THE CONNECTIVE K-THEORY OF THE EILENBERG-MACLANE SPACE $K(\mathbb{Z}_2, 2)$

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Abstract. We compute $ku^*(K(\mathbb{Z}_2, 2))$ and $ku_*(K(\mathbb{Z}_2, 2))$, the connective KU-cohomology and connective KU-homology groups of the mod 2 Eilenberg-MacLane space $K(\mathbb{Z}_2, 2)$, using the Adams spectral sequence. The mod-2 connective KU-cohomology groups, $k(1)^*(K(\mathbb{Z}_2, 2))$, computed elsewhere, are needed in order to establish higher differentials and exotic extensions in the integral groups.

1. Main results

In [11] and [5], the authors initiated a partial computation of the connective KU-homology groups, $ku_*(K_2)$, of the mod 2 Eilenberg-MacLane space $K_2 = K(\mathbb{Z}_2, 2)$ in separate studies of Stiefel-Whitney classes of manifolds. We eventually turned to the associated cohomology groups, $ku^*(K_2)$, and here we give a complete determination, via the Adams spectral sequence (ASS). Subsequently the first author noticed a duality result ([4]) relating these homology and cohomology groups, and in Section 2, we discuss the resulting $ku_*(K_2)$.

The bulk of this introductory section is a discussion of the result of our ASS computation of (reduced) $ku^*(K_2)$. There are nice families of exotic extensions. We depict the ASS with cohomological (co)degrees increasing from right-to-left. The Bott element $v \in ku^* = \mathbb{Z}_{(2)}[v]$ decreases grading by 2.

In $ku^*(K_2)$, there is an infinite family of split $\mathbb{Z}_2$'s whose Poincaré series is described at the end of Section 3. **Ignoring these from now on**, as a ku^*-module, $ku^*(K_2)$ is generated by certain products of elements of $E_2^{0,*}$, x_4, x_9, and x_8, with $|x_i| = i$, and z_j for $j \geq 3$ with $|z_j| = 2^j + 2$. We let Λ_j denote the exterior algebra $E[z_i : i \geq j]$, and $\overline{\Lambda}$ and $\overline{E}$ the augmentation ideal in an exterior algebra.

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We will show that there are closely-related ku^*-modules A_k and B_k for $k \geq 3$ such that in even gradings[1] there is an isomorphism of ku^*-modules

$$ku^{\mathrm{ev}}(K_2) \approx \bigoplus_{k\geq 3} \mathbb{Z}_2[x_4^{2^{k-2}}] \otimes (A_k \oplus x_4^{2^{k-3}} B_k z_k \Lambda_{k+1} \oplus B_k \overline{\Lambda}_{k+1}). \tag{1.1}$$

The notation $x_4^{2^{k-3}} B_k z_k \Lambda_{k+1}$ means that all elements of B_k are multiplied by $x_4^{2^{k-3}} z_k$, and this is tensored with Λ_{k+1}. Note that B_k never appears alone.

We give three descriptions of A_k and B_k, and discuss how Figure 1.10 depicts A_k and B_k for all $k \leq 7$, and enables one to envision them for all k. As a preview, the dashed lines in Figure 1.10 connect elements of A_k which are not in B_k, and the red lines (sometimes slightly curved) are exotic extensions (multiplication by 2, not seen in Ext).

We first give an inductive description. Let $B_3 = 0$, and A_3 have as its only nonzero classes[2] x_8, z_3, and $2x_8 = vz_3$. Let

$$z_{i,j} = z_i^2 z_{i+1} \cdots z_{j-1} \text{ for } 4 \leq i \leq j-1, \tag{1.2}$$

while $z_{j,j} = z_j$. These classes occur in consecutive even gradings from $2^j + 2j - 6$ down to $2^j + 2$ as i goes from 4 to j. For $k \geq 4$, there are ku^*-modules T_k^A and T_k^B generated by $z_{j,k}$ for $4 \leq j \leq k$, with relations

$$2z_{j,k} = vz_{j-1,k} \text{ for } 5 \leq j \leq k, \tag{1.3}$$

$2z_{4,k} = 0$, $v^{2^{k-2}} z_{k,k} = 0$ in T_k^A, and otherwise $v^{2^{j-2}-(j-2)} z_{j,k} = 0$ in both T_k^A and T_k^B. In Figure 1.10, the batch of v-towers going up from gradings 130 to 136 are T_7^A and T_7^B, with the dashed part (whose slope was changed for typographical reasons) representing the elements $v^i z_7$ for $27 \leq i \leq 31$, which are in T_7^A, but not in T_7^B.

The inductive description is that, for $k \geq 4$, there are short exact sequences of ku^*-modules

$$0 \to T_k^B \to B_k \to \bigoplus_{j=4}^{k-1} x_4^{2^{j-3}} B_j z_{j+1} \cdots z_{k-1} \to 0 \tag{1.4}$$

and

$$0 \to T_k^A \to A_k \to x_4^{2^{k-4}} A_{k-1} \oplus \bigoplus_{j=4}^{k-2} x_4^{2^{j-3}} B_j z_{j+1} \cdots z_{k-1} \to 0 \tag{1.5}$$

[1]Odd gradings will be described in (1.11).

[2]The three elements in the lower right corner of Figure 1.10 are $x_4^{15} A_3$.

with extensions given by

$$\begin{aligned}
(2\cdot x_8 = vz_3) &\quad\otimes\quad P[x_4]\\
(2\cdot z_3 = 0) &\quad\otimes\quad P[x_4^2]\\
(2\cdot x_4z_3 = v^2z_4) &\quad\otimes\quad P[x_4^2]\\
(2\cdot z_4 = 0) &\quad\otimes\quad P[x_4^4]\otimes\Lambda_4\\
(2\cdot z_j = vz_{j-1}^2) &\quad\otimes\quad P[x_4^{2^{j-2}}]\otimes\Lambda_j,\quad j\ge 5\\
(2\cdot x_4^2z_4 = v^4z_5) &\quad\otimes\quad P[x_4^4]\otimes\Lambda_5\\
(2\cdot x_4^{2^{j-3}}z_j = vx_4^{2^{j-3}}z_{j-1}^2 + v^{2^{j-2}}z_{j+1}) &\quad\otimes\quad P[x_4^{2^{j-2}}]\otimes\Lambda_{j+1},\quad j\ge 5.
\end{aligned}\tag{1.6}$$

These formulas can be also multiplied by powers of v, as long as the elements are nonzero. The extension formulas can be visualized in Figure 1.10. For example, in grading 116, $2x_4^4z_5z_6 = vx_4^4z_4^2z_6 + v^8z_6^2$, and in grading 114, $vx_4^4z_5z_6 + v^8z_7$ has order 2, and v^{23} times it is nonzero in A_7. As another example, Figure 1.10 shows that A_7 contributes a $\mathbb{Z}_8\oplus\mathbb{Z}_2$ summand to $ku^{126}(K_2)$ with generators v^2z_7 and $x_4^2z_4z_5z_6+v^3z_6^2$.

