4, Example 3.4], the following result was proved.

**Theorem 1.16.** *There is an isomorphism of  $ku_*$ -modules  $ku_*(K_2) \approx (ku^{*+2p}K_2)^\vee$ .*

Here  $M^\vee = \text{Hom}(M, \mathbb{Z}/p^\infty)$ , the Pontryagin dual, localized at  $p$ . A homotopy chart for  $ku_*(K_2)$  could be thought of as a shifted version of the homotopy chart of  $ku^*(K_2)$  viewed upside-down and backwards. For example, the element of  $ku^{108}(K_2)^\vee$  dual to the element  $v^4 y_3 z_3 z_4$  in Figure 1.6 corresponds to the generator of a  $\mathbb{Z}_4$  in  $ku_{104}(K_2)$  on which  $v^4$  acts nontrivially. This element can be seen in Figure 1.18.

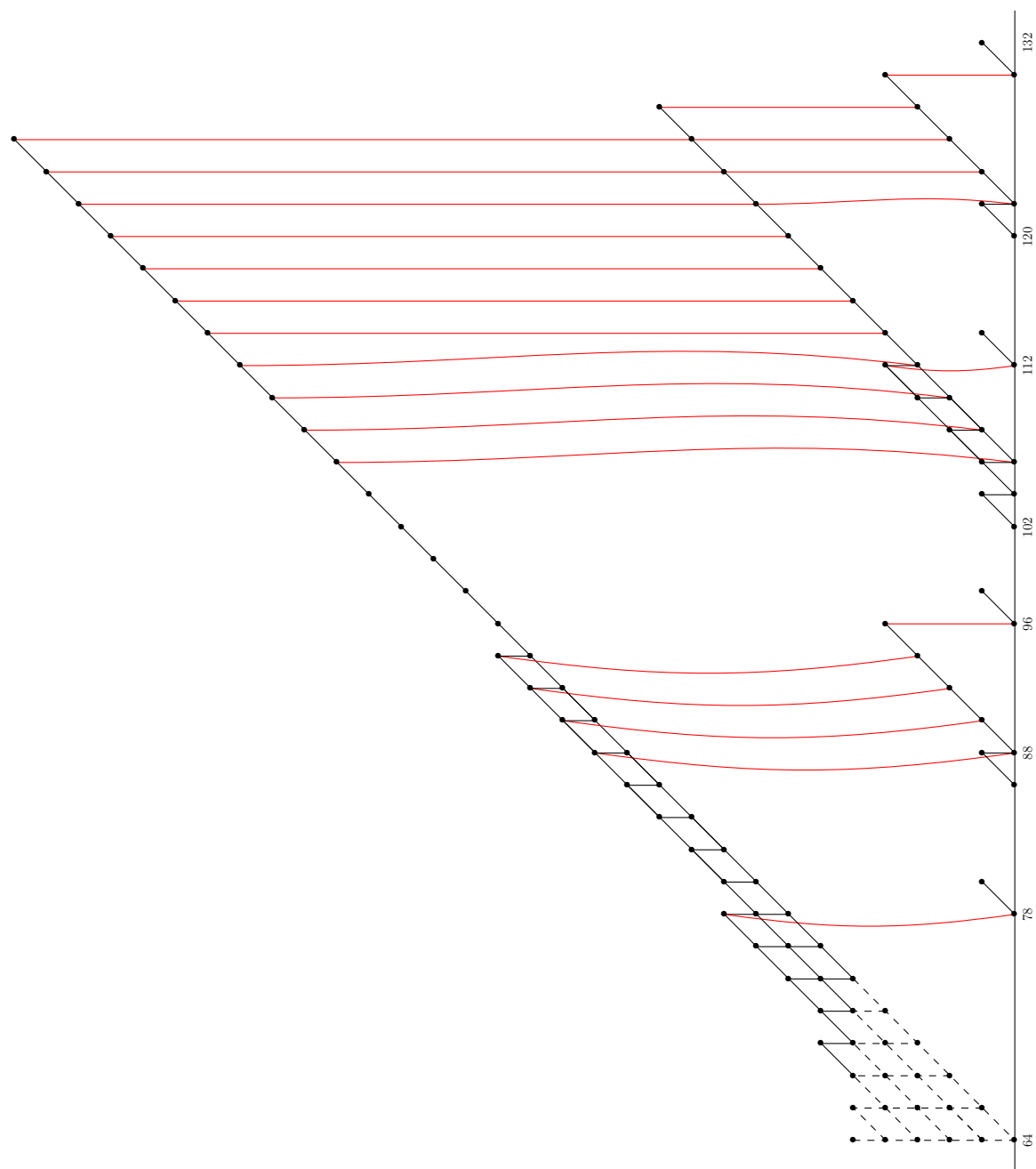
A remarkable property, for which one explanation is given in Section 7, is that  $B_k$  is self-dual as a  $ku^*$ -module. One way of stating this is to let  $\tilde{B}_k$  denote  $B_k$  with its indices negated. Then there is an isomorphism of  $ku_*$ -modules

$$\Sigma^{2(p^{k+1}+p^k+(k+1)p-k+1)} \tilde{B}_k \approx B_k^\vee. \quad (1.17)$$

For example, with  $p = 2$ , the second smallest generator  $Y$  of  $\Sigma^{208} \tilde{B}_5$  is in grading  $208 - 134 = 74$  and has  $2Y \neq 0$  and  $v^4 Y \neq 0$ . (See Figure 1.6.) The second generator  $Z$  of  $B_5^\vee$  is dual to the class in position  $(74, 4)$  in Figure 1.6, and also satisfies  $2Z \neq 0$  and  $v^4 Z \neq 0$ . The isomorphism (1.17) can be proved by induction on  $k$  using Definition 1.1.

A complete description of the  $ku_*$ -module  $ku_*(K_2)$  is immediate from Theorems 1.4, 1.11, and 1.16. However, one might like a complete description of its ASS. We can write formulas for the  $E_2$ -term and differentials, but will not do so here. In Theorem 1.19 we give a complete description of the  $E_\infty$ -term of the ASS of  $ku_*(K_2)$  with exotic extensions included, in terms of the charts described in Section 1.

In [4], a comparison was made of a chart for  $A_3$  and its  $ku_*$  analogue. Here we present in Figure 1.18 the  $ku_*$  analogue of Figure 1.6. This presents the portion of the ASS of  $ku_*(K_2)$  dual to  $A_5$  with  $p = 2$  under the isomorphism of Theorem 1.16. The chart dual to  $B_5$  is obtained from this by removing the classes connected by dashed lines, and lowering the remaining tower so that the bottom is in filtration 0. The resulting chart is isomorphic to the  $B_5$  part of Figure 1.6.

**Figure 1.18.** Portion of  $ku_*(K_2)$  corresponding to  $B_5$  and  $A_5$ .



We observe that in even gradings of the ASS for  $ku_*(K_2)$ ,  $h_0$ -extensions exactly correspond to exotic extensions in the ASS of  $ku^{*+2p}(K_2)$ , and vice versa. As a typical example of the duality, the summands of  $ku^{82}(K_2)$ ,  $ku^{82}(K_2)^\vee$ , and  $ku_{78}(K_2)$  in Figures 1.6 and 1.18 are all isomorphic to  $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ . But for the  $ku_*$ -module structure, it is  $ku^{82}(K_2)^\vee$  and  $ku_{78}(K_2)$  that correspond, since in both, the element that is divisible by 4, in position  $(82, 0)$  and  $(78, 7)$ , resp., is also divisible by  $v^7$  for  $A_5$  and by  $v^4$  for  $B_5$ .

**Theorem 1.19.** *The  $E_\infty$ -term of the ASS of  $ku_*(K_2)$  with exotic extensions included contains exactly the following.*

- *There is a trivial  $ku_*$ -module, which when  $p = 2$  has generators corresponding to those enumerated at the end of Section 2 with gradings decreased by 4, and similarly when  $p$  is odd.*
- *For every  $S_{k,\ell}$  occurring in a summand of Theorem 1.11, there is a chart of the same form as Figure 1.10 with  $v$ -towers of height  $k + 1$  on generators in gradings  $2p^{\ell+1} + 2(p - 1)(i - k_0 - 1)$  for  $1 \leq i \leq \ell - k$ . One must add to this the grading of the other factors accompanying  $S_{k,\ell}$  in Theorem 1.11.*
- *For each occurrence of  $B_k$  in Theorem 1.4, there is a summand*

$$\Sigma^{2(p^{k+1}+p^k+kp-k+1)} \widetilde{B}_k$$

*with gradings increased by those of other factors accompanying  $B_k$  in 1.4. Here  $\widetilde{B}_k$  is as defined prior to (1.17).*

- *For each summand  $y_k^e A_k$  in Theorem 1.4, there is a variant of  $\Sigma^{2(p^{k+1}+p^k+kp-k+1)} \widetilde{B}_k$  with gradings increased by  $2ep^k$ . In this variant, the initial  $v$ -towers are pushed up by  $k$  filtrations and surrounded with a triangle of classes of the sort appearing in the lower left corner of Figure 1.18. See Remark 1.20.*

*Proof.* Theorem 1.16 and our results for  $ku^*(K_2)$  give the  $ku_*$ -module structure of  $ku_*(K_2)$ , but that is not the same as the ASS picture. Expanding on work done in [5] and [12] and using methods such as those in Section 2, we were able to write the  $E_2$ -term of the ASS for  $ku_*(K_2)$ , and had conjectured the differentials (but not the extensions) prior to embarking on our  $ku$ -cohomology project. We were unable

to *prove* the differentials, probably because we had not taken sufficient advantage of the exact sequence with  $k(1)_*(K_2)$ . Now that we know the 2-orders and  $v$ -heights of generators (by grading, at least, if not by name), it is straightforward to see that the differentials must be as we expected. The isomorphism (1.17) plays an important role here; the left hand side gives the ASS form of the right hand side. ■

**Remark 1.20.** Regarding the unusual portion of the ASS chart for part of  $ku_*(K_2)$  in the lower left of Figure 1.18, this is obtained from [5, Fig. 4.2] with  $d_6$ -differentials on all odd-graded towers. For  $A_k$ , it will be a triangle going up to filtration  $k$ , with all but the first two dots on the top row being part of  $B_k$ .

The structure of the rest of the paper is as follows. In Section 2, we compute the  $E_2$ -term of the ASS for  $ku^*(K_2)$ . In Section 3 we determine the differentials in this ASS. In order to do so, we need to compare with  $k(1)^*(K_2)$ , where  $k(1)$  is the spectrum for mod- $p$  connective  $KU$ -theory, using the exact sequence

$$\rightarrow k(1)^{*-1}(K_2) \rightarrow ku^*(K_2) \xrightarrow{p} ku^*(K_2) \rightarrow k(1)^*(K_2) \rightarrow ku^{*+1}(K_2) \xrightarrow{p} . \quad (1.21)$$

In Section 3, we restate results about  $k(1)^*(K_2)$  from [7]. At the end of Section 3, we show how the descriptions of  $ku^*(K_2)$  in Theorems 1.4 and 1.11 are obtained once we know the differentials. This exact sequence is also used in determining the exotic extensions of (1.2), which is done in Section 4. In Section 5, we propose complete formulas for the exact sequence (1.21), and then in Section 6, we show that our proposed formulas account for all elements of  $k(1)^*(K_2)$  exactly once.

The main point of Section 6 is to prove that there are no additional exotic extensions in  $ku^*(K_2)$ . An exotic extension  $p \cdot A = B$  implies that  $A$  is not in the image from  $k(1)^{*-1}(K_2)$ , and  $B$  does not map nontrivially to  $k(1)^*(K_2)$ , so once we have shown that all elements are accounted for, there can be no more extensions. Many of our formulas in Section 5 are forced by naturality. However, many others occur in regular families, but with surprising filtration jumps. We could probably prove that the homomorphisms *must* be as we claim, by showing that there are no other possibilities, but we prefer to forgo doing that. In the optional Section 7, we discuss in more detail how the charts are obtained and provide an explanation for the duality result (1.17).

2. THE  $E_2$ -TERM OF THE ASS FOR  $ku^*(K_2)$ 

We will need some notation. By  $H^*K_2$ , we understand  $H^*(K(\mathbb{Z}_p, 2); \mathbb{Z}_p)$ . Let  $E$  denote an exterior algebra,  $P$  a polynomial algebra, and  $TP_n[x] = P[x]/(x^n)$  the truncated polynomial algebra. In all cases these will be over  $\mathbb{Z}_p$ , the integers mod  $p$ . Let  $\bar{E}$  denote the augmentation ideal of an exterior algebra, and  $E_1 = E[Q_0, Q_1]$ , where  $Q_i$  are the Milnor primitives. Because  $Q_i^2 = 0$  we have homology groups,  $H_*(-; Q_i)$ , defined for  $E_1$ -modules. We let  $\langle y_1, y_2, \dots \rangle$  denote the  $\mathbb{Z}_p$ -span of classes  $y_i$ .

The Adams spectral sequence (ASS) for  $ku^*(K_2)$  has  $E_2^{s,t} = \text{Ext}_{\mathcal{A}}^{s,t}(H^*(bu), H^*K_2)$ , where  $\mathcal{A}$  is the mod  $p$  Steenrod algebra and  $H^*(bu) \approx \mathcal{A}/\mathcal{A}(Q_0, Q_1)$ . Using a standard change of rings theorem, [8], this is  $\text{Ext}_{E_1}^{s,t}(\mathbb{Z}_p, H^*K_2)$ . This converges to  $ku^{-(t-s)}(K_2)$ . We depict this with  $E_2^{s,t}$  in position  $(t-s, s)$  as usual, but label the axis with codegrees, the negative of the homotopical degree, so the left side of the chart will have positive gradings and refer to cohomological grading. In an attempt to avoid confusion, we rewrite this as  $G_2^{-(t-s), s}$ . With this notation, the differentials are  $d_r : G_r^{a,b} \rightarrow G_r^{a+1, b+r}$ , multiplication by the element  $v \in ku^{-2(p-1)}$  (also considered in  $G_r^{-2(p-1), 1}$ ), is  $v : G_r^{a,b} \rightarrow G_r^{a-2(p-1), b+1}$ , and multiplication by the element representing  $p \in ku^0$ , ( $h_0 \in G_r^{0,1}$ ), is  $h_0 : G_r^{a,b} \rightarrow G_r^{a, b+1}$ .

We will later define elements  $z_j \in G_2^{2(p^{j+1}+1), 0}$  for  $j \geq 0$  and elements

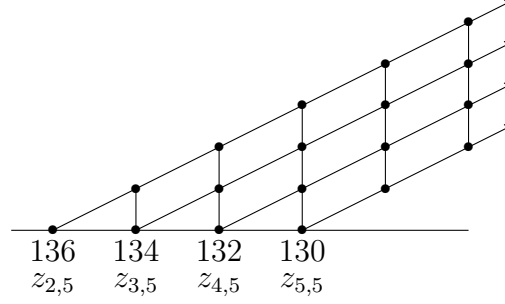
$$z_{i,j} \in G_2^{2(p^{j+1}+1+(p-1)(j-i)), 0}$$

as in (1.8) satisfying the properties in Definition 1.9. For  $j \geq k_0$ , we define  $W_j = \langle z_{j,j}, z_{j-1,j}, \dots, z_{k_0,j} \rangle$ . We also have  $y_i \in G_2^{2p^i, 0}$  for  $i \geq 0$ , and

$$q \in G_2^{9,0} \text{ if } p = 2, \text{ and in } G_2^{4p-1,0} \text{ if } p \text{ is odd.} \quad (2.1)$$

One last definition, let  $\Lambda_{j+1} = TP_p[z_i : i \geq j+1]$ .

A picture of  $P[v] \otimes W_5$  as a  $P[v, h_0]$ -module with  $p = 2$  appears in Figure 2.2.

**Figure 2.2.** A depiction of  $P[v] \otimes W_5$ 

The remainder of this section is devoted to the proof of the following result.

**Theorem 2.3.** *The  $E_2$  term of the Adams spectral sequence for the reduced  $ku^*(K_2)$  is isomorphic as a  $P[h_0, v]$ -module to*

$$P[v, y_1] \otimes E[q] \otimes \left( \bigoplus_{j \geq k_0} (W_j \otimes TP_{p-1}[z_j] \otimes \Lambda_{j+1}) \right)$$

$$\oplus (P[h_0, v, y_1] \otimes E[v^{k_0} q]) \oplus \left( P[y_1] \otimes \begin{cases} \langle y_0^{p-1} z_0 \rangle & p \text{ odd} \\ \langle y_0 z_0, z_1, h_0 y_0 z_0 = v z_1 \rangle & p = 2. \end{cases} \right)$$

plus a trivial  $P[h_0, v]$ -module.