In Figure 1.10, the v-towers emanating from gradings ≤ 102 comprise A_6 (if dashed arrows are included) and B_6 (if not), after dividing the labels by x_4^8. Those from gradings ≤ 84 are A_5 and B_5 after dividing by x_4^{12}.

Remark 1.7. A simpler inductive description is that B_k (resp. A_k) is built from

$$\begin{aligned}
&B_{k-1}z_{k-1},\ \langle z_k\rangle/(2, v^{2^{k-2}-(k-2)}),\ \text{and } x_4^{2^{k-4}}B_{k-1}\\
\text{resp.}\quad &B_{k-1}z_{k-1},\ \langle z_k\rangle/(2, v^{2^{k-2}}),\ \text{and } x_4^{2^{k-4}}A_{k-1},
\end{aligned}$$

with exotic extensions from $v^ix_4^{2^{k-4}}z_{k-1}$ to $v^{i+2^{k-3}}z_k$, $0\le i\le 2^{k-3}-(k-1)$ (resp. $0\le i\le 2^{k-3}-1$), and h_0-extensions from v^iz_k to $v^{i+1}z_{k-1}^2$, $0\le i\le 2^{k-3}-(k-1)$.

The non-inductive analogue of (1.4) is

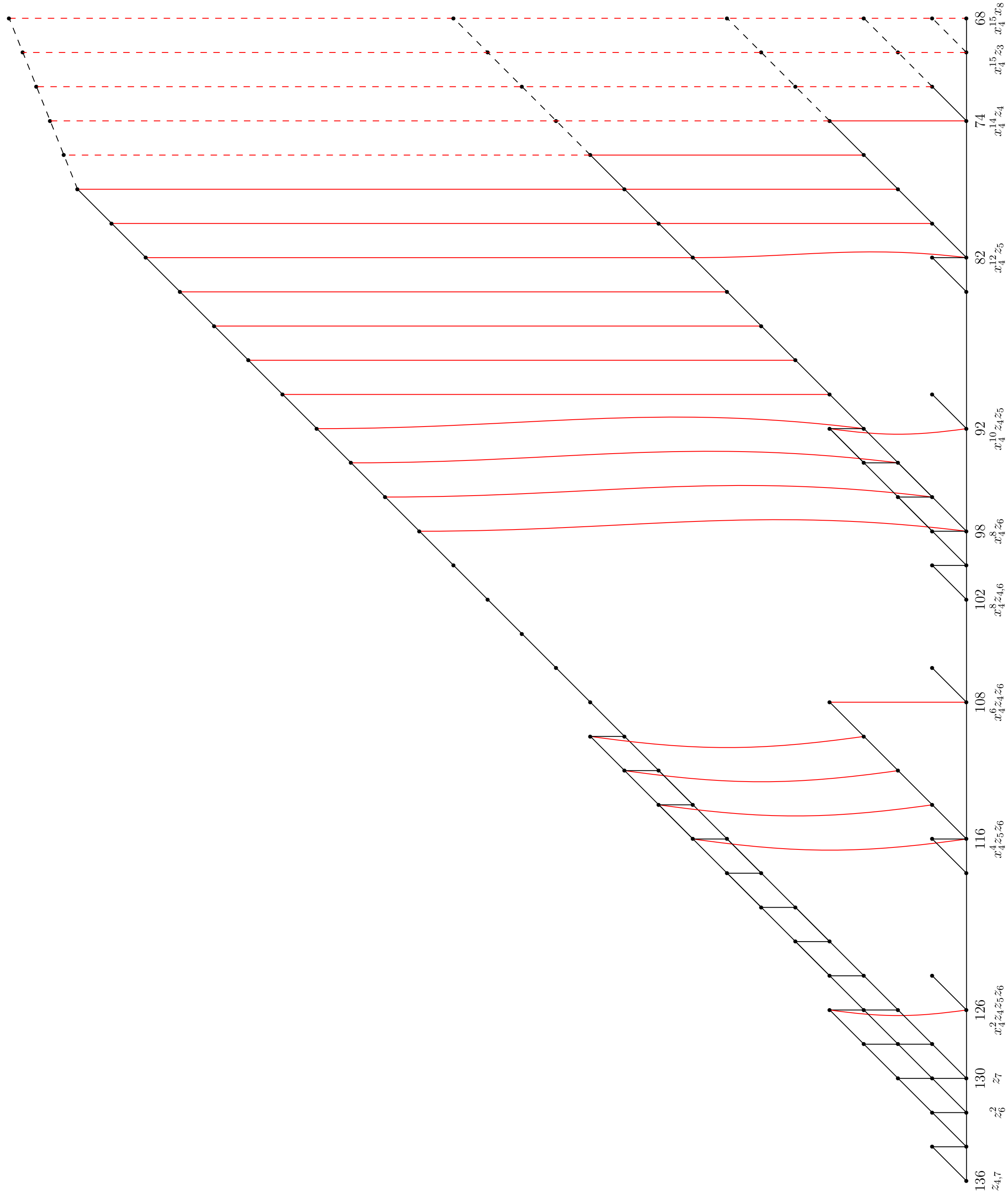
$$B_k = T_k^B \oplus \bigoplus_{i=4}^{k-1} x_4^{2^{i-3}} \bigoplus \prod_{j=i+1}^{k-1}\{z_j, x_4^{2^{j-3}}\}\cdot T_i^B,\tag{1.8}$$

with extensions from T_i^B to T_{i+1}^B determined by (1.6). Here $\bigoplus\prod_{j=i+1}^{k-1}\{z_j,x_4^{2^{j-3}}\}$ is the sum over all ways of choosing one or the other of the two expressions and taking the product of the selected expressions. For example, this says that

$$B_7 = T_7^B\oplus x_4^8T_6^B\oplus x_4^4z_6T_5^B\oplus x_4^{12}T_5^B\oplus x_4^2z_5z_6T_4^B\oplus x_4^6z_6T_4^B\oplus x_4^{10}z_5T_4^B\oplus x_4^{14}T_4^B,\tag{1.9}$$

as can be seen in Figure 1.10. The analogue of (1.8) for A_k is that T_i^B is replaced by T_i^A whenever no z_j's accompany it, and there is an additional $x_4^{2^{k-3}-1}A_3$.

Figure 1.10. B_7 **and** A_7**.**



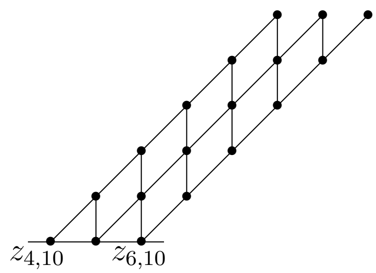
We now define ku^*-modules $S_{k,\ell}$ for $3 \le k < \ell$ such that the odd-grading portion of $ku^*(K_2)$ is

$$ku^{\mathrm{od}}(K_2) = \bigoplus_{k\ge 3}\bigoplus_{\ell>k} x_4^{2^{k-3}-1}\mathbb{Z}_2[x_4^{2^{k-2}}]x_9 S_{k,\ell}\Lambda_{\ell+1}. \tag{1.11}$$

Definition 1.12. *For $3 \le k < \ell$, the ku^*-module $S_{k,\ell}$ has v-towers of v-height $k-1$ with generators $z_{i,\ell}$ for $4 \le i \le \ell - k + 3$, with h_0 (the Ext analogue of multiplication by 2) nonzero wherever possible.*

Thus $2v^m z_{i,\ell} = v^{m+1} z_{i-1,\ell}$ iff $i > 4$ and $m \le k-3$. For example, $S_{7,10}$ is depicted in Figure 1.13.

Figure 1.13. $S_{7,10}$



Recapitulating into theorem form, our main result is

Theorem 1.14. *In addition to the split $\mathbb{Z}_2$'s, which are enumerated at the end of Section 3, the ku^*-module $ku^*(K_2)$ is as in (1.1) and (1.11), where A_k and B_k are given either inductively or explicitly as above, and $S_{k,\ell}$ is as in Definition 1.12.*

The non-visual, formulaic form of our result is as follows, where $TP_m[v] = P[v]/(v^m)$.

Theorem 1.15. *The ku^*-module $ku^*(K_2)$ is isomorphic to a trivial ku^*-module plus*

$$P[x_4]x_8 \oplus \bigoplus_{t\ge 0} TP_{2^{t+1}}[v] \otimes P[x_4^{2^t}]z_{t+3} \tag{1.16}$$

$$\oplus \bigoplus_{t\ge 1} TP_{2^{t+1}-t-1}[v] \otimes P[x_4^{2^t}]z_{t+3}\Lambda_{t+3} \tag{1.17}$$

$$\oplus \bigoplus_{e\ge 1} TP_{e+1}[v] \otimes P[x_4^{2^e}]x_4^{2^{e-1}-1}x_9 \otimes \bigoplus_{j\ge 4} z_j Z_j^{j+e-2}\Lambda_{j+e-1}, \tag{1.18}$$

where $Z_j^{j+e-2} = z_j \cdots z_{j+e-2}$*. Multiplication by 2 in (1.16) and (1.17) is given in (1.6), while in (1.18) it is determined by*

$$2 \cdot z_j M = \begin{cases} v z_{j-1}^2 M & j \geq 5 \\ 0 & j = 4 \end{cases} \text{ for } M \in \Lambda_j.$$

The most direct route to this result is via the right-hand-side of equations (4.3), (4.4), and (4.5).

The structure of the rest of the paper is as follows. As already noted, Section 2 presents the results for $ku_*(K_2)$. In Section 3, we compute the E_2-term of the ASS for $ku^*(K_2)$. In Section 4 we determine the differentials in this ASS. In order to do so, we need to compare with $k(1)^*(K_2)$, where $k(1)$ is the spectrum for mod-2 connective KU-theory, using the exact sequence

$$\to k(1)^{*-1}(K_2) \to ku^*(K_2) \xrightarrow{2} ku^*(K_2) \to k(1)^*(K_2) \to ku^{*+1}(K_2) \xrightarrow{2} . \quad (1.19)$$

In Section 4, we restate results about $k(1)^*(K_2)$ from [6]. At the end of Section 4, we show how the descriptions of $ku^*(K_2)$ in (1.1) and (1.11) are obtained once we know the differentials. This exact sequence is also used in determining the exotic extensions of (1.6), which is done in Section 5. In Section 6, we propose complete formulas for the exact sequence (1.19), and then in Section 7, we show that our proposed formulas exactly account for all elements of $k(1)^*(K_2)$. In the optional Section 8, we discuss in more detail how the charts are obtained and explain a surprising duality in the B_k charts.

The main point of Section 7 is to prove that there are no additional exotic extensions in $ku^*(K_2)$. An exotic extension $2 \cdot A = B$ implies that A is not in the image from $k(1)^{*-1}(K_2)$, and B does not map nontrivially to $k(1)^*(K_2)$, so once we have shown that all elements are accounted for, there can be no more extensions. Many of our formulas in Section 6 are forced by naturality. However, many others occur in regular families, but with surprising filtration jumps. We could probably show that the homomorphisms *must* be as we claim, by showing that there are no other possibilities, but we prefer to forgo doing that.

2. Results for $ku_*(K_2)$

Our initial interest in this project was $ku_*(K_2)$ ([11],[5]), but here we first achieved success in computing $ku^*(K_2)$. In [4, Corollary 1.3], the first author proved the following result.

Theorem 2.1. *There is an isomorphism of ku_*-modules $ku_*(K_2) \approx (ku^{*+4}K_2)^\vee$.*

Here $M^\vee = \mathrm{Hom}(M, \mathbb{Z}/2^\infty)$, the Pontryagin dual, localized at 2. A homotopy chart for $ku_*(K_2)$ could be thought of as a shifted version of the homotopy chart of $ku^*(K_2)$ viewed upside-down and backwards.