Some of the algebra structure of this  $E_2$  will be useful later. For example, the product structure among the  $z_j$ 's will be clear, and also the formula

$$(v^2 q)^2 = v^4 z_2, \quad (2.4)$$

holds when  $p = 2$  since, as we shall see, in  $H^*(K_2)$ ,  $x_9^2 - Q_0 x_{17} \in \text{im}(Q_1)$ .

We will give a detailed proof when  $p = 2$ , and then sketch the minor changes for odd  $p$ . There are two parts to proving this theorem. First, we must give a complete description of the  $E_1$ -module structure of  $H^* K_2$ . Second, we have to compute  $\text{Ext}_{E_1}^{*,*}(\mathbb{Z}_2, -)$  of this. We begin the first part.

Serre ([9]) showed that  $H^* K_2$  is a polynomial algebra on classes  $u_{2^j+1}$  in degree  $2^j + 1$  for  $j \geq 0$  defined by  $u_2 = \iota_2$  and  $u_{2^{j+1}+1} = \text{Sq}^{2^j} u_{2^j+1}$  for  $j \geq 0$ . We easily have

$$Q_0(u_2) = u_3, \quad Q_0(u_3) = 0, \quad Q_0(u_{2^j+1}) = u_{2^{j-1}+1}^2 \text{ for } j \geq 2,$$

and

$$Q_1(u_2) = u_5, \quad Q_1(u_3) = u_3^2, \quad Q_1(u_5) = 0, \quad Q_1(u_{2^j+1}) = u_{2^{j-2}+1}^4 \text{ for } j \geq 3.$$

Let  $x_5 = u_5 + u_2u_3$  and write  $H^*K_2$  as an associated graded object:

$$P[u_2^2] \otimes E[x_5] \otimes (E[u_2] \otimes P[u_3]) \otimes_{j \geq 2} (E[u_{2^j+1+1}] \otimes P[(u_{2^j+1})^2])$$

From this, we can read off

**Lemma 2.5.**

$$H_*(H^*K_2; Q_0) = P[u_2^2] \otimes E[x_5]$$

Letting  $x_9 = u_9 + u_3^3$  and  $x_{17} = u_{17} + u_2u_5^3$ , we rewrite again as

$$P[u_2^2] \otimes TP_4[x_9] \otimes TP_4[x_{17}] \otimes_{j > 4} E[(u_{2^j+1})^2] \\ \otimes (E[u_2] \otimes P[u_5]) \otimes (E[u_3] \otimes P[u_3^2]) \otimes_{j > 4} (E[u_{2^j+1}] \otimes P[(u_{2^j-2+1})^4]).$$

Again we read off

**Lemma 2.6.**

$$H_*(H^*K_2; Q_1) = P[u_2^2] \otimes TP_4[x_9] \otimes TP_4[x_{17}] \otimes_{j > 4} E[(u_{2^j+1})^2]$$

An associated graded version of this is

**Lemma 2.7.**

$$H_*(H^*K_2; Q_1) = P[u_2^2] \otimes E[x_9] \otimes E[x_{17}] \otimes_{j > 2} E[(u_{2^j+1})^2]$$

The bulk of the work here is finding a nice splitting of  $H^*K_2$  as an  $E_1$ -module.

Let  $N$  be the  $E_1$ -submodule with single nonzero elements in gradings 5, 7, 8, 9, and 10 with generators  $x_5 = u_5 + u_2u_3$ ,  $x_7 = u_2u_5$ , and  $x_9 = u_9 + u_3^3$ , satisfying  $Q_0x_7 = Q_1x_5$  and  $Q_0x_9 = Q_1x_7 = x_{10}$ . It has a  $Q_0$ -homology class  $x_5$  and a  $Q_1$ -homology class  $x_9$ . This class  $x_9$  is called  $q$  in Theorem 2.3 and in all other sections. A picture of  $N$  is in Figure 2.8. In pictures such as this, straight lines indicate  $Q_0 = \text{Sq}^1$  and curved lines  $Q_1$ .

**Figure 2.8.** An  $E_1$ -module  $N$ .



The  $E_1$ -submodule  $P[u_2^2] \oplus P[u_2^2] \otimes N$  carries the  $Q_0$ -homology of  $H^*K_2$ , while the remaining  $Q_1$ -homology is, written in our usual way as an associated graded version,

$$P[u_2^2] \otimes E[x_9] \otimes \overline{E}[x_{17}, u_{2^{j+1}}^2, j > 2]. \quad (2.9)$$

We will exhibit a  $Q_0$ -free  $E_1$ -submodule  $R$  whose  $Q_1$ -homology is exactly this  $\overline{E}$ . Moreover,  $N \otimes R$  contains an  $E_1$ -split summand  $S$  which maps isomorphically to  $\langle x_9 \rangle \otimes R$ .

It is premature to state this because we haven't defined  $R$  and  $S$  yet, but for the record:

**Proposition 2.10.** *As an  $E_1$  module,  $\tilde{H}^*K_2$  is isomorphic to  $T \oplus F$  where  $F$  is free over  $E_1$  and  $T$  is*

$$P[u_2^2] \otimes (\langle u_2^2 \rangle \oplus N \oplus R \oplus S)$$

**A start on  $R$  and  $S$ .**

For this to make sense, we need to find  $R$  and  $S$ . The module  $R$  is a direct sum of shifted versions of modules  $L_k$ ,  $k \geq 0$ , which have generators  $g_{2i}$ ,  $0 \leq i \leq k$ , with  $Q_1 g_{2i} = Q_0 g_{2i+2}$  for  $0 \leq i < k$ ,  $Q_0 g_0 \neq 0$ , and  $Q_1 g_{2k} = 0$ . For example,  $L_3$  is depicted in Figure 2.11.

**Figure 2.11.** The  $E_1$ -module  $L_3$ .



A splitting map,  $\langle x_9 \rangle \otimes L_k \longrightarrow N \otimes L_k$ , for the epimorphism  $N \otimes L_k \rightarrow \langle x_9 \rangle \otimes L_k$  is defined by

$$x_9 g_{2i} \mapsto x_9 \otimes g_{2i} + x_7 \otimes g_{2i+2} + x_5 \otimes g_{2i+4} \text{ for } 0 \leq i \leq k-2,$$

$$x_9 g_{2k-2} \mapsto x_9 \otimes g_{2k-2} + x_7 \otimes g_{2k}, \text{ and } x_9 \otimes g_{2k} \mapsto x_9 \otimes g_{2k}.$$

**The  $E_1$ -module  $M_j$**

Let

$$x_{2j+1} = u_{2j+1} + \begin{cases} u_2 u_5^3 & j = 4 \\ u_2 u_3 u_5^2 u_9^2 & j = 5 \\ u_3 u_5^2 u_9^2 u_{17}^2 & j = 6 \\ 0 & j > 6 \end{cases} \text{ and } w_{2j-1} = \begin{cases} u_2 u_3 u_5^2 & j = 4 \\ u_3 u_5^2 u_9^2 & j = 5 \\ 0 & j > 5. \end{cases}$$

Then  $Q_0x_{2j+1} = u_{2^{j-1}+1}^2 + Q_1w_{2j-1}$ , so  $Q_0x_{2j+1}$  and  $u_{2^{j-1}+1}^2$  represent the same  $Q_1$ -homology class. Define  $E_1$ -modules  $M_j$  inductively by  $M_3 = 0$ , and for  $j \geq 4$  there is a short exact sequence of  $E_1$ -modules

$$0 \rightarrow u_{2^{j-2}+1}^2 M_{j-1} \rightarrow M_j \rightarrow M'_j \rightarrow 0, \quad (2.12)$$

where  $M'_j = \langle x_{2j+1}, Q_0x_{2j+1} \rangle$  and  $Q_1x_{2j+1} = u_{2^{j-2}+1}^2 Q_0x_{2^{j-1}+1}$ . The above definitions of the  $x_{2j+1}$  are necessary to get this formula to work right.

There is an isomorphism of  $E_1$ -modules  $M_j \approx \Sigma^{2^j+1} L_{j-4}$  given by

$$\Sigma^{2^j+1} g_{2i} \mapsto \begin{cases} x_{2j+1} & i = 0 \\ u_{2^{j-2}+1}^2 x_{2^{j-1}+1} & i = 1 \\ u_{2^{j-2}+1}^2 u_{2^{j-3}+1}^2 x_{2^{j-2}+1} & i = 2 \\ u_{2^{j-2}+1}^2 u_{2^{j-3}+1}^2 \cdots u_{2^{j-i-1}+1}^2 x_{2^{j-i}+1} & 2 < i \leq j-4 \end{cases} \quad (2.13)$$

And we have

$$H_*(M_j; Q_1) = \begin{cases} \langle u_9^2, u_{17} \rangle & j = 4 \\ \langle u_{17}^2, u_9^2 u_{17} \rangle & j = 5 \\ \langle u_{33}^2, u_{17}^2 u_9^2 u_{17} \rangle & j = 6 \\ \langle u_{2^{j-1}+1}^2, u_{2^{j-2}+1}^2 \cdots u_9^2 x_{17} \rangle & j > 6 \end{cases} \quad (2.14)$$

### The $E_1$ -module $R$

Let

$$R = \bigoplus_{j \geq 4} M_j \otimes E[u_{2^j+1}^2, u_{2^{j+1}+1}^2, \dots]. \quad (2.15)$$

Then  $H_*(R; Q_1) = \overline{E}[x_{17}, u_9^2, u_{17}^2, \dots]$ , since monomials in  $\overline{E}$  without  $x_{17}$  appear from a first term (of the two in (2.14)) in  $H_*(M_j \otimes E; Q_1)$ , where  $j$  is minimal such that  $u_{2^{j-1}+1}^2$  appears in the monomial, while those with  $x_{17}$ , and also containing a product  $u_9^2 \cdots u_{2^{j-2}+1}^2$  of maximal length, occur as a second term in  $H_*(M_j \otimes E; Q_1)$ .

*Proof of Proposition 2.10.* We have the  $E_1$ -submodule  $T$  given in Proposition 2.10. Because this contains all of the  $Q_0$  and  $Q_1$  homology, what remains must be free over  $E_1$  by [11]. ■

*Proof of Theorem 2.3.* We compute  $\text{Ext}_{E_1}(\mathbb{Z}_2, T)$  with  $T$  as in Proposition 2.10. We will not be concerned with the free  $E_1$ -module  $F$  but later we will give the Poincaré series for it. Each copy of  $E_1$  in  $F$  gives a  $\mathbb{Z}_2$  in  $G^{*,0}$  that corresponds to  $Q_0Q_1$ .

That

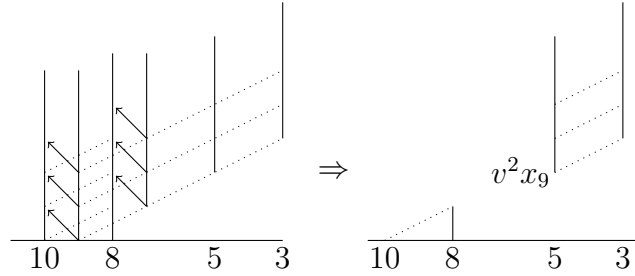
$$\text{Ext}_{E_1}^{*,*}(\mathbb{Z}_2, P[u_2^2]) = P[v, h_0, y_1]$$

with  $y_1 \in G_2^{4,0}$  should be clear, given our labeling conventions. We normally work with the reduced cohomologies, so the  $y_1^0$  generator above would be ignored. The  $y_1$  notation is particularly useful when we consider all primes  $p$ . It is  $y_0^{p^1}$  where  $y_0 \in G_2^{2,0}$ . So  $|y_1| = 2p$ .

We compute  $\text{Ext}_{E_1}(\mathbb{Z}_2, N)$  in two ways using two different filtrations of  $N$ . From this we see that the generator of the towers can be thought of either as  $v^2x_9$  or  $h_0^2x_5$ .

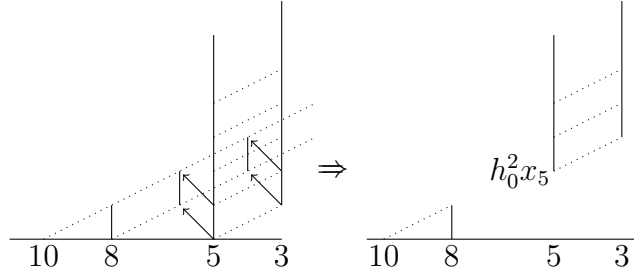
Using Figure 2.8 as our guide, our first filtration is  $\langle x_5, x_8 \rangle$ ,  $\langle x_7, x_{10} \rangle$ , and  $\langle x_9 \rangle$ . The Ext on  $x_9 \in G^{9,0}$  is just  $P[v, h_0]$ . For the other two, we get  $h_0$ -towers on  $x_{10} \in G^{10,0}$  and  $x_8 \in G^{8,0}$ . The extensions in  $N$  show these two  $h_0$ -towers are connected by multiplication by  $v$ . In addition, a  $d_1$  is forced on us by the extensions. Figure 2.16 describes this completely.

**Figure 2.16. The first computation of  $\text{Ext}_{E_1}(\mathbb{Z}_2, N)$**



Again referring to Figure 2.8, our second filtration is  $\langle x_9, x_{10} \rangle$ ,  $\langle x_7, x_8 \rangle$ , and  $\langle x_5 \rangle$ . Now our Ext groups are  $P[v, h_0]$  on  $x_5 \in G^{5,0}$ ,  $P[v]$  on  $x_8 \in G^{8,0}$  and  $x_{10} \in G^{10,0}$ . Again, the  $d_1$  is forced by the extensions in  $N$ . Figure 2.17 describes the result.



**Figure 2.17.** The second computation of  $\text{Ext}_{E_1}(\mathbb{Z}_2, N)$ 

This concludes the computation of  $\text{Ext}$  for  $P[u_2^2] \otimes (\langle u_2^2 \rangle \oplus N)$  of Proposition 2.10. The result is the second line of Theorem 2.3.

We need to compute  $\text{Ext}$  for  $P[u_2^2] \otimes (R \oplus S)$  and show it is the same as the top line in Theorem 2.3. Since  $S \approx \langle x_9 \rangle \otimes R$ , all we need to do is  $P[u_2^2] \otimes R$  and ignore the  $E[x_9]$ . Similarly we can ignore the  $P[u_2^2]$  and the  $P[y_1]$  because for every power of  $u_2^2$  we will have a copy of the answer indexed by powers of  $y_1$ . All we have left now is  $R$ , but  $R$  is just many copies of the various  $M_j$  and the indexing for the number of copies is given by the  $\Lambda_{j+1}$ .

All that remains is to show that  $\text{Ext}_{E_1}(\mathbb{Z}_2, M_j) \approx P[v] \otimes W_{j-2}$ .<sup>1</sup> Recall that  $M_j = \Sigma^{2^j+1} L_{j-4}$ . We can filter  $L_{j-4}$  into pairs of elements  $g_{2i}, Q_0 g_{2i}$ , for  $0 \leq i \leq j-4$ .  $\text{Ext}$  for each of these gives a  $P[v]$  on the element  $Q_0 g_{2i}$  represented by  $z_{j-i-2, j-2} \in G^{2^j+2+2i, 0}$ . There is no  $d_1$ , but undoing the filtration does solve the extension problem and gives us  $h_0 z_{k, j-2} = v z_{k-1, j-2}$ . This completes our computation and thus our proof. ■

**Remark 2.18.** To illustrate the last computation in the proof, consider the generators of the  $v$ -towers for  $\text{Ext}_{E_1}(\mathbb{Z}_2, M_7)$ . They are  $z_5, z_4^2, z_3^2 z_4$ , and  $z_2^2 z_3 z_4$ , which is what we have called  $z_{5,5}, z_{4,5}, z_{3,5}$ , and  $z_{2,5}$ , as pictured in Figure 2.2. For future reference, we note that (with  $\sim$  meaning homologous)

$$z_j = Q_0 x_{2^j+2+1} \sim u_{2^j+1+1}^2 = Q_0 u_{2^j+2+1} = Q_0 Q_{j+2} \iota_2 = Q_{j+2} Q_0 \iota_2. \quad (2.19)$$

<sup>1</sup>The reason for this awkward shift is that the gradings for  $z_j$  which give the elegant statements in Definition 1.1 and elsewhere are not particularly convenient in developing the  $E_2$  statement.