A remarkable property, for which one explanation is given in Section 8, is that B_k is self-dual as a ku^*-module. One way of stating this is to let $\widetilde{B}_k$ denote B_k with its indices negated. Then there is an isomorphism of ku_*-modules

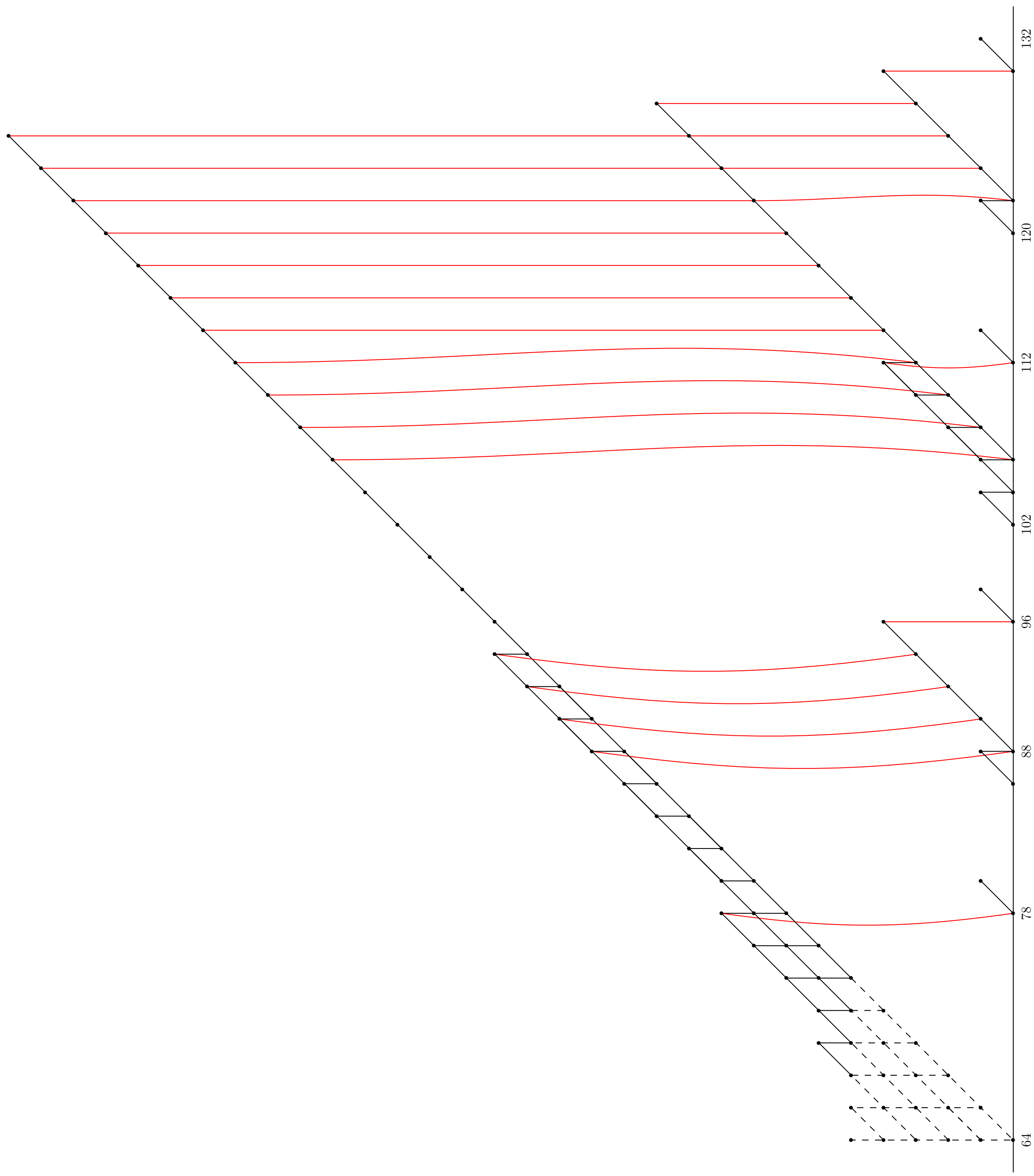
$$\Sigma^{2^k+2^{k-1}+2k+2}\widetilde{B}_k \approx B_k^\vee. \tag{2.2}$$

For example, the second generator Y of $\Sigma^{208}\widetilde{B}_7$ is in grading $208 - 134 = 74$ and has $2Y \neq 0$ and $v^4 Y \neq 0$. (See Figure 1.10.) The second generator Z of $B_7^\vee$ is dual to the class in position $(74, 4)$ in Figure 1.10, and also satisfies $2Z \neq 0$ and $v^4 Z \neq 0$. The isomorphism (2.2) can be proved by induction on k using Remark 1.7.

A complete description of the ku_*-module $ku_*(K_2)$ is immediate from Theorems 1.14 and 2.1. However, one might like a complete description of its ASS. We can write formulas for the E_2-term and differentials, but will not do so here. In Theorem 2.4 we give a complete description of the E_∞-term of the ASS of $ku_*(K_2)$ with exotic extensions included, in terms of the charts described in Section 1.

In [4], a comparison was made of the chart for A_5 and its ku_* analogue. Here we present in Figure 2.3 the ku_* analogue of Figure 1.10. This presents the portion of the ASS of $ku_*(K_2)$ dual to A_7 under the isomorphism of Theorem 2.1. The chart dual to B_7 is obtained from this by removing the classes connected by dashed lines, and lowering the remaining tower so that the bottom is in filtration 0. The resulting chart is isomorphic to the B_7 part of Figure 1.10.

Figure 2.3. Portion of $ku_*(K_2)$ corresponding to B_7 and A_7.



We observe that in even gradings of the ASS for $ku_*(K_2)$, h_0-extensions exactly correspond to exotic extensions in the ASS of $ku^{*+4}(K_2)$, and vice versa. As a typical example of the duality, the summands of $ku^{82}(K_2)$, $ku^{82}(K_2)^\vee$, and $ku_{78}(K_2)$ in Figures 1.10 and 2.3 are all isomorphic to $\mathbb{Z}_8 \oplus \mathbb{Z}_2$. But for the ku_*-module structure, it is $ku^{82}(K_2)^\vee$ and $ku_{78}(K_2)$ that correspond, since in both, the element that is divisible by 4, in position $(82, 0)$ and $(78, 7)$, resp., is also divisible by v^7 for A_7 and by v^4 for B_7..

Theorem 2.4. *The E_∞-term of the ASS of $ku_*(K_2)$ with exotic extensions included contains exactly the following.*

- *There are $\mathbb{Z}_2$'s annihilated by v corresponding to those enumerated at the end of Section 3 with gradings decreased by 4.*
- *For every summand of (1.11), there is a chart of the same form as Figure 1.13 with v-towers of height $k-1$ on generators in gradings described as follows. Corresponding to the factor $S_{k,\ell}$ itself, they are in gradings $2^\ell + 2i - 4$ for $0 \le i \le \ell - k - 1$. One must add to this the grading of the other factors accompanying $S_{k,\ell}$ in (1.11).*
- *For each occurrence of B_k in (1.1), there is a summand $\Sigma^{2^k+2^{k-1}+2k-2}\widetilde{B}_k$ with gradings increased by those of other factors accompanying B_k in (1.1). Here $\widetilde{B}_k$ is as defined prior to (2.2).*
- *For each summand $x_4^{c2^{k-2}} A_k$ in (1.1), there is a variant of $\Sigma^{2^k+2^{k-1}+2k-2}\widetilde{B}_k$ with gradings increased by $c2^k$. In this variant, the initial T_k^B is pushed up by $k-2$ filtrations and surrounded with a triangle of classes of the sort appearing in the lower left corner of Figure 2.3. See Remark 2.5.*

Proof. Theorem 2.1 and our results for $ku^*(K_2)$ give the ku_*-module structure of $ku_*(K_2)$, but that is not the same as the ASS picture. Expanding on work done in [5] and [11] and using methods such as those in Section 3, we were able to write the E_2-term of the ASS for $ku_*(K_2)$, and had conjectured the differentials (but not the extensions) prior to embarking on our ku-cohomology project. We were unable to *prove* the differentials, probably because we had not taken sufficient advantage of the exact sequence with $k(1)_*(K_2)$. Now that we know the 2-orders and v-heights of generators (by grading, at least, if not by name), it is straightforward to see that the differentials and extensions must be as claimed. The isomorphism (2.2) plays an important role here; the left hand side gives the ASS form of the right hand side. ■

Remark 2.5. Regarding the unusual portion of the ASS chart for part of $ku_*(K_2)$ in the lower left of Figure 2.3, this is obtained from [5, Fig. 4.2] with d_6-differentials on all odd-graded towers. For A_k, it will be a triangle going up to filtration $k-2$, with all but the first two dots on the top row being part of B_k.

3. The E_2-term of the ASS for $ku^*(K_2)$

We will need some notation. By H^*K_2, we understand $H^*(K(\mathbb{Z}_2, 2); \mathbb{Z}_2)$. Let E denote an exterior algebra, P a polynomial algebra, and $TP_n[x] = P[x]/(x^n)$ the truncated polynomial algebra. In all cases these will be over $\mathbb{Z}_2$, the integers mod 2, and we also use $\mathbb{Z}_2[-]$ notation for polynomial algebras. Let $\overline{E}$ denote the augmentation ideal of an exterior algebra, and $E_1 = E[Q_0, Q_1]$, where $Q_0 = \mathrm{Sq}^1$ and $Q_1 = \mathrm{Sq}^2 \mathrm{Sq}^1 + \mathrm{Sq}^1 \mathrm{Sq}^2$. Because $Q_i^2 = 0$ we have homology groups, $H_*(-; Q_i)$, defined for E_1-modules. We let $\langle y_1, y_2, \ldots \rangle$ denote the $\mathbb{Z}_2$-span of classes y_i.

The ASS for $ku^*(K_2)$ has $E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(bu), H^*K_2)$, where $\mathcal{A}$ is the mod 2 Steenrod algebra and $H^*(bu) \approx \mathcal{A}/\mathcal{A}(Q_0, Q_1)$. Using a standard change of rings theorem, [7], this is $\mathrm{Ext}_{E_1}^{s,t}(\mathbb{Z}_2, H^*K_2)$. This converges to $ku^{-(t-s)}(K_2)$. We depict this with $E_2^{s,t}$ in position $(t-s, s)$ as usual, but label the axis with codegrees, the negative of the homotopical degree, so the left side of the chart will have positive gradings. In an attempt to avoid confusion, we rewrite this as $G_2^{-(t-s),s}$. With this notation, the differentials are $d_r : G_r^{a,b} \longrightarrow G_r^{a+1,b+r}$, multiplication by the element $v \in ku^{-2}$ (also considered in $G_r^{-2,1}$), is $v : G_r^{a,b} \longrightarrow G_r^{a-2,b+1}$, and multiplication by the element representing $2 \in ku^0$, ($h_0 \in G_r^{0,1}$), is $h_0 : G_r^{a,b} \longrightarrow G_r^{a,b+1}$.

We will later define elements $z_j \in G_2^{2^j+2,0}$ for $j \geq 4$ and elements $z_{i,j} \in G_2^{2^j+2+2(j-i),0}$ as

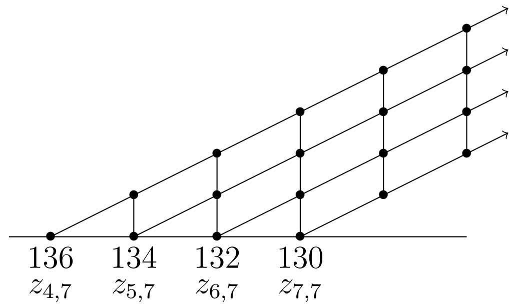
$$z_{i,j} = z_i^2 \prod_{t=1}^{j-i-1} z_{i+t}$$

for $4 \leq i \leq j$ with $z_{j,j} = z_j$, the Ext analogues of (1.2). They will have the properties: $h_0 z_j = v z_{j-1}^2$ for $j \geq 5$, and $h_0 z_4 = 0$. Additionally, $h_0 z_{i,j} = v z_{i-1,j}$, and $h_0 z_{4,j} = 0$.

For $j \geq 4$, we define $W_j = \langle z_{j,j}, z_{j-1,j}, \ldots, z_{4,j}\rangle$. We also have $x_i \in G_2^{i,0}$ for $i = 4, 8, 9, 10$. One last definition, let $\Lambda_{j+1} = E[z_i : i \geq j+1]$.

A picture of $P[v] \otimes W_7$ as a $P[v, h_0]$-module appears in Figure 3.1.

Figure 3.1. A depiction of $P[v] \otimes W_7$



The remainder of this section is devoted to proving the following result.

Theorem 3.2. *The E_2 term of the Adams spectral sequence for the reduced $ku^*(K_2)$ is isomorphic as a $P[h_0, v]$-module to*

$$P[v, x_4] \otimes E[x_9] \otimes \Big(\bigoplus_{j \geq 4} (W_j \otimes \Lambda_{j+1})\Big)$$

$$\oplus \big(P[h_0, v, x_4] \otimes E[v^2 x_9]\big) \oplus \big(P[x_4] \otimes \langle x_8, x_{10}, h_0 x_8 = v x_{10}\rangle\big)$$

plus the family of filtration-0 $\mathbb{Z}_2$'s annihilated by h_0 and v enumerated at the end of this section.