We now describe briefly the changes required when  $p$  is odd. We have

$$H^*(K_2) = P[y_0] \otimes P[g_1, g_2, \dots] \otimes E[u_0, u_1, \dots],$$

with  $|y_0| = 2$ ,  $|g_j| = 2(p^j + 1)$ ,  $Q_0 y_0 = u_0$ ,  $Q_0 u_i = g_i$ ,  $Q_1 y_0 = u_1$ ,  $Q_1 u_i = g_{i-1}^p$ ,  $i \geq 2$ . Let  $y_1 = y_0^p$ . Then, similarly to the case  $p = 2$ ,

$$H_*(H^* K_2, Q_0) = P[y_1] \otimes E[y_0^{p-1} u_0].$$

Let  $N = \langle y_0^{p-1} u_0, q = y_0^{p-1} u_1, Q_0 q = Q_1(y_0^{p-1} u_0) \rangle$ . Then  $P[y_1] \oplus P[y_1] \otimes N$  carries the  $Q_0$ -homology and part of the  $Q_1$ -homology. Similarly to (2.9), the rest of the  $Q_1$ -homology is

$$P[y_1] \otimes E[q] \otimes \overline{E[w_1] \otimes TP_p[g_2, g_3, \dots]},$$

where  $w_1 = u_2 + u_0 g_1^{p-1}$ . There are  $E_1$ -submodules  $M_j$  for  $j \geq 2$ , defined inductively by  $M_2 = \langle w_1, g_2 = Q_0 w_1 \rangle$ ,  $M'_j = \langle u_j, g_j = Q_0 u_j \rangle$  for  $j \geq 3$ , and for  $j \geq 3$ , there exists a short exact sequence of  $E_1$ -modules

$$0 \rightarrow g_{j-1}^{p-1} M_{j-1} \rightarrow M_j \rightarrow M'_j \rightarrow 0,$$

with  $Q_1 u_j = g_{j-1}^p$ . There is an isomorphism of  $E_1$ -modules  $M_j \approx \Sigma^{2p^j+1} L_{j-2}$ , where  $L_j$  is similar to Figure 2.11, but with  $i$ th generator ( $i \geq 0$ ) in grading  $2(p-1)i$  rather than  $2i$ .

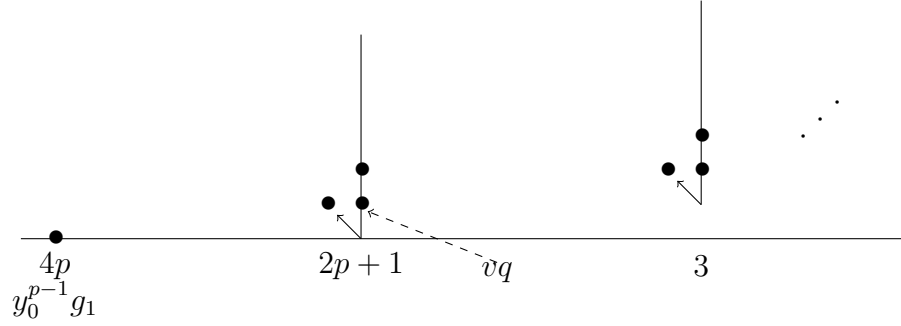
Let

$$R = \bigoplus_{j \geq 2} M_j \otimes TP_{p-1}[g_j] \otimes TP_p[g_{j+1}, \dots].$$

Then  $H_*(R; Q_1) = \overline{E[w_1] \otimes TP_p[g_2, g_3, \dots]}$ , and so, similarly to Proposition 2.10, up to free  $E_1$ -modules

$$H^* K_2 \approx P[y_1] \otimes (\langle y_1 \rangle \oplus N \oplus R \oplus qR). \quad (2.20)$$

Similarly to Figure 2.17,  $\text{Ext}_{E_1}(\mathbb{Z}_p, N)$  can be read off from Figure 2.21. This gives the third summand and  $vq$  part of the second summand in Theorem 2.3, while the  $\langle y_1 \rangle$  part of (2.20) gives the non- $vq$  part of the second summand. For the first summand in Theorem 2.3, we replace  $g_j$  by  $z_{j-1}$ , and then note that  $\text{Ext}_{E_1}(\mathbb{Z}_p, M_j) \approx P[v] \otimes W_{j-1}$ , similar to Figure 2.2. For example,  $M_3$  has  $v$ -towers on  $g_3$  and  $g_2^p$ , which are renamed  $z_2 = z_{2,2}$  and  $z_1^p = z_{1,2}$ , the generators of the  $v$ -towers of  $W_2$ . This completes our sketch of proof of Theorem 2.3 when  $p$  is odd.

**Figure 2.21.** Computation of  $\text{Ext}_{E_1}(\mathbb{Z}_p, N)$ 

We explain here the reason for the  $k_0$  in Definition 1.1. In Theorem 2.3,  $y_0^{p-1}z_0$  and  $z_1$  are in the part that is not multiplied by higher  $z$ 's when  $p = 2$ , but when  $p$  is odd, they form the module  $M_2$ , whose  $\text{Ext}$  is  $P[v] \otimes W_1$ , which is multiplied by higher  $z$ 's. Since  $B_k$ 's are multiplied by higher  $z$ 's, but  $A_k$ 's are not, this explains why  $z_1$  is in  $B_1$  when  $p$  is odd, but not when  $p = 2$ . The reason for the split in Theorem 2.3 is the difference in the submodules  $N$ . Its second class is  $y_0^{p-1}Q_1y_0$  in each. Applying  $Q_1$  yields  $y_0^{p-2}(Q_1y_0)^2$ . This is 0 when  $p$  is odd, but not when  $p = 2$ . The reason that the portion of  $\text{Ext}$  corresponding to  $N$  is not multiplied by higher  $z$ 's is that it gives part of the  $Q_0$ -homology, and this is not multiplied by higher  $z$ 's.

We close this section with enumeration of the unimportant  $\mathbb{Z}_2$ -classes in  $ku^*(K_2)$  when  $p = 2$ .

### More on the $E_1$ -free part when $p = 2$

If we compute the  $\text{Ext}_{E_1}(\mathbb{Z}_2, F)$  for the  $E_1$  free part of  $H^*K_2$ , we just get a  $\mathbb{Z}_2$  corresponding to the top element for each copy of  $E_1$ . If we find the Poincaré series (PS) for the free part, all we have to do to get the PS for these elements is multiply by  $\frac{x^4}{(1+x)(1+x^3)}$ . The Poincaré series for free part is obtained by subtracting the PS for the non-free part of Proposition 2.10 from that of  $H^*K_2$ . This is:

$$\prod_{k \geq 0} \frac{1}{(1 - x^{2^{k+1}})} - \frac{1}{(1 - x^4)} (1 + x^5 + x^7 + x^8 + x^9 + x^{10})$$

$$- \frac{1}{(1 - x^2)(1 - x^4)} \left( \bigoplus_{j \geq 4} (x^{2^j+1}(1+x^9)(1+x)(1-x^{2^j-6}) \prod_{k \geq j} (1+x^{2^{k+1}+2})) \right)$$

The first term is the PS for  $H^*K_2$ . The second is the PS for  $P[u_2^2] \otimes (\langle 1 \rangle \oplus N)$ . The last term is more complicated but does the  $S$  and  $R$  terms. The  $(1 - x^4)$  in the denominator is for the  $P[u_2^2]$ . The  $x^9$  is the shift that takes  $R$  to  $S$ . The  $(1 + x)$  is because they are  $Q_0$  free. The  $x^{2^j+1}(1 - x^{2^j-6})/(1 - x^2)$  is for the odd part of  $M_j$  and the remainder is for  $\Lambda$ .

This is easy to put into a computer and calculate. For example, the number of free generators in degree 79 is 245.

### 3. DIFFERENTIALS IN THE ASS OF $ku^*(K_2)$

The main theorem of this section determines the differentials in the ASS for  $ku^*(K_2)$ .

**Theorem 3.1.** *The differentials in the spectral sequence whose  $E_2$ -term was given in Theorem 2.3 are as follows. All  $v$ -towers are involved, either as source or target, in exactly one of these. Here  $M$  refers to any monomial (possibly = 1) in the specified algebra. Recall that  $\Lambda_j = TP_p[z_i : i \geq j]$ , which is an exterior algebra if  $p = 2$ . Also, recall  $y_t = y_1^{2^{t-1}}$ . We give reference numbers to the differentials when  $p$  is odd, but references to these also apply to the corresponding differential when  $p = 2$ , as the proofs are extremely similar.*

*First with  $p = 2$ .*

$$\begin{aligned}
 d_{\nu(i)+2}(y_1^i) &= h_0^{\nu(i)} v^2 q y_1^{i-1}, \quad i \geq 1; \\
 d_{\nu(i)+2}(y_1^i z_j M) &= v^{\nu(i)+2} q y_1^{i-1} z_{j-\nu(i),j} M, \\
 &\quad j \geq \nu(i) + 2, \quad M \in \Lambda_j; \\
 d_{2^t-t}(h_0^{t-2} v^2 q y_1^{2^{t-1}-1} M) &= v^{2^t} z_t M, \\
 &\quad t \geq 2, \quad M \in P[y_t]; \\
 d_{2^t-t}(q y_1^{2^{t-1}-1} z_{j-(t-2),j} M) &= v^{2^t-t} z_t z_j M, \\
 &\quad j \geq t \geq 2, \quad M \in P[y_t] \otimes \Lambda_{j+1}.
 \end{aligned}$$

Now with  $p$  odd.

$$d_{\nu(i)+2}(y_1^i) = h_0^{\nu(i)+1} v q y_1^{i-1}, \quad i \geq 1; \quad (3.2)$$

$$\begin{aligned} d_{\nu(i)+2}(y_1^i z_j M) &= v^{\nu(i)+2} q y_1^{i-1} z_{j-\nu(i)-1,j} M, \\ j &\geq \nu(i) + 2, \quad M \in \Lambda_j; \end{aligned} \quad (3.3)$$

$$\begin{aligned} d_{p^t-t}(h_0^{t-1} v q y_1^{p^{t-1}-1} M) &= v^{p^t} z_t M, \\ t &\geq 1, \quad M \in P[y_t]; \end{aligned} \quad (3.4)$$

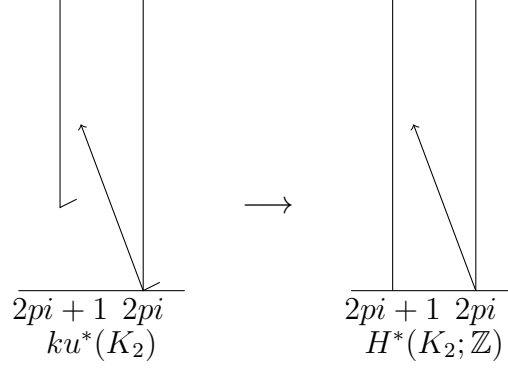
$$\begin{aligned} d_{p^t-t}(q y_1^{p^{t-1}-1} z_{j-(t-1),j} M) &= v^{p^t-t} z_t z_j M, \\ j &\geq t \geq 1, \quad M \in P[y_t] \otimes TP_{p-1}[z_j] \otimes \Lambda_{j+1}. \end{aligned} \quad (3.5)$$

The proof occupies the rest of this section, except that at the end of the section we explain briefly how this leads to our description of  $ku^*(K_2)$  in Section 1, except for the exotic extensions.

By [10, Theorem A],  $Q_j Q_0 \iota_2$  is in the image from  $BP^*(K_2)$ , and hence must be a permanent cycle in our ASS. Thus by (2.19),  $z_j$  is a permanent cycle, and so (3.3) follows from (3.2), and (3.5) follows from (3.4), using  $p z_{i,\ell} = v z_{i-1,\ell}$ , as noted in 1.9.

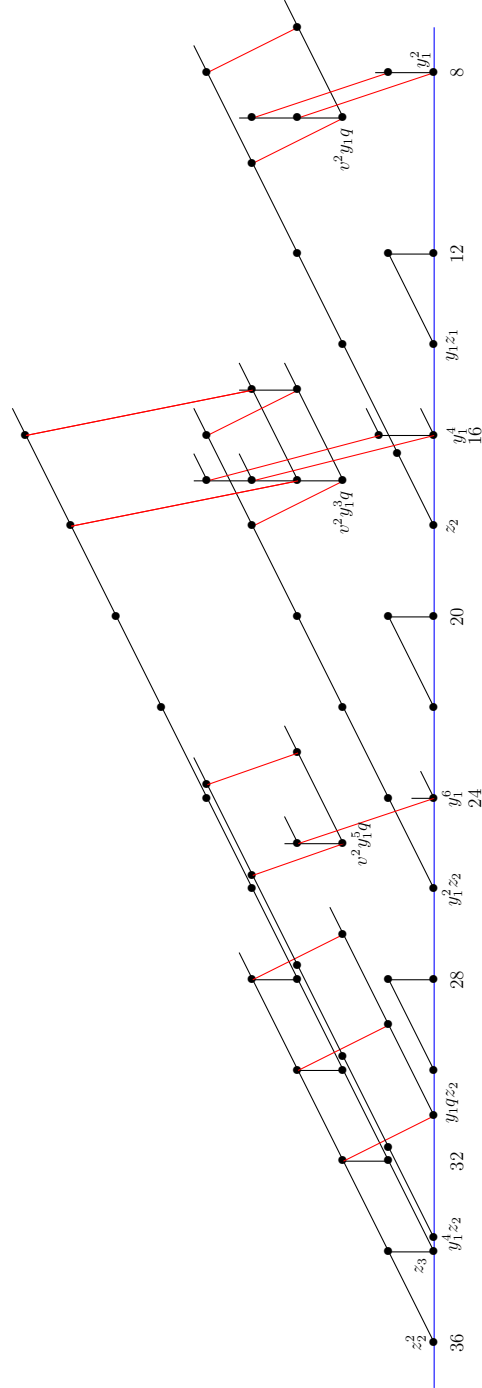
The differentials (3.2) follow from the result of [2] that  $H^{2pi+1}(K_2; \mathbb{Z}) \approx \mathbb{Z}/p^{\nu(i)+2} \oplus \bigoplus \mathbb{Z}_p$ . See also [3, Proposition 1.3.5] when  $p = 2$ . The ASS converging to  $H^*(K_2; \mathbb{Z})$  has  $E_2 = \text{Ext}_{A_0}(\mathbb{Z}_2, H^* K_2)$ , where  $A_0 = \langle 1, Q_0 \rangle$ . We depict this  $E_2$  similarly to our ASS for  $ku^*(K_2)$ . It has an  $h_0$ -tower for each element of  $H_*(H^* K_2, Q_0)$ , which was described in Lemma 2.5. These come in pairs in grading  $2pi$  and  $2pi+1$  corresponding to  $y_1^i$  and  $y_1^{i-1} y_0^{p-1} u_0$ . In order to get the  $\mathbb{Z}/p^{\nu(i)+2}$ , there must be a  $d_{\nu(i)+2}$ -differential, as pictured on the right hand side of Figure 3.6.

Similarly to Figures 2.16 and 2.17, we have, for  $p = 2$  and  $i \geq 1$ , an  $h_0$ -tower in the ASS for  $ku^*(K_2)$  arising from  $G^{4i+1,2}$ , called either  $h_0^2 y_1^{i-1} x_5$  or  $v^2 y_1^{i-1} q$ . There is also an  $h_0$ -tower arising from  $y_1^i \in G^{4i,0}$ . The classes  $y_1$  and  $x_5$  correspond to cohomology classes  $u_2^2$  and  $u_5 + u_2 u_3$ . Under the morphism  $ku^*(K_2) \rightarrow H^*(K_2; \mathbb{Z})$ , these towers map across, as suggested in Figure 3.6. We deduce the  $d_{\nu(i)+2}$ -differential claimed in (3.2), promulgated by the action of  $v$ . Note that  $x_9 = q$ .

**Figure 3.6.**  $ku^*(K_2) \rightarrow H^*(K_2; \mathbb{Z})$ 

The situation when  $p$  is odd is extremely similar, using Figure 2.21. The difference is that the  $h_0$ -tower in  $2pi + 1$  in the  $ku^*$  ASS starts in filtration 1 rather than 2. Its generator can be called  $vy_1^{i-1}q$ .