Some of the algebra structure of this E_2 will be useful later. For example, the product structure among the z_j's will be clear, and also the formula

$$(v^2 x_9)^2 = v^4 z_4, \tag{3.3}$$

holds since, as we shall see, in $H^*(K_2)$, $x_9^2 - Q_0 x_{17} \in \mathrm{im}(Q_1)$.

There are two parts to proving this theorem. First, we must give a complete description of the E_1-module structure of H^*K_2. Second, we have to compute $\mathrm{Ext}_{E_1}^{*,*}(\mathbb{Z}_2, -)$ of this. We begin the first part.

Serre ([8]) showed that H^*K_2 is a polynomial algebra on classes u_{2^j+1} in degree $2^j + 1$ for $j \geq 0$ defined by $u_2 = \iota_2$ and $u_{2^{j+1}+1} = \mathrm{Sq}^{2^j} u_{2^j+1}$ for $j \geq 0$. We easily have

$$Q_0(u_2) = u_3,\ Q_0(u_3) = 0,\ Q_0(u_{2^j+1}) = u_{2^{j-1}+1}^2 \text{ for } j \geq 2,$$

and

$$Q_1(u_2) = u_5,\ Q_1(u_3) = u_3^2,\ Q_1(u_5) = 0,\ Q_1(u_{2^j+1}) = u_{2^{j-2}+1}^4 \text{ for } j \geq 3.$$

Let $x_5 = u_5 + u_2u_3$ and write H^*K_2 as an associated graded object:

$$P[u_2^2] \otimes E[x_5] \otimes \big(E[u_2] \otimes P[u_3]\big) \otimes_{j\geq 2} \big(E[u_{2^{j+1}+1}] \otimes P[(u_{2^j+1})^2]\big)$$

From this, we can read off

Lemma 3.4.

$$H_*(H^*K_2; Q_0) = P[u_2^2] \otimes E[x_5]$$

Letting $x_9 = u_9 + u_3^3$ and $x_{17} = u_{17} + u_2u_5^3$, we rewrite again as

$$\begin{gathered} P[u_2^2] \otimes TP_4[x_9] \otimes TP_4[x_{17}] \otimes_{j>4} E[(u_{2^j+1})^2] \\ \otimes\big(E[u_2] \otimes P[u_5]\big) \otimes \big(E[u_3] \otimes P[u_3^2]\big) \otimes_{j>4} \big(E[u_{2^j+1}] \otimes P[(u_{2^{j-2}+1})^4]\big). \end{gathered}$$

Again we read off

Lemma 3.5.

$$H_*(H^*K_2; Q_1) = P[u_2^2] \otimes TP_4[x_9] \otimes TP_4[x_{17}] \otimes_{j>4} E[(u_{2^j+1})^2]$$

An associated graded version of this is

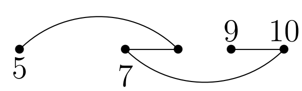
Lemma 3.6.

$$H_*(H^*K_2; Q_1) = P[u_2^2] \otimes E[x_9] \otimes E[x_{17}] \otimes_{j>2} E[(u_{2^j+1})^2]$$

The bulk of the work here is finding a nice splitting of H^*K_2 as an E_1-module.

Let N be the E_1-submodule with single nonzero elements in gradings 5, 7, 8, 9, and 10 with generators $x_5 = u_5 + u_2u_3$, $x_7 = u_2u_5$, and $x_9 = u_9 + u_3^3$, satisfying $Q_0x_7 = Q_1x_5$ and $Q_0x_9 = Q_1x_7 = x_{10}$. It has a Q_0-homology class x_5 and a Q_1-homology class x_9. A picture of N is in Figure 3.7.

Figure 3.7. An E_1-module N.



The E_1-submodule $P[u_2^2] \oplus P[u_2^2] \otimes N$ carries the Q_0-homology of H^*K_2, while the remaining Q_1-homology is, written in our usual way as an associated graded version,

$$P[u_2^2] \otimes E[x_9] \otimes \overline{E}[x_{17}, u_{2^j+1}^2,\ j > 2].$$

We will exhibit a Q_0-free E_1-submodule R whose Q_1-homology is exactly this $\overline{E}$. Moreover, $N \otimes R$ contains an E_1-split summand S which maps isomorphically to $\langle x_9 \rangle \otimes R$.

It is premature to state this because we haven't defined R and S yet, but for the record:

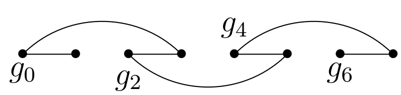
Proposition 3.8. *As an E_1 module, $\widetilde{H}^*K_2$ is isomorphic to $T \oplus F$ where F is a free over E_1 and T is*

$$P[u_2^2] \otimes \big(\langle u_2^2 \rangle \oplus N \oplus R \oplus S\big)$$

A start on R and S.

For this to make sense, we need to find R and S. The module R is a direct sum of shifted versions of modules L_k, $k \geq 0$, which have generators g_{2i}, $0 \leq i \leq k$, with $Q_1 g_{2i} = Q_0 g_{2i+2}$ for $0 \leq i < k$, $Q_0 g_0 \neq 0$, and $Q_1 g_{2k} = 0$. For example, L_3 is depicted in Figure 3.9.

Figure 3.9. The E_1-module L_3.



A splitting map, $\langle x_9 \rangle \otimes L_k \longrightarrow N \otimes L_k$, for the epimorphism $N \otimes L_k \to \langle x_9 \rangle \otimes L_k$ is defined by

$$x_9 g_{2i} \longrightarrow x_9 \otimes g_{2i} + x_7 \otimes g_{2i+2} + x_5 \otimes g_{2i+4} \text{ for } 0 \leq i \leq k-2,$$

$x_9 g_{2k-2} \longrightarrow x_9 \otimes g_{2k-2} + x_7 \otimes g_{2k}$, and $x_9 \otimes g_{2k} \longrightarrow x_9 \otimes g_{2k}$.