In Figure 3.7, we depict many of the differentials asserted in Theorem 3.1 in grading  $\leq 36$  when  $p = 2$ . Regarding the third (final) summand in Theorem 2.3, which is  $P[y_1] \otimes A_1$  when  $p = 2$ , we have included  $y_1 A_1$ ,  $y_1^3 A_1$ , and  $y_1^5 A_1$ . Not included are the portions involving (3.2) and (3.3) when  $i$  is odd, as this portion self-annihilates. What is shown is (3.2) for  $i = 2, 4$ , and  $6$ , (3.4) for  $(t, k) = (1, 0), (1, 1), (1, 2)$ , and  $(2, 0)$ , and (3.5) with  $t = 1$ ,  $k = 0$ , and  $j = 4$ .

**Figure 3.7.** Some differentials with  $p = 2$ 

In order to establish the remaining differentials, we will need the following description of  $k(1)^*(K_2)$ , which is proved in [7]. We shift by 1 the subscripts of the classes  $z_j$

and  $w_j$  used there. The formulas for  $r(j)$  and  $r'(j)$  are as in [7]. We recapitulate some of their properties. Those stated here but not there are easily proved by induction.

**Proposition 3.8.** [7] *For  $j \geq 0$ ,  $z_j$  is the reduction of the class in  $ku^*(K_2)$  and satisfies  $|z_j| = 2(p^{j+1} + 1)$ . The classes  $w_j$  satisfy  $|w_1| = 2p^2 + 1$ ,  $|w_2| = 2p^3 - 2p^2 + 6p - 3$ , and  $w_{j+2} = y_j^{p-1} w_j z_{j+1}^{p-1}$ . The integers  $r(j)$  and  $r'(j)$  satisfy the following properties.*

$$r(0) = 1, \quad r(1) = p, \quad r(j+2) = r(j) + p^{j+1}(p-1) + 1; \quad (3.9)$$

$$r'(0) = p-1, \quad r'(1) = p^2 - p,$$

$$r'(j+2) = r'(j) + p^{j+2}(p-1) - 1; \quad (3.10)$$

$$r(j) - r'(j-1) = j, \quad (3.11)$$

$$r(j) + r'(j) = p^{j+1}, \quad (3.12)$$

$$r(j+2) + r'(j) = p^{j+2} + 1, \quad (3.13)$$

$$(p-1)(r(j-1) + j-1) < p^j, \quad (3.14)$$

$$p^{j+1} - p^j \leq r'(j) < p^{j+1} - p^{j-1}. \quad (3.15)$$

**Theorem 3.16.** [7] *For any  $p$ ,  $k(1)^*(K_2)$  is an ignorable trivial  $k(1)^*$ -module plus*

$$\begin{aligned} & \bigoplus_{j>0} TP_{r(j)}[v] \otimes P[y_{j+1}] \otimes TP_{p-1}[y_j] \otimes \overline{E}[w_j] \otimes E[w_{j+1}] \otimes \Lambda_{j+1} \\ & \oplus \bigoplus_{j \geq 1} TP_{r'(j-1)}[v] \otimes P[y_j] \otimes E[w_j] \otimes \overline{TP}_p[z_j] \otimes \Lambda_{j+1} \\ & \oplus \bigoplus_{j \geq 1} P[y_1] \otimes E[q] \otimes \overline{E}[z_j^p] \otimes \Lambda_{j+1} \oplus P[y_1] \otimes \left( \overline{E}[y_0^{p-1} z_0] \oplus \begin{cases} \overline{E}[z_1] & p = 2 \\ 0 & p \text{ odd.} \end{cases} \right) \end{aligned}$$

The last line was not discussed in [7]; it is from free  $E[Q_1]$  summands which are not part of free  $E_1$  summands, and plays a very important role.

Now we continue the proof of Theorem 3.1. We have already proved (3.2) and (3.3). As already noted, the  $z_j$ 's are infinite cycles by [10], and so the differentials in (3.5) are implied as soon as the corresponding differential in (3.4) is proved.

As a warmup, we consider the cases  $t = 2$  and  $3$  of (3.4) when  $p = 2$ . We make extensive use of the exact sequence (1.21). Referring to Figure 3.7 is useful.

In even gradings  $\leq 14$ ,  $k(1)^*(K_2) = 0$  in positive filtration, by Theorem 3.16. Thus the map  $ku^*(K_2) \rightarrow k(1)^*(K_2)$  implies that in the ASS for  $ku^*(K_2)$ ,  $v^s z_2$  must be



hit by a differential or divisible by 2 for  $s \geq 2$ . In grading  $< 8$ , there is nothing that can divide it, and the only odd-grading  $v$ -tower in that range is on  $v^2 y_1 q$ . Thus  $d_2(v^2 y_1 q) = v^4 z_2$ , the case  $t = 2$ ,  $M = 1$  of (3.4). Since  $d_2(y_1^{2k}) = 0$  by (3.2), the case  $t = 2$  of (3.4) follows for any  $M$  by the derivation property. An analogous argument does not work at the odd primes.

Similarly  $v^s z_3$  must be hit or divisible for  $s \geq 4$ , and examination of options in Figure 3.7 shows that we must have  $d_5(h_0 v^2 y_1^3 q) = v^8 z_3$ , preceded by extensions. Since  $d_5(y_1^8) = h_0^3 v^2 y_1^7 q$ , we deduce the case  $t = 3$ ,  $M \in P[y_1^8]$  of (3.4) using the derivation property, (2.4) and  $h_0 z_2 = 0$ . We do not have *a priori* knowledge that  $y_1^4 z_3$  is a permanent cycle in the ASS of  $ku^*(K_2)$ . However, if it supported a nonzero differential, then the tower of  $v$ -height 4 on  $y_1^4 z_3$  in the ASS of  $k(1)^*(K_2)$  would have to map to  $v^t C$  for  $0 \leq t \leq 3$  for some  $C$  in positive filtration in grading 51 in the ASS of  $ku^*(K_2)$ . Then  $v^4 C$  must be  $d_r(B)$  with  $r \geq 5$  and  $B$  in filtration 0 in grading 42. ( $B$  cannot have higher filtration since everything is  $v$ -towers, and  $v^3 C$  cannot be hit.) But the only possible  $B$  is  $y_1^6 z_2$ , and we already know that  $v^4 y_1^6 z_2 \in \text{im}(d_4)$ . (Ordinarily this would not preclude the possibility of  $B$  supporting a differential, but it does since everything is  $v$ -towers.) Thus  $y_1^4 z_3$  is a permanent cycle, and consideration of its image in  $k(1)^*(K_2)$  implies that  $v^s y_1^4 z_3$  is hit by a differential for some  $s \geq 4$ . The only element in odd grading  $< 42$  not yet accounted for is  $h_0 v^2 y_1^7 q$  in grading 33, and so this must be the source of the differential. This is the case  $t = 3$ ,  $M = y_1^4$  of (3.4). The validity for all  $M = y_1^{8i+4}$  (and  $t = 3$ ) now follows similarly to what we did for  $M = y_1^{8i}$  at the beginning of this paragraph.

Now we switch our attention to the odd primes. The situation when  $p = 2$  is extremely similar. We want to prove the following version of (3.4).

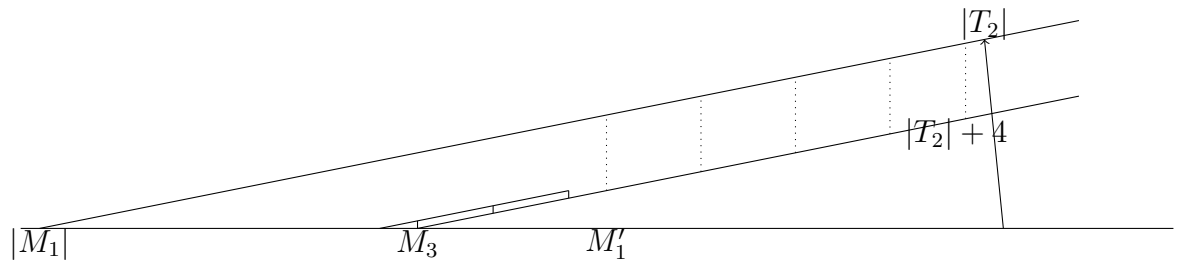
$$d_{p^t-t}(h_0^{t-1} v q y_1^{(i+1)p^{t-1}-1}) = v^{p^t} y_1^{ip^{t-1}} z_t. \quad (3.17)$$

Now we work toward proving this. We illustrate with  $p = 5$ , but it should be clear how it generalizes to an arbitrary prime. One new thing is the Divisibility Criterion as invoked in [7]. Each mod  $(p-1)$  value of  $i$  can be considered separately. We will consider (3.17) with  $p = 5$  and  $i = 4\ell$ ; other congruences follow similarly. We index the differential (3.17) by  $(\ell, t)$ . We write  $T$  (for vertical Tower) for the class  $h_0^{t-1} v \beta y_1^{(4\ell+1)5^{t-1}-1}$ , and  $M$  (for Monomial) is  $y_1^{4\ell 5^{t-1}} z_t$ . We write  $|T|$  for  $\frac{1}{2}(|T| + 1)$ . The  $\frac{1}{2}$  avoids extraneous factors of 2 that always cancel out. The  $+1$  is so that this

indicates the grading (times  $\frac{1}{2}$ ) of the class that it hits.  $|M|$  denotes  $\frac{1}{2}$  times the grading of  $M$ , and  $M'$  equals  $\frac{1}{2}$  times the grading of  $v^h M$ , where  $h$  is the  $v$ -height of  $M$  in  $k(1)^*(K_2)$ . We wish to show that the differentials *must be* as claimed.

Constraint C1 is that if  $T_2 \rightarrow M_1$  (by which we mean a certain  $T$  hits a certain  $M$ , with the possibility  $T_2 = T_1$  allowed), then  $|T_2| \leq M'_1$ . (This says that the  $v$ -tower on  $M_1$  cannot be hit while its image in  $k(1)^*(K_2)$  is nonzero.) Constraint C2 says that if  $T(5\ell + 1, t - 1) \rightarrow M_1$  and  $T(\ell, t) \rightarrow M_2$ , then  $|M_2| > |M_1|$ . Since  $|T(5\ell + 1, t - 1)| = |T(\ell, t)|$ , this says that as you move up an  $h_0$  tower, differentials must get longer (unless they are hitting into an  $h_0$  tower, which is not the case here.) Constraint C3 says that if  $T_2 \rightarrow M_1$ , then there exists  $M_3$  with  $|M_3| \geq M'_1$  and  $M'_3 \leq |T_2|$ . Alternatively, if  $T_3 \rightarrow M_3$  has already been proved, then  $|T_3| \leq |T_2|$  (and  $|M_3| \geq M'_1$ ). The reason for C3 is that there must be extensions into the  $M_1$ -tower from grading  $M'_1$  to  $|T_2| + 4$ . The nonzero classes on the  $v$ -tower (on  $M_3$ ) supporting the extensions must go to at least  $|T_2| + 4$ , and it has nonzero classes at least to  $M'_3 + 4$ , and if  $T_3 \rightarrow M_3$  was already proved, it has nonzero classes to  $|T_3| + 4$ . Note that we are saying that the  $v$ -tower on  $M_1$  maps to 0 in  $k(1)^*(K_2)$  once we get to grading  $M'_1$  (and hence in gradings  $\leq M'_1$  it is either hit by differentials or is divisible by  $p$ ). There might be classes of higher filtration in  $k(1)^*(K_2)$  to which it could map, but, if so, we can modify the generator of the  $M_1$  tower by the class on the tower sitting above it. Also note that it is possible that extensions from the tower  $M_3$  above don't start from the generator, if there are  $h_0$ -extensions on the tower for awhile. See Figure 3.18. There is an exception to the C3 requirement for  $T(\ell, 1) \rightarrow M(\ell, 1)$ . Here the extension into  $v^4 y_1^{4\ell} z_1$  is obtained from the special class  $y_1^{4\ell} y_0^4 z_0$ .

**Figure 3.18. The role of  $M_3$**



With the above conventions, we have  $|T| = 5^t(4\ell + 1) + 1$ ,  $|M| = 5^t(4\ell + 5) + 1$ , and  $M' = |M| - 4r'(t - 1)$ , where  $4r'(t - 1)$  has the values 16, 80, 412, and 2076 for  $t = 1, 2, 3$ , and 4. Increasing from  $t$  to  $t + 2$  increases this by  $4^2 \cdot 5^{t+1} - 4$ . We consider the cases in order of increasing  $|M|$  and, for equal values of  $|M|$ , increasing  $\ell$ . We tabulate a representative sample in Table 1. We omit listing values of  $\ell \equiv 3, 4 \pmod{5}$  because they behave similarly to  $\ell \equiv 2$ .

$\ell$	$t$	$ T $	$ M $	$M'$	$\ell$	$t$	$ T $	$ M $	$M'$
0	1	6	26	10	36	1	726	746	730
1	1	26	46	30	37	1	746	766	750
2	1	46	66	50	7	2	726	826	746
0	2	26	126	46	40	1	806	826	810
5	1	106	126	110	41	1	826	846	830
6	1	126	146	130	42	1	846	866	850
7	1	146	166	150	8	2	826	926	846
1	2	126	226	146	45	1	906	926	910
10	1	206	226	210	46	1	926	946	930
11	1	226	246	230	47	1	946	966	950
12	1	246	266	250	9	2	926	1026	946
2	2	226	326	246	50	1	1006	1026	1010
15	1	306	326	310	51	1	1026	1046	1030
16	1	326	346	330	52	1	1046	1066	1050
17	1	346	366	350	1	3	626	1126	714
3	2	326	426	346	10	2	1026	1126	1046
20	1	406	426	410	55	1	1106	1126	1110
21	1	426	446	430	56	1	1126	1146	1130
22	1	446	466	450	57	1	1146	1166	1150
4	2	426	526	446	11	2	1126	1226	1146
25	1	506	526	510	60	1	1206	1226	1210
26	1	526	546	530	61	1	1226	1246	1230
27	1	546	566	550	62	1	1246	1266	1250
0	3	126	626	214			$\vdots$		
5	2	526	626	546	154	1	3086	3106	3090
30	1	606	626	610	0	4	626	3126	1050
31	1	626	646	630	5	3	2626	3126	2714
32	1	646	666	650	30	2	3026	3126	3046
6	2	626	726	646	155	1	3106	3126	3110
35	1	706	726	710	156	1	3126	3146	3130

Table 1: Cases in order

We begin by noting that C1, C2, and C3 are satisfied by  $T(\ell, t) \rightarrow M(\ell, t)$ . C1 is quite clear from Table 1 and follows in general from  $r'(t-1) < p^t$ . C2 is immediate

from the construction of the table. C3 when  $t = 1$  was handled by the exception noted previously. C3 for  $T(\ell, t) \rightarrow M(\ell, t)$  is satisfied by  $M_3 = M(5\ell + 1, t - 1)$ , as  $|T(5\ell + 1, t - 1)| = |T(\ell, t)|$  and  $|M(5\ell + 1, t - 1)| \geq M'(\ell, t)$  follows from  $r'(t - 1) \geq p^t - p^{t-1}$ .

We now seek to show that all other possibilities are eliminated. We first note that if  $t \geq 3$  there are many cases  $M$  occurring between  $(5\ell + 1, t - 1)$  and  $(\ell, t)$  with  $M' > |T(\ell, t)|$  so that C1 allows the possibility of  $T(\ell, t) \rightarrow M$ . For example, if  $(\ell, t) = (1, 3)$ ,  $M$  could be any case between  $(36, 1)$  and  $(52, 1)$ . Assume that we have handled all cases through  $(5\ell + 1, t - 1)$ ; i.e., we have proved that  $T \rightarrow M$  must be as in the table. We seek to show that  $T(\ell, t)$  cannot hit any  $M$  preceding it in the list. By C2,  $T(\ell, t)$  cannot hit any  $M$  with  $|M| = |M(5\ell + 1, t - 1)|$ , and it cannot hit anything before  $(5\ell + 1, t - 1)$  because those cases have already been handled. We will use C3 to show it cannot hit anything above it with  $|M| > |M(5\ell + 1, t - 1)|$ . First note that  $M'_3 \leq |T(\ell, t)|$  is not satisfied by any  $M$ 's after  $(5\ell + 1, t - 1)$  since  $M' > |T| > |T(\ell, t)|$ , so those cannot work as  $M_3$ . The cases with  $|M| \leq |M(5\ell + 1, t - 1)|$  cannot work as  $M_3$  because they do not satisfy the  $|M_3| \geq M'_1$  criterion for C3.