The E_1-module M_j

Let

$$x_{2^j+1} = u_{2^j+1} + \begin{cases} u_2 u_5^3 & j = 4 \\ u_2 u_3 u_5^2 u_9^2 & j = 5 \\ u_3 u_5^2 u_9^2 u_{17}^2 & j = 6 \\ 0 & j > 6 \end{cases} \text{ and } w_{2^j-1} = \begin{cases} u_2 u_3 u_5^2 & j = 4 \\ u_3 u_5^2 u_9^2 & j = 5 \\ 0 & j > 5. \end{cases}$$

Then $Q_0 x_{2^j+1} = u^2_{2^{j-1}+1} + Q_1 w_{2^j-1}$, so $Q_0 x_{2^j+1}$ and $u^2_{2^{j-1}+1}$ represent the same Q_1-homology class. Define E_1-modules M_j inductively by $M_3 = 0$, and for $j \geq 4$ there is a short exact sequence of E_1-modules

$$0 \to u^2_{2^{j-2}+1} M_{j-1} \to M_j \to M'_j \to 0, \tag{3.10}$$

where $M'_j = \langle x_{2^j+1}, Q_0 x_{2^j+1}\rangle$ and $Q_1 x_{2^j+1} = u^2_{2^{j-2}+1} Q_0 x_{2^{j-1}+1}$. The above definitions of the x_{2^j+1} are necessary to get this formula to work right.

There is an isomorphism of E_1-modules $M_j \approx \Sigma^{2^j+1} L_{j-4}$ given by

$$\Sigma^{2^j+1} g_{2i} \longrightarrow \begin{cases} x_{2^j+1} & i = 0 \\ u^2_{2^{j-2}+1} x_{2^{j-1}+1} & i = 1 \\ u^2_{2^{j-2}+1} u^2_{2^{j-3}+1} x_{2^{j-2}+1} & i = 2 \\ u^2_{2^{j-2}+1} u^2_{2^{j-3}+1} \cdots u^2_{2^{j-i-1}+1} x_{2^{j-i}+1} & 2 < i \leq j-4 \end{cases} \tag{3.11}$$

And we have

$$H_*(M_j; Q_1) = \begin{cases} \langle u_9^2, u_{17}\rangle & j = 4 \\ \langle u_{17}^2, u_9^2 u_{17}\rangle & j = 5 \\ \langle u_{33}^2, u_{17}^2 u_9^2 u_{17}\rangle & j = 6 \\ \langle u^2_{2^{j-1}+1}, u^2_{2^{j-2}+1} \cdots u_9^2 x_{17}\rangle & j > 6 \end{cases} \tag{3.12}$$

The E_1-module R

Let

$$R = \bigoplus_{j \geq 4} M_j \otimes E[u^2_{2^j+1}, u^2_{2^{j+1}+1}, \ldots]. \tag{3.13}$$

Then $H_*(R; Q_1) = \overline{E}[x_{17}, u_9^2, u_{17}^2, \ldots]$, since monomials in $\overline{E}$ without x_{17} appear from a first term (of the two in (3.12)) in $H_*(M_j \otimes E; Q_1)$, where j is minimal such that $u^2_{2^{j-1}+1}$ appears in the monomial, while those with x_{17}, and also containing a product $u_9^2 \cdots u^2_{2^{j-2}+1}$ of maximal length, occur as a second term in $H_*(M_j \otimes E; Q_1)$.

Proof of Proposition 3.8. We have the E_1-submodule given in Proposition 3.8. Because this contains all of the Q_0 and Q_1 homology, what remains must be free over E_1 by [10]. ■

Proof of Theorem 3.2. We compute $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, T)$ with T as in Proposition 3.8. We will not be concerned with the free E_1-module F but later we will give the Poincaré series for it. Each copy of E_1 in F gives a $\mathbb{Z}_2$ in $G^{*,0}$ that corresponds to Q_0Q_1.

That

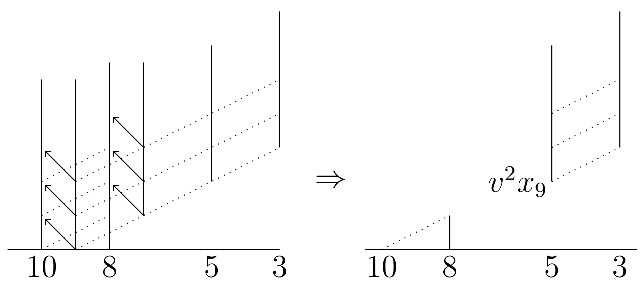
$$\mathrm{Ext}_{E_1}^{*,*}(\mathbb{Z}_2, P[u_2^2]) = P[v, h_0, x_4]$$

with $x_4 \in G_2^{4,0}$ should be clear, given our labeling conventions. We normally work with the reduced cohomologies, so the x_4^0 generator above would be ignored.

We compute $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, N)$ in two ways using two different filtrations of N. From this we see that the generator of the towers can be thought of either as v^2x_9 or $h_0^2x_5$.

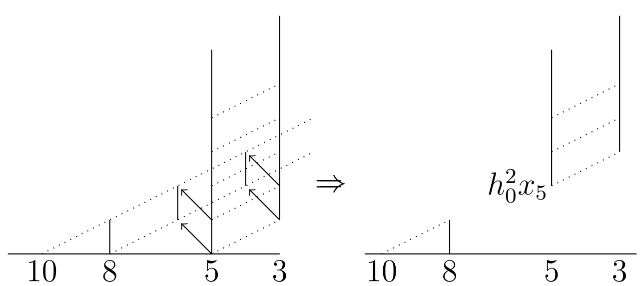
Using Figure (3.7) as our guide, our first filtration is $\langle x_5, x_8\rangle$, $\langle x_7, x_{10}\rangle$, and $\langle x_9\rangle$. The Ext on $x_9 \in G^{9,0}$ is just $P[v, h_0]$. For the other two, we get h_0-towers on $x_{10} \in G^{10,0}$ and $x_8 \in G^{8,0}$. The extensions in N show these two h_0-towers are connected by multiplication by v. In addition, a d_1 is forced on us by the extensions. Figure 3.14 describes this completely.

Figure 3.14. The first computation of $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, N)$



Again referring to the picture (3.7), our second filtration is $\langle x_9, x_{10}\rangle$, $\langle x_7, x_8\rangle$, and $\langle x_5\rangle$. Now our Ext groups are $P[v, h_0]$ on $x_5 \in G^{5,0}$, $P[v]$ on $x_8 \in G^{8,0}$ and $x_{10} \in G^{10,0}$. Again, the d_1 is forced by the extensions in N. Figure 3.15 describes the result.