Thus we have shown that if we are done through  $(5\ell + 1, t - 1)$ , then  $T(\ell, t)$  cannot hit any  $M$  preceding it in the list. Consider the smallest  $|M|$  such that, for some  $(\ell, t)$ ,  $T(\ell, t) \rightarrow M$  with  $M \neq M(\ell, t)$ . Then we have just seen that  $M$  cannot be between  $M(5\ell + 1, t - 1)$  and  $M(\ell, t)$ . It cannot be on or before  $M(5\ell + 1, t - 1)$  by C2, since  $|T(5\ell + 1, t - 1)| = |T(\ell, t)|$ . And it cannot come after  $M(\ell, t)$ , for then  $M(\ell, t)$  would not be hit by  $T(\ell, t)$ , contradicting minimality of  $|M|$ . This completes most of the proof of (3.17) and hence of Theorem 3.1.

Underlying the above analysis has been an assumption that the  $M$ -classes are always hit by  $T$ -classes. We show now that it could not have occurred that an  $M$ -class supported a differential. Assume that  $M = y_1^{ip^{t-1}} z_t$  is the  $M$ -class of lowest grading which supports a differential.

In  $k(1)^*(K_2)$ ,  $M$  supports a  $v$ -tower of  $v$ -height  $r'(t - 1)$  by 3.16. We will show at the end of the proof that there is a number  $\Delta \leq t$  such that  $v^i M$  maps nontrivially to  $ku^{*+1}(K_2)$  if and only if  $i \leq r'(t - 1) - \Delta$ . (Usually  $\Delta = 1$ .) The image of  $M$  in  $ku^{|M|+1}(K_2)$  is a class  $C$  of positive filtration such that  $v^{r'(t-1)-\Delta} C \neq 0$  and  $v^{r'(t-1)-\Delta+1} C = 0 \in ku^*(K_2)$ , so there must be a differential in the ASS of  $ku^*(K_2)$

from a filtration-0 class hitting a class of filtration  $\geq r'(t-1) - \Delta + 2$  in grading  $|M| + 1 - 2(p-1)(r'(t-1) - \Delta + 1)$ . (The reason that the differential must start from filtration 0 is that in even gradings,  $E_2$  consists entirely of  $v$ -towers starting in filtration 0.) This differential cannot come from another such  $M$  because of our lowest-grading assumption, and it cannot come from an  $M$  of higher grading because its grading is too small. It cannot come from a product of one or more  $z$ 's times one of these  $M$ 's because  $z$ 's are infinite cycles. We must rule out the possibility that this differential is one of type (3.3). They are distinguished by having the smallest  $z$ -subscript at least 2 greater than the  $p$ -exponent of the exponent of  $y_1$ .

The differential on  $C$  has subscript  $\geq r'(t-1) - \Delta + 2$ , and so the class in (3.3) would be  $y_1^{\ell p^{r'(t-1)-\Delta}} Z$  for some positive integer  $\ell$ , where  $Z$  is a product of  $z_j$ 's with  $j \geq r'(t-1) - \Delta + 2$ , and each  $j$  appears at most  $p-1$  times, except that the smallest  $j$  might appear  $p$  times. Equating this grading with  $|M| - 2(p-1)(r'(t-1) - \Delta + 1)$ , and cancelling a common factor 2 from all terms yields

$$\ell p^{r'(t-1)-\Delta+1} + \sum_j (p^{j+1} + 1) = ip^t + p^{t+1} + 1 - (p-1)(r'(t-1) - \Delta + 1). \quad (3.19)$$

Using (3.12) and (3.14) and  $\Delta \leq t$ , the right hand side of (3.19) equals  $p^t(i+1) + (p-1)(r(t-1) + \Delta - 1) + 1 \equiv (p-1)(r(t-1) + \Delta - 1) + 1 \pmod{p^t}$ , with  $(p-1)(r(t-1) + \Delta - 1) + 1 \leq p^t$  (strict if  $t > 2$ ). Since  $r'(t-1) - \Delta > t$ , this implies that the  $\sum_j$  on the left hand side of (3.19) must contain at least  $(p-1)(r(t-1) + \Delta - 1) + 1$  summands. We obtain

$$\begin{aligned} \sum p^j &\geq p \cdot p^{r'(t-1)-\Delta+2} + (p-1)(p^{r'(t-1)-\Delta+3} + \dots + p^{r'(t-1)+r(t-1)}) \\ &= p^{r'(t-1)+r(t-1)+1} = p^{p^t+1}, \end{aligned}$$

so  $\sum p^{j+1} \geq p^{p^t+2}$ , and hence  $p^t(i+1) > p^{p^t+2}$ . Thus  $i \geq p^{p^t-t+2} > p^{p^t-2t}$ .

Since  $d_{p^t-t+1}(y_1^{p^{p^t-t-1}})$  is defined,

$$d_r(y_1^{p^{p^t-t-1}}) = 0 \text{ for } r \leq p^t - t, \quad (3.20)$$

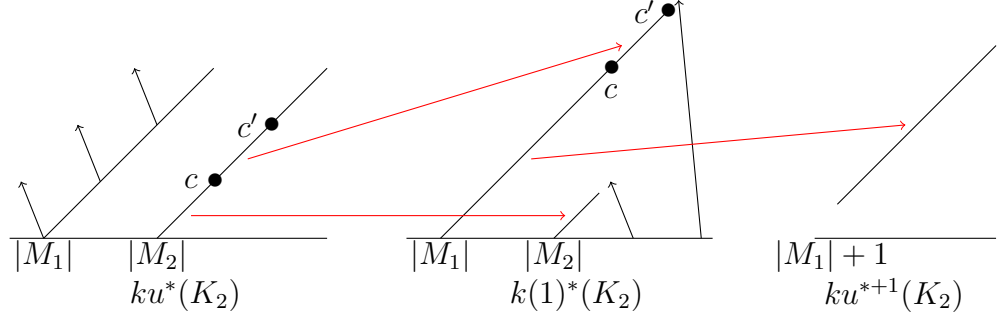
and by the lowest-grading assumption,  $d_{p^t-t}(h_0^{t-1} v q y_1^{(i-p^{p^t-2t}+1)p^{t-1}-1}) = v^{p^t} y_1^{(i-p^{p^t-2t})p^{t-1}} z_t$  and  $y_1^{(i-p^{p^t-2t})p^{t-1}} z_t$  is a permanent cycle. Since

$$y_1^{ip^{t-1}} z_t = y_1^{(i-p^{p^t-2t})p^{t-1}} z_t \cdot y_1^{p^{p^t-t-1}},$$

we deduce that  $y_1^{ip^{t-1}} z_t$  survives to  $E_{p^t-t}$  and (3.17), using the derivation property of differentials.

Now we consider the need for  $\Delta$  in the above argument. The worry is that maybe part of the  $v$ -tower on  $M$  in  $k(1)^*(K_2)$  might be in the image from  $ku^*(K_2)$ , due to a filtration jump from a lower tower, as sketched in Figure 3.21, so that only a smaller part of the  $M$ -tower in  $k(1)^*(K_2)$  maps to  $ku^{*+1}(K_2)$ .

**Figure 3.21. An unwanted possibility**



The monomials  $M_\varepsilon = y_{t_\varepsilon}^{i_\varepsilon} z_{t_\varepsilon}$  ( $\varepsilon = 1, 2$ ) have  $|M_\varepsilon| = 2(p^{t_\varepsilon}(i_\varepsilon + p) + 1)$  and are truncated in  $k(1)^*(K_2)$  in grading  $M'_\varepsilon = |M_\varepsilon| - 2(p-1)r'(t_\varepsilon - 1)$ . In  $ku^*(K_2)$ ,  $M_2$  is truncated in grading  $|T_2| = |v^{p^{t_2}} M_2| = 2(p^{t_2}(i_2 + 1) + 1)$ . In Figure 3.21, elements  $c$  are in grading  $M'_2$ , and  $c'$  is in grading  $M'_1 + 2(p-1)$ . The necessary condition for nontrivial image in  $k(1)^*(K_2)$  (and hence  $\Delta > 1$ ) is

$$|T_2| + 2(p-1) \leq M'_1 + 2(p-1) \leq M'_2. \quad (3.22)$$

If this occurs, then we might have  $\Delta$  as large as  $\frac{M'_2 - M'_1}{2(p-1)} + 1$ . We now show in Lemma 3.23 that if (3.22) holds, then  $(M'_2 - M'_1)/(2(p-1)) < t$ , establishing the claim made earlier about  $\Delta \leq t$ .

We restrict to  $p = 5$ ,  $i = 4\ell$  for simplicity, and so that the reader can refer to Table 1 as an aid. The argument easily generalizes to any prime and any congruence. We divide everything by 2 as was done above, and also subtract off the  $+1$  which occurs in formulas for  $|M|$  and  $|T|$ , so the numbers will be 1 smaller than those in the table.

**Lemma 3.23.** *If  $t_1 > t_2$  and*

$$5^{t_2}(4\ell_2 + 1) + 4 \leq 5^{t_1}(4\ell_1 + 5) - 4r'(t_1 - 1) + 4 \leq 5^{t_2}(4\ell_2 + 5) - 4r'(t_2 - 1),$$

then

$$\frac{1}{4}(5^{t_2}(4\ell_2 + 5) - 4r'(t_2 - 1) - (5^{t_1}(4\ell_1 + 5) - 4r'(t_1 - 1))) < t_1 - 1.$$

*Proof.* If there is a counterexample to this, then there is one with  $\ell_1 = 0$ , since  $\ell_2$  could be decreased by  $5^{t_1-t_2}\ell_1$ , so it suffices to use  $\ell_1 = 0$ . Let  $Q(k) = (5^{2k} - 1)/24$  (called  $q(k)$  in [7, Lemma 5.3]). Then, using [7, Lemma 5.5], for  $t = 2k + \delta$  with  $\delta = 1$  or  $2$ ,

$$5^{t+1} - 4r'(t - 1) = 5^{2k+\delta} + 16 \cdot 5^\delta Q(k) + 4k + 4 \cdot 5^{\delta-1}.$$

Since  $16 \cdot 5^\delta Q(k) + 4k + 4 \cdot 5^{\delta-1} < 3 \cdot 5^{2k+\delta}$ , the hypothesis of the lemma says that  $5^{t_1+1} - 4r'(t_1 - 1) \bmod 4 \cdot 5^{t_2}$  lies in the mod- $(4 \cdot 5^{t_2})$  interval  $[5^{t_2}, 5^{t_2+1} - 4r'(t_2 - 1) - 4]$ .

Let  $t_1 = 2k_1 + \delta_1$  and  $t_2 = 2k_2 + \delta_2$ . The condition is restated as

$$5^{2k_1+\delta_1} + 16 \cdot 5^{\delta_1} Q(k_1) + 4k_1 + 4 \cdot 5^{\delta_1-1} \quad (3.24)$$

lies in the mod- $(4 \cdot 5^{t_2})$  interval

$$[5^{t_2}, 5^{t_2} + 16 \cdot 5^{\delta_2} Q(k_2) + 4k_2 + 4 \cdot 5^{\delta_2-1} - 4]. \quad (3.25)$$

Let  $\delta_2 = 1$ . The reduction mod  $4 \cdot 5^{t_2}$  of (3.24) is

$$5^{t_2} + 16 \cdot 5^{\delta_1} Q(k_2) + 4k_1 + 4 \cdot 5^{\delta_1-1}. \quad (3.26)$$

Let  $\delta_1 = 2$ . Then  $5^{t_2} + 16 \cdot 5^{\delta_1} Q(k_2) > 4 \cdot 5^{t_2}$  and equals  $5^{2k_2+2} - (2000Q(k_2 - 1) + 100)$ , so (3.26) will first be in the interval (3.25) when  $4k_1 + 20 = 2000Q(k_2 - 1) + 100$ , hence  $k_1 = 500Q(k_2 - 1) + 20$ , so  $t_1 = 1000Q(k_2 - 1) + 42$ . The left hand side of the conclusion of the lemma is  $\frac{1}{8}(M'_2 - M'_1)$  with  $M'_1$  and  $M'_2$  as in (3.22). For  $k_1 = 500Q(k_2 - 1) + 20$ , the value of  $M'_1$  is at the left end of the interval (3.25), and so  $\frac{1}{8}(M'_2 - M'_1)$  equals  $\frac{1}{4}$  times the length plus 4 of (3.25), which is

$$20Q(k_2) + k_2 + 1 = 500Q(k_2 - 1) + k_2 + 21 = \frac{1}{2}t_1 + k_2.$$

Since  $k_2 \ll t_1$ , this is less than  $t_1 - 1$ , as claimed. If  $k_1$  is increased from the value  $500Q(k_2 - 1) + 20$ , the value of  $t_1$  increases, while  $M'_2 - M'_1$  decreases, since  $M'_1$  is moving through the interval, so the inequality asserted in the lemma is satisfied more strongly.



Now, with  $\delta_2 = 1$  continuing, let  $\delta_1 = 1$ . Since  $k_1 > k_2$ , (3.26) lies outside the interval (3.25) until  $80Q(k_2) + 4k_1 + 4 = 4 \cdot 5^{t_2}$ , so

$$k_1 = 5^{2k_2+1} - 20Q(k_2) - 1 = 100Q(k_2) + 4$$

and  $t_1 = 200Q(k_2) + 9$ . Again  $\frac{1}{8}(M'_2 - M'_1) = 20Q(k_2) + k_2 + 1 \approx \frac{1}{10}t_2 + k_2$ , so the conclusion of the lemma is satisfied more strongly.

A similar analysis works when  $\delta_2 = 2$ . In this case  $\frac{1}{8}(M'_2 - M'_1) \approx \frac{1}{2}t_1 + k_2$  if  $\delta_1 = 1$ , and  $\frac{1}{8}(M'_2 - M'_1) \approx \frac{1}{10}t_1 + k_2$  if  $\delta_1 = 2$ . ■

We close this section by explaining how Theorems 2.3 and 3.1 lead to the descriptions of  $ku^*(K_2)$  given in Theorems 1.4 and 1.11, modulo exotic extensions. We begin with the portion in even gradings and restrict our attention to odd  $p$ . All elements in the  $P[h_0, v, y_1]$  part of Theorem 2.3 support differentials of type (3.2). Note that  $y_0^{p^k-1} = y_0^{p-1}y_1^{p^{k-1}-1} = \prod_{j=0}^{k-1} y_j^{p-1}$ . The first is easiest to write, the second occurs in Theorem 2.3, and the third in 1.1 and Figure 1.7. From 1.1,  $y_0^{p^k-1}z_0$  is in  $A_k$  for  $k \geq 1$ , the bottom right element in Figure 1.7. Then

$$P[y_1]y_0^{p-1}z_0 = \bigoplus \mathcal{M}_k^A \cdot y_0^{p^k-1}z_0 \subset \bigoplus \mathcal{M}_k^A A_k. \quad (3.27)$$

The first part occurs in Theorem 2.3 and the last part in Theorem 1.4.

Now we consider  $P[y_1] \otimes \bigoplus_{j \geq 1} W_j \otimes TP_{p-1}[z_j] \otimes \Lambda_{j+1}$  in Theorem 2.3. The  $\bigoplus$  part is all monomials  $z_\ell M$  with  $\ell \geq 1$  and  $M \in \Lambda_\ell$ . From Theorem 3.1,  $y_1^i z_\ell M$  supports a differential (3.3) if  $\ell \geq \nu(i) + 2$ , while those with  $\nu(i) \geq \ell - 1$  are hit by differentials (3.4) and (3.5), yielding  $v$ -towers with heights as given in 1.1. These are all monomials in  $\bigoplus_{\ell \geq 1} P[y_\ell, y_{\ell+1}, \dots] z_\ell \Lambda_\ell$ . From 1.1 or (1.3), the generators of the  $v$ -towers in  $B_k$  are all

$$z_j \prod_{i=j}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}, \quad 1 \leq j \leq k.$$

Let  $(z_\ell M)_i$  be the  $y_i^e z_i^{e'}$  factors of  $M$ . Then  $\mathcal{M}_k B_k$  consists of all monomials  $z_\ell M$  such that  $(z_\ell M)_i$  equals  $y_i^{p-1}$  or  $z_i^{p-1}$  for  $i < k$ , but not for  $\ell \leq i = k$ , and so every monomial  $z_\ell M$  is in a unique  $\mathcal{M}_k B_k$ . From Theorem 3.1,  $z_\ell M$  has  $v$ -height  $p^\ell$  if and only if  $M$  contains no  $z$ -factors, which explains the split into  $\mathcal{M}_k^A$  and  $\mathcal{M}_k^B$  in Theorem 1.4.

Now we address the odd gradings. The  $P[h_0, v, y_1]vq$  part of Theorem 2.3 is totally removed either as sources (3.4) or targets (3.2) of differentials. See grading 17 in Figure 3.7 for a nice illustration. The  $qy_1^{i-1}S_{\nu(i)+1,\ell}$  part of Theorem 1.11 is formed from  $TP_{\nu(i)+2}[v]qy_1^{i-1}W_\ell$  in 2.3 using (3.3). The generators of  $S_{\nu(i)+1,\ell}$  are  $z_{1,\ell}, \dots, z_{\ell-\nu(i)-1,\ell}$ , but to see the differential from (3.3), one should write  $z_{t,\ell} = z_{t,t+\nu(i)+1}Z_{t+\nu(i)+1}^\ell$ , where

$$Z_i^j = (z_i \cdots z_{j-1})^{p-1} \text{ for } j > i, \text{ with } Z_i^i = 1. \quad (3.28)$$

The remaining generators of  $qy_1^{i-1}W_\ell$ , namely  $qy_1^{i-1}z_{j,\ell}$  with  $\ell - \nu(i) \leq j \leq \ell$ , support differentials (3.5).

#### 4. EXOTIC EXTENSIONS

In this section, we prove the following expansion of (1.2).

**Theorem 4.1.** *If  $i \geq 0$  and  $k \geq k_0$ ,*

$$py_k^i y_{k-1}^{p-1} z_{k-1} = v^{p^{k-1}(p-1)} y_k^i z_k$$

*with an additional term  $vy_k^i y_{k-1}^{p-1} z_{k-2}^p$  if  $k \geq k_0 + 2$ .*

The additional term is seen in Ext, and will be ignored in the rest of this section. We have included the factor  $y_k^i$ , which is not automatic since  $y_k^i$  is not a permanent cycle. Since, for example,  $y_{k+1} = y_k^p$ , we need not consider  $y_i$  for  $i > k$ . It is automatic that this formula can be multiplied by  $z_j$ 's, since they do survive the spectral sequence.

The extension is deduced from the exact sequence

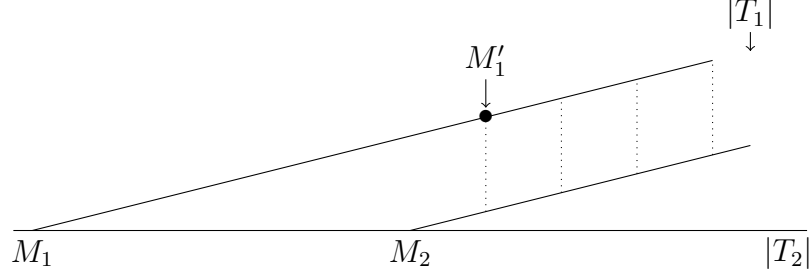
$$ku^*(K_2) \xrightarrow{p} ku^*(K_2) \longrightarrow k(1)^*(K_2)$$

and the fact that  $v^{r'(k-1)} y_k^i z_k = 0$  in  $k(1)^*(K_2)$  with  $r'(k-1) \geq p^k(p-1)$ . Thus  $v^{r'(k-1)} y_k^i z_k$  must be divisible by  $p$  in  $ku^*(K_2)$ , and, as we will show, the  $v$ -tower on  $y_k^i y_{k-1}^{p-1} z_{k-1}$  provides the only classes that can do the dividing. Once we know the division formula toward the end of the  $v$ -tower, we can deduce that it holds earlier in the tower, as well. For example,  $r'(2) = p^3 - p^2 + p - 2$ , which is the height in the top  $v$ -tower in Figure 1.7 where the extensions into it do not also involve an  $h_0$ -extension. We deduce the extensions from the earlier part of the  $v$ -tower on  $y_2^{p-1} z_2$  by naturality.

We illustrate in Figure 4.2, using the notation of the preceding section. Thus  $T_i$  is the class satisfying  $d_r(T_i) = v^r M_i$ . Here the portion of the top tower to the

right of  $M'_1$  must be divisible by  $p$ . The tower providing the extension must have  $M'_1 \leq |M_2| < |M_1|$  and  $|T_2| \leq |T_1|$ .

**Figure 4.2. Conditions for extension**



As we did for the differentials in the previous section, we will perform the argument for  $p = 5$ . It will be clear that it generalizes to an arbitrary odd prime, and with minor modification to  $p = 2$ . Also, we use  $i = 4\ell$  in Theorem 4.1. If instead we used  $i = 4\ell + d$  for  $1 \leq d \leq 3$ , it will just add the same amount to the quantities  $|M|$ ,  $|T|$ , and  $M'$  involved in the argument. We can use Table 1 to envision the analysis, with the  $t$  there replaced by  $k$ . For a monomial  $M(\ell, k) = y_k^{4\ell} z_k$ , we have, after dividing by 2,  $|M| = 5^k(4\ell+5)+1$ ,  $|T| = 5^k(4\ell+1)+1$ , and  $5^k(4\ell+1.16)+1 < M' \leq 5^k(4\ell+1.8)+1$ , using (3.15). With  $M_1$  and  $M_2$  as in Figure 4.2, we will show that  $M_2(5\ell + 1, k - 1)$  is the unique monomial satisfying the inequalities stated just before Figure 4.2 for  $M_1(\ell, k)$ . Note that  $M(5\ell + 1, k - 1) = y_k^{4\ell} y_{k-1}^4 z_{k-1}$ . We omit the  $+1$  in all the formulas.

The inequalities are satisfied by  $M_2(5\ell + 1, k - 1)$  since

$$5^k(4\ell+1.8) \leq 5^{k-1}(4(5\ell+1)+5) < 5^k(4\ell+5) \text{ and } 5^{k-1}(4(5\ell+1)+1) \leq 5^k(5\ell+1).$$

If  $k_2 \geq k$ , then the first inequality, after dividing by  $5^k$ , becomes

$$4\ell + 1.8 \leq 5^{k_2-k}(4\ell_2 + 5) < 4\ell + 5,$$

which cannot be satisfied since the middle term is  $\equiv 1 \pmod{4}$ . If  $k_2 < k - 1$ , then

$$M'_1 - |T_1| > 5^k \cdot .16 \geq 4 \cdot 5^{k_2} = |M_2| - |T_2|,$$

which is inconsistent with two of the inequalities. Let  $k_2 = k - 1$ . If  $\ell_2 < 5\ell + 1$ , then

$$|M_2| = 5^{k-1}(4\ell_2 + 5) \leq 5^{k-1}(4 \cdot 5\ell + 5) < 5^k(4\ell + 1.16) < M'_1,$$

contradicting one of the inequalities. If  $k_2 = k - 1$  and  $\ell_2 > 5\ell + 1$ , then

$$|T_2| \geq 5^{k-1}(4(5\ell + 2) + 1) > 5^k(4\ell + 1) = |T_1|,$$

contradicting one of the inequalities.

We deduce that  $M_2 = y_k^{4\ell} y_{k-1}^4 z_{k-1}$ , as claimed. We should perhaps have noted that the extensions could not have come from classes with more than one  $z_j$ -factor, because these are  $z_j$  times a class on which the extensions have already been determined.

## 5. PROPOSED FORMULAS FOR THE EXACT SEQUENCE (1.21)

In this section we propose what we feel must be the correct complete formulas for the exact sequence (1.21). Some homomorphisms are forced by naturality, but many others involve significant filtration jumps. However, they all occur in several families with nice properties. The 10-term exact sequence (5.2) shows how the  $S_{k,\ell}$  portions and the exotic extensions yield compatibility of the differing  $v$ -tower heights in  $ku^*(K_2)$  and  $k(1)^*(K_2)$ . In Section 6, we show that all elements of  $k(1)^*(K_2)$  are accounted for exactly once in these homomorphisms, which implies that there can be no more exotic extensions. This does not require us to prove that our homomorphism formulas are actually correct, as discussed at the end of Section 1. We will focus on the case when  $p$  is odd. We could incorporate all primes together at the expense of involving the parameter  $k_0$ , but things are complicated enough without that. In an earlier version of this paper ([6]), a thorough analysis when  $p = 2$  was performed.

We propose that (1.21) can be split into exact sequences of length 4 and 10 (not including 0's at the end). There are subgroups of  $k(1)^*(K_2)$  called  $G_k^1$  and  $G_k^2$  for  $k \geq 1$  and  $G_{k,\ell}^i$  for  $3 \leq i \leq 6$  and  $1 \leq k < \ell$  such that there are exact sequences

$$0 \rightarrow G_k^1 \rightarrow A_k \xrightarrow{p} A_k \rightarrow G_k^2 \rightarrow 0 \quad (5.1)$$

for  $k \geq 1$ , and, for  $1 \leq k < \ell$ ,

$$\begin{aligned} 0 &\rightarrow G_{k,\ell}^3 \rightarrow y_k B_k Z_k^\ell \xrightarrow{p} y_k B_k Z_k^\ell \rightarrow G_{k,\ell}^4 \rightarrow y_1^{p^{k-1}-1} q S_{k,\ell} \\ &\xrightarrow{p} y_1^{p^{k-1}-1} q S_{k,\ell} \rightarrow G_{k,\ell}^5 \rightarrow B_k z_\ell \xrightarrow{p} B_k z_\ell \rightarrow G_{k,\ell}^6 \rightarrow 0, \end{aligned} \quad (5.2)$$

with  $Z_k^\ell$  as defined in (3.28). The sequence (5.1) can be tensored with  $TP_{p-1}[y_k] \otimes P[y_{k+1}]$ , while (5.2) can be tensored with  $TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}$ . If

$p$  is odd, there are also exact sequences

$$0 \rightarrow G_{k,e}^7 \rightarrow B_k z_k^e \xrightarrow{p} B_k z_k^e \rightarrow G_{k,e}^8 \rightarrow 0 \quad (5.3)$$

for  $k \geq 1$  and  $1 \leq e \leq p-2$ . This can be tensored with  $P[y_k] \otimes \Lambda_{k+1}$ .

One can verify that the totality of  $A_k$  and  $B_k$  groups in these exact sequences agrees with that in Theorem 1.4. We will study these exact sequences by breaking them up into short exact sequences and isomorphisms involving kernels and cokernels of  $\cdot p$ .

Let  $K_k^A = \ker(\cdot p|A_k)$ ,  $K_k^B = \ker(\cdot p|B_k)$ ,  $C_k^A = \text{coker}(\cdot p|A_k)$ , and  $C_k^B = \text{coker}(\cdot p|B_k)$ . There are important elements  $g_k \in K_k^A$  and  $K_k^B$  defined (up to unit coefficients) by  $g_1 = z_1$ ,  $g_2 = v^{p-2}z_2$ , and, for  $k \geq 1$ ,

$$g_{k+2} = v^{r'(k)-1}z_{k+2} + g_k y_k^{p-1} z_{k+1}^{p-1}. \quad (5.4)$$

To see that this is in  $\ker(\cdot p)$ , we use (1.2) to see that  $p \cdot v^{r'(k)-1}z_{k+2} = v^{r'(k)}z_{k+1}^p$ , and that the  $v^{r'(k-2)-1}z_k$  term in  $g_k$  yields  $v^{r'(k-2)-1}v^{p^k(p-1)}z_{k+1}z_{k+1}^{p-1}$  in  $p \cdot g_k y_k^{p-1} z_{k+1}^{p-1}$ . Using (3.10), these terms cancel. Other terms in  $p \cdot g_k y_k^{p-1} z_{k+1}^{p-1}$  yield 0 since  $g_k \in \ker(\cdot p)$ .

The  $v$ -towers in  $K_k^A$  are generated by

$$g_k \text{ and } g_j z_j^{p-1} \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}, \quad 1 \leq j \leq k-1. \quad (5.5)$$

For example, using Figure 1.7 when  $k = 3$ , these are  $g_3 = v^{p^2-p-1}z_3 + y_1^{p-1}z_1z_2^{p-1}$ ,  $g_2z_2^{p-1} = v^{p-2}z_2^p$ ,  $g_1z_1^{p-1}z_2^{p-1}$ , and  $g_1z_1^{p-1}y_2^{p-1}$ . The  $v$ -heights are  $p^k - (r'(k-2) - 1)$  for  $g_k$ , and  $p^j - j - (r'(j-2) - 1)$  for the others, since they are determined by  $v$ -heights of  $z_j$  in  $B_k$ . The map  $G_k^1 \rightarrow K_k^A$  sends  $w_k$  to  $g_k$  and

$$w_j P \mapsto g_j P \text{ for } P = z_j^{p-1} \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}, \quad (5.6)$$

with  $w_j$  as in 3.8 and 3.16. The  $v$ -height of  $w_j$  is  $r(j)$  if it is not accompanied by  $z_j$ , and  $r'(j-1)$  if it is. By (3.13) and ((3.12) and (3.11)) the  $v$ -heights agree, so (5.6) is an isomorphism on  $v$ -towers.

For  $L = K_k^A$  or  $K_k^B$  or  $C_k^A$  or  $C_k^B$ , we say that a  $\mathbb{Z}_p$  in  $L$  is a class of  $v$ -height 1 in  $L$  which is not part of a larger  $v$ -tower in  $L$ . There is one  $\mathbb{Z}_p$  in  $K_3^A$ , as can be seen in Figure 1.7. This is the element  $v^{p-2}y_1^{p-1}z_1z_2^{p-1}$ . Note that for  $i < p-1$ ,  $v^i y_1^{p-1} z_1 z_2^{p-1} + v^{i+p^2-p-1} z_3$  is part of a  $v$ -tower in  $K_3^A$ , which continues with the

elements  $v^i z_3$  for  $i > p^2 - 3$ , but  $v^i y_1^{p-1} z_1 z_2^{p-1}$  itself is in  $K_3^A$  only for  $i = p - 2$ . Using 1.1, we find that the  $\mathbb{Z}_p$ 's in  $K_k^A$  are

$$v^{p^t-t-1}(y_t \cdots y_{j-1})^{p-1} z_t z_j^{p-1} \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\} \text{ for } 1 \leq t < j < k. \quad (5.7)$$

For example, the elements  $v^{p-2}(y_1 y_2)^{p-1} z_1$  and  $v^{p^2-3} y_2^{p-1} z_2$  in Figure 1.7 yield elements in  $K_4^A$  after being multiplied by  $z_3^{p-1}$ . The basic formula for the homomorphism from part of  $k(1)^*(K_2)$  to  $\mathbb{Z}_p$ 's in various  $K_k^A$  and  $K_k^B$ , possibly tensored with other classes as in Theorem 1.4, is

$$(q(y_1 \cdots y_t)^{p-1} z_{j-t,j} \mapsto v^{p^t-t-1} y_t^{p-1} z_t z_j) \otimes P[y_j] \otimes TP_{p-1}[z_j] \otimes \Lambda_{j+1} \text{ for } j > t \geq 1. \quad (5.8)$$

The domain elements are in the first half of the third line of Theorem 3.16. The ones that are in  $G_k^1$  in the isomorphism  $G_k^1 \rightarrow K_k^A$  can be extracted using (5.7).

The isomorphism  $G_{k,\ell}^3 \rightarrow y_k K_k^B Z_k^\ell$  in (5.2) is given using formulas analogous to (5.6) and (5.8). There are several minor differences. One is that the  $v$ -tower on  $y_k g_k Z_k^\ell$  is truncated due to  $v^{p^k-k} z_k = 0$  in  $B_k$  (as opposed to  $v^{p^k} z_k = 0$  in  $A_k$ ). This is compatible with the fact that the  $v$ -height of  $w_k z_k$  in  $k(1)^*(K_2)$  is  $k$  less than that of  $w_k$ , using Theorem 3.16 and (3.11). The other is that  $K_k^B$  has additional  $\mathbb{Z}_p$ 's

$$v^{p^t-t-1}(y_t \cdots y_{k-1})^{p-1} z_t \text{ for } 1 \leq t \leq k-1, \quad (5.9)$$

as seen in Figure 1.7 when  $k = 3$ , but these are always multiplied by higher  $z$ 's, and so (5.8) applies.

The isomorphisms  $C_k^A \rightarrow G_k^2$  and  $C_k^B z_\ell \rightarrow G_{k,\ell}^6$  are defined simply by sending an element to one with the same name. Moreover  $C_k^A = C_k^B$  except for  $(y_0 \cdots y_{k-1})^{p-1} z_0 \in C_k^A - C_k^B$ . When  $k = 3$ , we see that the  $\mathbb{Z}_p$ 's in  $C_k^B$  are  $\{z_1^p z_2^{p-1}, z_2^p, y_2^{p-1} z_1^p\}$  in Figure 1.7.<sup>2</sup> For future reference,

$$\mathbb{Z}_p \text{'s in } C_k^B \text{ are } \{z_t^p \prod_{i=t+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\} : 1 \leq t < k\}. \quad (5.10)$$

The corresponding elements in  $k(1)^*(K_2)$  are from the third line of 3.16.

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<sup>2</sup>The class  $y_2^{p-1} z_1^{p-1}$  should really be called  $y_2^{p-1} z_1^{p-1} + v^{p^2(p-1)-1} z_3$  so that  $v$  times it is divisible by  $p$ , hence 0 in  $C_k^B$ , but we will ignore this fine-tuning.

The  $v$ -towers in  $C_k^A = C_k^B$  are generated by

$$z_k \text{ and } y_t^{p-1} z_t \prod_{i=t+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}, \quad 1 \leq t < k. \quad (5.11)$$

We will show that the  $v$ -height of  $z_k$  in  $C_k^B$  is  $r'(k-1)$ , which equals its  $v$ -height in  $k(1)^*(K_2)$ . It follows from 1.1 that the  $v$ -height of  $y_t^{p-1} z_t \prod_{i=t+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}$  equals  $r'(t-1)$ , establishing the isomorphisms out of  $C_k^A$  and  $C_k^B z_\ell$ . In Figure 1.7, the  $v$ -height of  $z_3$  equalling  $p^3 - p^2 + p - 2 = r'(2)$  is apparent.

The proof of the claim about  $v$ -heights is by induction. By (3.10),  $r'(k-1) - r'(k-3) = p^{k-1}(p-1) - 1$ . Let  $D = (|z_k| - |y_{k-1}^{p-1} z_{k-1}|)/(2(p-1)) = p^{k-1}(p-1)$ . This is the filtration on the  $z_k$ -tower above the element  $y_{k-1}^{p-1} z_{k-1}$ . We show that  $v^{i-1+D} z_k$  is divisible by  $p$  if and only if  $v^i z_{k-2}$  is divisible by  $p$ . Thus the difference of the  $v$ -heights in cokernels equals the difference of the corresponding  $r'$  values. From Theorem 4.1, we have

$$p v^{i-1} y_{k-1}^{p-1} z_{k-1} = v^{i-1+D} z_k + v^i y_{k-1}^{p-1} z_{k-2}^p.$$

The claim follows, since  $v^i y_{k-1}^{p-1} z_{k-2}^p$  is divisible by  $p$  if and only if  $v^i z_{k-2}$  is, by 1.1.

The analysis of (5.3) is extremely similar.

Now  $S_{k,\ell}$  becomes involved. Let  $S_{k,\ell}^K = \ker(\cdot p | S_{k,\ell})$  and  $S_{k,\ell}^C = \text{coker}(\cdot p | S_{k,\ell})$ . Then  $S_{k,\ell}^K$  consists of  $TP_{k+1}[v]\langle z_{1,\ell} \rangle$  plus  $\mathbb{Z}_p$ 's on  $v^k z_{i,\ell}$  for  $2 \leq i \leq \ell - k$ , while  $S_{k,\ell}^C$  has  $TP_{k+1}[v]\langle z_{\ell-k,\ell} \rangle$  plus  $\mathbb{Z}_p$ 's on  $z_{i,\ell}$  for  $1 \leq i < \ell - k$ . Next we consider the short exact sequence

$$0 \rightarrow y_k C_k^B Z_k^\ell \xrightarrow{\phi} G_{k,\ell}^4 \xrightarrow{\psi} y_1^{p^{k-1}-1} q S_{k,\ell}^K \rightarrow 0. \quad (5.12)$$

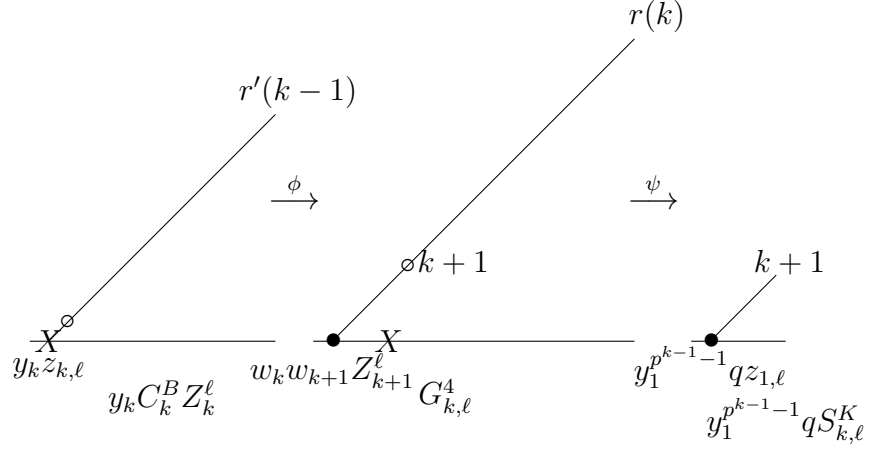
The map  $\phi$  sends everything except the  $v$ -tower on  $y_k z_k Z_k^\ell$  to classes with the same name, and the heights of these  $v$ -towers agree, as seen above. The class  $y_k z_k Z_k^\ell = y_k z_{k,\ell}$  maps to a  $\mathbb{Z}_p$  with the same name in  $k(1)^*(K_2)$ . We have  $\psi(w_k w_{k+1} Z_{k+1}^\ell) = q y_1^{p^{k-1}-1} z_{1,\ell}$ . Then  $v^{k+1} w_k w_{k+1} Z_{k+1}^\ell \in \ker(\psi)$ , and we have

$$\phi(v y_k z_{k,\ell}) = v^{k+1} w_k w_{k+1} Z_{k+1}^\ell.$$

We illustrate this in the schematic Figure 5.13, in which  $X$ ,  $\circ$ , and  $\bullet$  map to elements with the same symbol. The expressions at the end of the  $v$ -towers are their  $v$ -heights. In particular,  $v^{r'(k-1)} y_k z_{k,\ell} = 0$  in  $y_k C_k^B Z_k^\ell$ . The  $v$ -heights agree by (3.11),

and the gradings match by an induction proof. The  $\mathbb{Z}_p$ 's in  $y_1^{p^{k-1}-1}qS_{k,\ell}^K$  are hit by  $\psi(y_k z_{i+k-1,\ell}) = y_1^{p^{k-1}-1}qv^k z_{i,\ell}$ ,  $2 \leq i \leq \ell - k$ , another interesting filtration jump.

**Figure 5.13. Towers in exact sequence.**



Finally we consider the short exact sequence

$$0 \rightarrow y_1^{p^{k-1}-1}qS_{k,\ell}^C \xrightarrow{\phi'} G_{k,\ell}^5 \xrightarrow{\psi'} K_k^B z_\ell \rightarrow 0. \quad (5.14)$$

Similarly to (5.5), the generators of  $v$ -towers in  $K_k^B$  are  $g_k$  and, for  $1 \leq j < k$ , elements of the form  $g_j z_j^{p-1} \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}$ . The morphism  $\psi'$  is determined by  $w_j \mapsto g_j$ . The  $v$ -heights of the corresponding elements in  $k(1)^*(K_2)$  and  $K_k^B$  both equal  $r'(j-1)$  for  $j < k$ . However, the  $v$ -height of  $w_k z_\ell$  is  $r(k)$ , which is  $k$  greater than  $r'(k-1)$ . We have  $\phi'(y_1^{p^{k-1}-1}qz_{\ell-k,\ell}) = v^{r'(k-1)}w_k z_\ell$ . The class  $y_1^{p^{k-1}-1}qz_{\ell-k,\ell}$  at the base of the  $v$ -tower maps to a  $\mathbb{Z}_p$  with the same name. The picture is quite similar to Figure 5.13 with  $k+1$  and  $r'(k-1)$  interchanged.

The  $\mathbb{Z}_p$  classes  $y_1^{p^{k-1}-1}qz_{i,\ell}$  for  $1 \leq i < \ell - k$  are mapped by  $\phi'$  to classes with the same name in  $G_{k,\ell}^5 \subset k(1)^*(K_2)$ . The  $\mathbb{Z}_p$ 's in  $K_k^B z_\ell$  are of the same form as in (5.7), and are hit by analogues of (5.8).

## 6. ALL ACCOUNTED FOR

In this section, we show that all elements of  $k(1)^*(K_2)$  are involved in exactly one of the homomorphisms involving some  $G$ -group described in the preceding section. As discussed earlier, this implies that there can be no exotic extensions in  $ku^*(K_2)$  other than those in (1.2), because an additional extension would decrease the number



of elements in  $\ker(\cdot p|ku^*(K_2))$  and  $\text{coker}(\cdot p|ku^*(K_2))$ , and these must correspond to elements of  $k(1)^*(K_2)$ . It also provides an excellent check on our analysis.

Let  $p$  be odd, and

$$G^i = \begin{cases} \bigoplus_{k \geq 1} G_k^i \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] & 1 \leq i \leq 2 \\ \bigoplus_{1 \leq k < \ell} G_{k,\ell}^i \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1} & 3 \leq i \leq 6 \\ \bigoplus_{k \geq 1} \bigoplus_{e=1}^{p-2} G_{k,e}^i \otimes P[y_k] \otimes \Lambda_{k+1} & 7 \leq i \leq 8. \end{cases}$$

**Theorem 6.1.**  $G^1 \oplus \cdots \oplus G^8$  equals  $k(1)^*(K_2)$ , as described in Theorem 3.16.

As throughout the paper,  $\mathbb{Z}_p$ 's coming from  $E_1$ -submodules of  $H^*(K_2)$  are ignored here. The remainder of this section is devoted to the proof of Theorem 6.1. There are four parts of Theorem 3.16. We deal with them one-at-a-time.

**Case 1.**  $P[y_1]y_0^{p-1}z_0$ . In (3.27), it is shown that these classes form a subset of  $\bigoplus \mathcal{M}_k^A A_k$ , and they map to classes with the same name in  $G^2$ .

**Case 2.**  $\bigoplus_{j>0} TP_{r(j)}[v] \otimes P[y_{j+1}] \otimes TP_{p-1}[y_j] \otimes \bar{E}[w_j] \otimes E[w_{j+1}] \otimes \Lambda_{j+1}$ . The generators of  $v$ -towers of height  $r(j)$  occur in  $G^1$ ,  $G^4$ , and  $G^5$ . From (5.6), only  $w_j$  is in  $G_j^1$ . So  $G^1$  has  $TP_{p-1}[y_j] \otimes P[y_{j+1}]w_j$ . From Figure 5.13,  $G_{j,\ell}^4$  has  $w_j w_{j+1} Z_{j+1}^\ell$ . Note that  $\bigoplus_\ell Z_{j+1}^\ell TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1} = \Lambda_{j+1}$ , since the  $\ell$ -component gives the monomials whose smallest non- $(p-1)$ -power is a power of  $z_\ell$ , so  $G^4$  contains  $P[y_{j+1}] \otimes TP_{p-1}[y_j]w_j w_{j+1} \otimes \Lambda_{j+1}$ . From the analysis following (5.14),  $G_{j,\ell}^5$  has only  $w_j z_\ell$  of  $v$ -height  $r(j)$ , so  $G^5$  will have  $P[y_{j+1}] \otimes TP_{p-1}[y_j]w_j \otimes \bar{\Lambda}_{j+1}$ . Thus  $G^1 \oplus G^5$  contains the part without  $w_{j+1}$ , while  $G^4$  contains the part with  $w_{j+1}$ .

**Case 3.**  $\bigoplus_{j \geq 1} TP_{r'(j-1)}[v] \otimes P[y_j] \otimes E[w_j] \otimes \overline{TP}_p[z_j] \otimes \Lambda_{j+1}$ . The generators of  $v$ -towers of height  $r'(j-1)$  occur in each  $G^i$  as follows.

$G^1$ :  $w_j z_j^{p-1} \bigoplus_{k \geq j+1} TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes \bigoplus_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}$ . This can be deduced from (5.6).

$G^2$ : From (5.11),

$$z_j TP_{p-1}[y_j] \otimes P[y_{j+1}] \oplus y_j^{p-1} z_j \bigoplus_{k \geq j+1} TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}.$$

$G^3$ : We use (5.5) and (5.6) and adapt some arguments used in Case 2 to obtain

$$w_j z_j^{p-1} \left( \overline{TP}_p[y_j] \otimes P[y_{j+1}] \otimes \Lambda_{j+1} \oplus \bigoplus_{k \geq j+1} \overline{TP}_p[y_k] P[y_{k+1}] z_k^{p-1} \Lambda_{k+1} \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\} \right).$$

$G^4$ : We use (5.11) and (5.12) to obtain

$$y_j^{p-1} z_j \bigoplus_{k \geq j+1} \overline{TP}_p[y_k] \otimes P[y_{k+1}] z_k^{p-1} \Lambda_{k+1} \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}.$$

$G^5$ : We use (5.14) and  $\bigoplus_{\ell > k} z_\ell TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1} \approx \overline{\Lambda}_{k+1}$  to obtain

$$w_j z_j^{p-1} \bigoplus_{k \geq j+1} TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes \overline{\Lambda}_{k+1} \otimes \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}.$$

$G^6$ : We combine the analysis for  $G^2$  and the observation used for  $G^5$  to obtain

$$\begin{aligned} & z_j TP_{p-1}[y_j] \otimes P[y_{j+1}] \otimes \overline{\Lambda}_{j+1} \\ & \oplus y_j^{p-1} z_j \bigoplus_{k \geq j+1} TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes \overline{\Lambda}_{k+1} \otimes \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\} \end{aligned}$$

$G^7$ : Similarly to  $G^3$ , we have

$$\bigoplus_{e=1}^{p-2} \left( w_j z_j^e \otimes P[y_j] \otimes \Lambda_{j+1} \oplus w_j z_j^{p-1} \bigoplus_{k \geq j+1} z_k^e \otimes P[y_k] \otimes \Lambda_{k+1} \otimes \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\} \right).$$

$G^8$ : Using (5.11), we get

$$\bigoplus_{e=1}^{p-2} \left( z_j^e \otimes P[y_j] \otimes \Lambda_{j+1} \oplus y_j^{p-1} z_j \bigoplus_{k \geq j+1} z_k^e \otimes P[y_k] \otimes \Lambda_{k+1} \otimes \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\} \right).$$

We begin by analyzing the portion including the factor  $w_j$ . We will show that

$$G^1 \oplus G^3 \oplus G^5 \oplus G^7 = P[y_j] w_j \otimes \overline{TP}_p[z_j] \otimes \Lambda_{j+1}.$$

Here, and in the remainder of our analysis of Case 3,  $G^i$  refers just to the relevant portion of  $G^i$ , here the part with  $TP_{r'(j-1)}[v]w_j$ . The first part of  $G^7$  gives all terms with  $z_j^e$  for  $1 \leq e \leq p-2$ . The remaining part has factors  $w_j z_j^{p-1}$ , which we will omit writing. Combining  $G^1$  and  $G^5$  removes the bar in  $G^5$ . The first part of  $G^3$  gives the part with positive exponent of  $y_j$ , which we now omit.

Let  $E_\ell = P[y_\ell] \otimes \Lambda_\ell$ , thought of as monomials in  $y_i$  and  $z_i$  for  $i \geq \ell$  with exponents  $\leq p-1$ . The remaining parts of the  $G^i$ 's under consideration combine to

$$\bigoplus_{k \geq j+1} \left( TP_{p-1}[y_k] \oplus y_k z_k^{p-1} TP_{p-1}[y_k] \oplus \bigoplus_{e=1}^{p-2} z_k^e TP_p[y_k] \right) \otimes E_{k+1} \otimes \prod_{i=j+1}^{k-1} \{z_i^{p-1}, y_i^{p-1}\}. \quad (6.2)$$

We wish to show this equals  $E_{j+1}$ . The portion in parentheses is all monomials in  $TP_p[y_k, z_k]$  except  $y_k^{p-1}$  and  $z_k^{p-1}$ . For a monomial  $M$  in  $E_{j+1}$ , let  $M_i$  denote its  $y_i^s z_i^t$  factor. The  $k$ -summand in (6.2) is all monomials  $M$  in  $E_{j+1}$  for which  $k$  is the smallest  $i$  such that  $M_i$  is neither  $y_i^{p-1}$  nor  $z_i^{p-1}$ . Thus the sum over all  $k$  yields all of  $E_{j+1}$ , as claimed.

A very similar argument shows that the  $G^2 \oplus G^4 \oplus G^6 \oplus G^8$  part for Case 3 equals the portion which includes just the 1 in  $E[w_j]$ ; i.e.,  $P[y_j] \otimes \overline{TP}_p[z_j] \otimes \Lambda_{j+1}$ .

**Case 4.**  $\bigoplus_{j \geq 1} P[y_1] \otimes E[q] \otimes \overline{E}[z_j^p] \otimes \Lambda_{j+1}$ . We first consider the part without the  $q$ , and fix  $j$  and omit writing the  $z_j^p$ . The desired answer is  $P[y_1] \otimes \Lambda_{j+1}$ . These come from the  $\mathbb{Z}_p$ 's in  $G^2 \oplus G^4 \oplus G^6 \oplus G^8$ . Similarly to Case 3,  $G^2$  and  $G^6$  combine to give

$$\bigoplus_{k \geq j+1} TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes \Lambda_{k+1} \otimes \prod_{i=k+1}^{j-1} \{z_i^{p-1}, y_i^{p-1}\}.$$

This, together with the portion of  $G^4$  from  $\text{im}(\phi)$  in (5.12) obtained using (5.10), and the  $\mathbb{Z}_p$ 's in  $G^8$  obtained using (5.10) give exactly (6.2), which we showed equals  $P[y_{j+1}] \otimes \Lambda_{j+1}$ .<sup>3</sup> The element  $X$  in Figure 5.13 with  $k$  replaced by  $j$  yields, from  $G^4$ ,

$$\begin{aligned} & y_j TP_{p-1}[y_j] \otimes P[y_{j+1}] \otimes \bigoplus_{\ell > j} Z_{j+1}^\ell TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1} \\ &= y_j TP_{p-1}[y_j] \otimes P[y_{j+1}] \otimes \Lambda_{j+1}, \end{aligned}$$

which combines with the portion just obtained to yield  $P[y_j] \otimes \Lambda_{j+1}$ .

<sup>3</sup>Here the classes in (6.2) are  $\mathbb{Z}_p$ 's and are multiplied by  $z_j^p$ , whereas in Case 3 they were multiplied by  $w_j z_j^{p-1}$  and were generators of  $v$ -towers of height  $r'(j-1)$ .

The last line of the  $G_{k,\ell}^4$  discussion in Section 5 describes  $\mathbb{Z}_p$ 's in  $G^4$  mapped by  $\psi$  in (5.12). Those with a  $z_j^p$  factor yield

$$\begin{aligned}
& \bigoplus_{k=1}^{j-1} y_k TP_{p-1}[y_k] P[y_{k+1}] \bigoplus_{\ell > j} Z_{j+1}^\ell TP_{p-1}[z_\ell] \Lambda_{\ell+1} \\
&= \bigoplus_{k=1}^{j-1} (P[y_k] - P[y_{k+1}]) \otimes \Lambda_{j+1} \\
&= (P[y_1] - P[y_j]) \otimes \Lambda_{j+1}.
\end{aligned}$$

Combining this with the result of the preceding paragraph yields the desired  $P[y_1] \otimes \Lambda_{j+1}$ .

We finish this section by showing that the  $\mathbb{Z}_p$ 's including a factor  $q$  are obtained exactly once. We omit writing the  $q$ . The classes which we must obtain are  $P[y_1] \bigoplus_{j \geq 1} z_j^p \Lambda_{j+1}$ . There are eight ways these appear in  $G^i$ -sets.

- (1) In  $G^1$ , using (5.7) and (5.8), for  $1 \leq i < j < k$ ,

$$y_1^{p^{j-1}-1} z_{i,j} z_j^{p-2} \prod_{s=j+1}^{k-1} \{z_s^{p-1}, y_s^{p-1}\} \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}].$$

- (2) In  $G^3$ , using (5.9) and (5.8), for  $1 \leq i < k < \ell$ ,

$$y_1^{p^{k-1}-1} y_k z_{i,k} z_k^{p-2} Z_{k+1}^\ell \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$

- (3) In  $G^3$ , using (5.7) and (5.8), for  $1 \leq i < j < k < \ell$ ,

$$y_1^{p^{j-1}-1} y_k z_{i,j} z_j^{p-2} \prod_{s=j+1}^{k-1} \{z_s^{p-1}, y_s^{p-1}\} Z_k^\ell \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$

- (4) From  $\text{im}(\phi')$  in (5.14), for  $1 \leq k < \ell$  and  $1 \leq i \leq \ell - k$ ,

$$y_1^{p^{k-1}-1} z_{i,\ell} \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$

- (5) From  $\psi'$  in (5.14), using (5.9) and (5.8), for  $k < \ell$  and  $\ell - k <$

$$i < \ell,$$

$$y_1^{p^{k-1}-1} z_{i,\ell} \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$

- (6) From  $\psi'$  in (5.14), using (5.7) and (5.8), for  $i < j < k < \ell$ ,

$$y_1^{p^{j-1}-1} z_{i,j} z_j^{p-2} \prod_{s=j+1}^{k-1} \{z_s^{p-1}, y_s^{p-1}\} \cdot z_\ell \otimes TP_{p-1}[y_k] \otimes P[y_{k+1}] \otimes TP_{p-1}[z_\ell] \otimes \Lambda_{\ell+1}.$$

(7) From (5.3), using (5.9) and (5.8), for  $i < k$  and  $1 \leq e \leq p-2$ ,

$$y_1^{p^{k-1}-1} z_{i,k} z_k^{e-1} P[y_k] \otimes \Lambda_{k+1}.$$

(8) From (5.3), using (5.7) and (5.8), for  $i < j < k$  and  $1 \leq e \leq p-2$ ,

$$y_1^{p^{j-1}-1} z_{i,j} z_j^{p-2} \prod_{s=j+1}^{k-1} \{z_s^{p-1}, y_s^{p-1}\} \cdot z_k^e P[y_k] \otimes \Lambda_{k+1}.$$

First combine (1)+(6) to put a  $\otimes \Lambda_{k+1}$  at the end of (1), and then, similarly to the simplification of (6.2), combine with (3)+(8) to get

$$\bigoplus_{i < j} y_1^{p^{j-1}-1} P[y_{j+1}] z_{i,j} z_j^{p-2} \Lambda_{j+1}. \quad (6.3)$$

We combine and relabel (4)+(5) to give

$$) \bigoplus_{i < j} y_1^{p^{j-1}-1} T P_{p-1}[y_j] P[y_{j+1}] z_{i,j+1} \Lambda_{j+1} \quad (6.4)$$

together with

$$\bigoplus_{i \geq j \geq 1} y_1^{p^{j-1}-1} T P_{p-1}[y_j] P[y_{j+1}] z_i^p \Lambda_{i+1}. \quad (6.5)$$

Let  $Y(s) = y_1^{p^s-1} T P_{p-1}[y_{s+1}] P[y_{s+2}] = \langle y_1^i : \nu(i+1) = s \rangle$ . Then (6.5) is

$$\bigoplus_{i > s \geq 0} Y(s) z_i^p \Lambda_{i+1}. \quad (6.6)$$

We simplify and relabel (2) to

$$\bigoplus_{i < j} y_1^{p^{j-1}-1} y_j T P_{p-1}[y_j] P[y_{j+1}] z_{i,j} z_j^{p-2} \Lambda_{j+1}. \quad (6.7)$$

(6.3), (6.7), and (7) combine to give

$$\bigoplus_{i < j} y_1^{p^{j-1}-1} P[y_j] z_{i,j} T P_{p-1}[z_j] \Lambda_{j+1} = \bigoplus_{i \leq j-1 \leq t} Y(t) z_{i,j} T P_{p-1}[z_j] \Lambda_{j+1}.$$

For any  $t \geq i$ , the coefficient of  $Y(t) z_i^p$  in (6.4) plus this is

$$Z_{i+1}^{t+2} \Lambda_{t+2} \oplus \bigoplus_{j=i+1}^{t+1} Z_{i+1}^j T P_{p-1}[z_j] \Lambda_{j+1} = \Lambda_{i+1},$$

as the second part has all monomials not divisible by  $Z_{i+1}^{t+2}$ . Combining this with (6.6) yields the desired result,

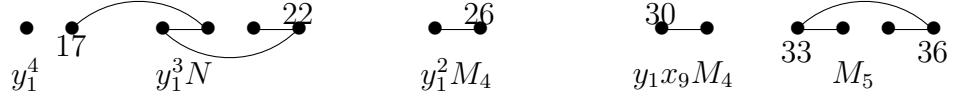
$$\bigoplus_{s \geq 0} Y(s) \bigoplus_{i \geq 1} z_i^p \Lambda_{i+1}.$$

## 7. AN EXPLANATION OF SELF-DUALITY OF $B_k$

In this optional section, we discuss some observations about the ASS of  $ku^*(K_2)$  and  $ku_*(K_2)$  which, among other things, provide an explanation of the self-dual nature of the  $B_k$  summands which occur in both  $ku^*(K_2)$  and  $ku_*(K_2)$ . We restrict to  $p = 2$ .

We first observe that, for  $k \geq 1$ , there is an  $E_1$ -submodule,  $\mathcal{M}_k$ , of  $H^*(K_2)$  such that  $\text{Ext}_{E_1}(\mathbb{Z}_2, \mathcal{M}_k)$  (resp.  $\text{Ext}_{E_1}(\mathcal{M}_k, \mathbb{Z}_2)$ ) is closed under the differentials in the ASS converging to  $ku^*(K_2)$  (resp.  $ku_*(K_2)$ ), yielding the chart  $A_k$  (resp. the  $ku$ -homology analogue of  $A_k$  discussed in Theorem 1.19). For example, with  $M_j$  as in (2.12) and  $N$  as in Figure 2.8,  $\mathcal{M}_3$  is as depicted in Figure 7.1.

**Figure 7.1.** The  $E_1$ -module  $\mathcal{M}_3$ .



The two ASSs for  $\mathcal{M}_3$  will yield the charts for  $A_3$  and its homology analogue pictured in [4].

The situation for  $B_k$  is slightly more complicated. There is no  $E_1$ -submodule of  $H^*(K_2)$  which, by itself, can give a chart  $B_k$  or  $B_k z_\ell$ . Some of the differentials that truncate  $v$ -towers in  $B_k z_\ell$  come from classes that are part of a summand that includes  $y_1^{2^{k-1}-1} q S_{k,\ell}$ . We find that, for  $2 \leq k < \ell$ , there is an  $E_1$ -submodule  $\mathcal{M}_{k,\ell}$  of  $H^* K_2$  such that  $\text{Ext}_{E_1}(\mathbb{Z}_2, \mathcal{M}_{k,\ell})$  is closed under the differentials in the ASS converging to  $ku^*(K_2)$  and yields the chart

$$B_k z_\ell \oplus y_1^{2^{k-1}-1} q S_{k,\ell} \oplus y_k B_k Z_k^\ell.$$

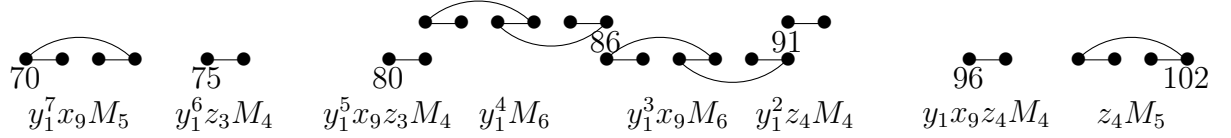
Note that these three subsets of  $ku^*(K_2)$  appeared together in the 10-term exact sequence (5.2).

This  $\mathcal{M}_{k,\ell}$  is symmetric; i.e., there is an integer  $D$  such that  $\Sigma^D \mathcal{M}_{k,\ell}^*$  and  $\mathcal{M}_{k,\ell}$  are isomorphic  $E_1$ -modules, where  $\mathcal{M}_{k,\ell}^*$  is obtained from  $\mathcal{M}_{k,\ell}$  by negating gradings and dualizing  $Q_0$  and  $Q_1$ . This implies that the  $v$ -towers in  $\text{Ext}_{E_1}(\mathbb{Z}_2, \mathcal{M}_{k,\ell})$  and

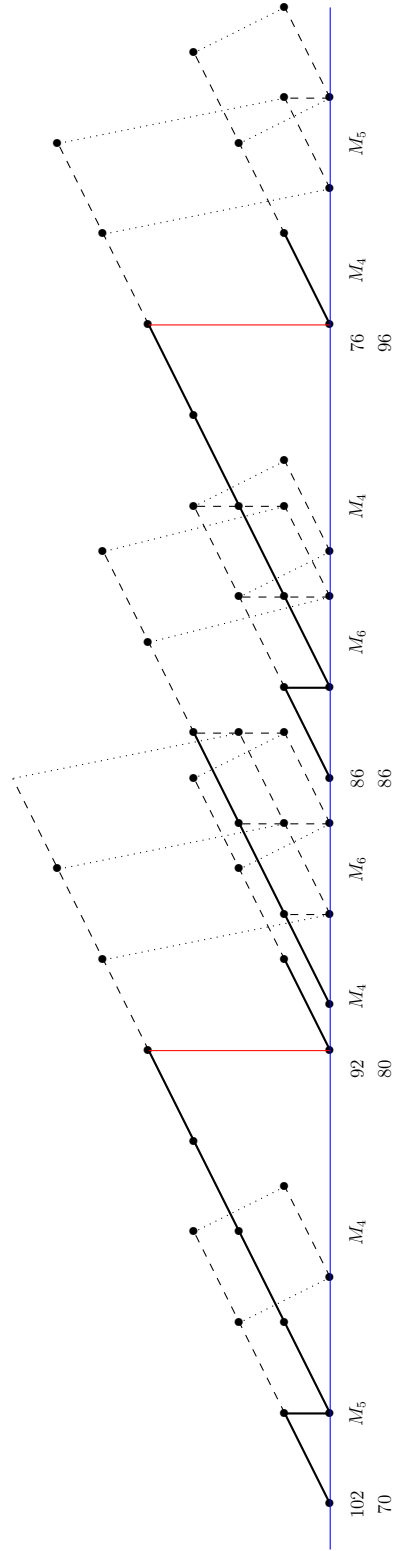
$\text{Ext}_{E_1}(\mathcal{M}_{k,\ell}, \mathbb{Z}_2)$  correspond nicely. Moreover, the differentials in the two ASSs correspond, too, obtaining isomorphic charts, although the gradings in one decrease from left to right, while in the other they increase.

We illustrate with an example,  $\mathcal{M}_{3,4}$ , and then discuss the implication for self-duality of  $B_k$ . In Figure 7.2, we depict  $\mathcal{M}_{3,4}$ .

**Figure 7.2.** The  $E_1$ -module  $\mathcal{M}_{3,4}$ .



In Figure 7.3, we depict the ASS chart for both  $\text{Ext}_{E_1}(\mathbb{Z}_2, \mathcal{M}_{3,4})$  and  $\text{Ext}_{E_1}(\mathcal{M}_{3,4}, \mathbb{Z}_2)$ . They are isomorphic except that, from left to right, the gradings start with 102 for the first and 70 for the second. We label the portions of the chart corresponding to the eight summands of  $\mathcal{M}_{3,4}$  just by the  $M$ -factor, since accompanying factors differ for the two versions. For example, the  $M_5$  on the left-hand side is  $z_4 M_5$  for the first spectral sequence, and is  $y_1^7 x_9 M_5$  for the second.

Figure 7.3. Two ASSs for  $\mathcal{M}_{2,3}$ .



For the  $ku^*(K_2)$  version,  $B_3z_4$  is on the left hand side of Figure 7.3, and  $y_3B_3z_3$  on the right hand side, with  $y_1^3qS_{3,4}$  separating them. The duality isomorphism in Theorem 1.16 says that the Pontryagin dual of  $B_3z_4$  is isomorphic as a  $ku_*$ -module to  $\Sigma^4$  of the right hand side of the  $ku_*(K_2)$  version of Figure 7.3, and we see that this is isomorphic to a shifted version of  $B_3$  with indices negated. This is the self-duality statement, that the Pontryagin dual of  $B_k$  is isomorphic as a  $ku_*$ -module to a shifted version of  $B_k$  with indices negated.

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