Figure 3.15. The second computation of $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, N)$



This concludes the computation of Ext for $P[u_2^2] \otimes (\langle u_2^2 \rangle \oplus N)$ of Proposition 3.8. The result is the second line of Theorem 3.2.

We need to compute Ext for $P[u_2^2] \otimes (R \oplus S)$ and show it is the same as the top line in Theorem 3.2. Since $S \approx \langle x_9 \rangle \otimes R$, all we need to do is $P[u_2^2] \otimes R$ and ignore the $E[x_9]$. Similarly we can ignore the $P[u_2^2]$ and the $P[x_4]$ because for every power of u_2^2 we will have a copy of the answer indexed by powers of x_4. All we have left now is R, but R is just many copies of the various M_j and the indexing for the number of copies is given by the Λ_{j+1}.

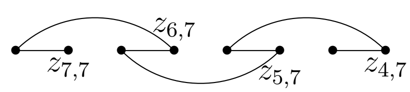
All that remains is to show that $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, M_j) \approx P[v] \otimes W_j$. Recall that $M_j = \Sigma^{2^j+1} L_{j-4}$. We can filter L_{j-4} into pairs of elements $g_{2i}, Q_0 g_{2i}$, for $0 \leq i \leq j-4$. Ext for each of these gives a $P[v]$ on the element $Q_0 g_{2i}$ represented by $z_{j-i,j} \in G^{2^j+2+2i,0}$. There is no d_1, but undoing the filtration does solve the extension problem and gives us $h_0 z_{k,j} = v z_{k-1,j}$. This completes our computation and thus our proof. ■

Remark 3.16. To illustrate the last computation in the proof, consider the generators of the v-towers for $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, M_7)$. They are z_7, z_6^2, $z_5^2 z_6$, and $z_4^2 z_5 z_6$, which is what we have called $z_{7,7}$, $z_{6,7}$, $z_{5,7}$, and $z_{4,7}$, as pictured in Figure 3.1. For future reference, we note that (with $\sim$ meaning homologous)

$$z_j = Q_0 x_{2^j+1} \sim u_{2^{j-1}+1}^2 = Q_0 u_{2^j+1} = Q_0 Q_j \iota_2 = Q_j Q_0 \iota_2. \tag{3.17}$$

We depict the E_1-module M_7 in Figure 3.18.

Figure 3.18. The E_1-module M_7.



More on the E_1-free part

If we compute the $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, F)$ for the E_1-free part of $H^* K_2$, we just get a $\mathbb{Z}_2$ corresponding to the top element for each copy of E_1. If we find the Poincaré series (PS) for the free part, all we have to do to get the PS for these elements is multiply by $\frac{x^4}{(1+x)(1+x^3)}$. The Poincaré series for free part is obtained by subtracting the PS for the non-free part of Proposition 3.8 from that of $H^* K_2$. This is:

$$\prod_{k\ge 0}\frac{1}{(1-x^{2^k+1})}-\frac{1}{(1-x^4)}\big(1+x^5+x^7+x^8+x^9+x^{10}\big)$$
$$-\frac{1}{(1-x^2)(1-x^4)}\Big(\bigoplus_{j\ge 4}\big(x^{2^j+1}(1+x^9)(1+x)(1-x^{2j-6})\prod_{k\ge j}(1+x^{2^{k+1}+2})\big)\Big)$$

The first term is the PS for H^*K_2. The second is the PS for $P[u_2^2]\otimes(\langle 1\rangle\oplus N)$. The last term is more complicated but does the S and R terms. The $(1-x^4)$ in the denominator is for the $P[u_2^2]$. The x^9 is the shift that takes R to S. The $(1+x)$ is because they are Q_0 free. The $x^{2^j+1}(1-x^{2j-6})/(1-x^2)$ is for the odd part of M_j and the remainder is for Λ.

This is easy to put into a computer and calculate. For example, the number of free generators in degree 79 is 245.

4. Differentials in the ASS of $ku^*(K_2)$

The main theorem of this section determines the differentials in the ASS for $ku^*(K_2)$.

Theorem 4.1. *The differentials in the spectral sequence whose E_2-term was given in Theorem 3.2 are as follows. All v-towers are involved, either as source or target, in exactly one of these. Here $\nu(i)$ denotes the exponent of 2 in the integer i, and M refers to any monomial (possibly $=1$) in the specified exterior algebra.*

$$\begin{aligned}
d_{\nu(i)+2}(x_4^i) &= h_0^{\nu(i)}v^2x_4^{i-1}x_9,\ i\ge 1. && (4.2)\\
d_{\nu(i)+2}(x_4^iz_jM) &= v^{\nu(i)+2}x_4^{i-1}x_9z_{j-\nu(i),j}M, && (4.3)\\
&\quad j\ge 4+\nu(i),\ M\in\Lambda_j.\\
d_{2^{t+1}-t-1}(h_0^{t-1}v^2x_4^{2^tk+2^t-1}x_9) &= v^{2^{t+1}}x_4^{2^tk}z_{t+3},\ t\ge 1,\ k\ge 0. && (4.4)\\
d_{2^{t+1}-t-1}(x_4^{2^tk+2^t-1}x_9z_{j-(t-1),j}M) &= v^{2^{t+1}-t-1}x_4^{2^tk}z_{t+3}z_jM, && (4.5)\\
&\quad j\ge t+3,\ M\in\Lambda_{j+1}.
\end{aligned}$$

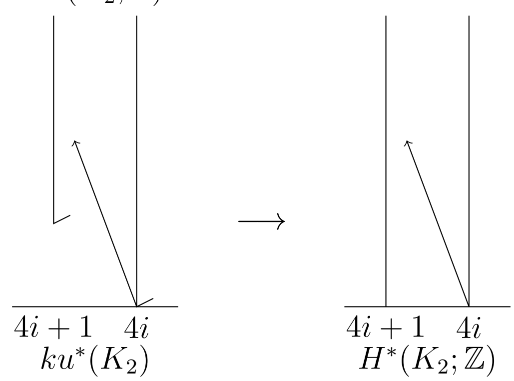
The proof occupies the rest of this section, except that at the end of the section we explain briefly how this leads to our description of $ku^*(K_2)$ in Section 1, except for the exotic extensions.

By [9, Theorem A], $Q_jQ_0\iota_2$ is in the image from $BP^*(K_2)$, and hence must be a permanent cycle in our ASS. Thus by (3.17), z_j is a permanent cycle, and so (4.3) follows from (4.2), and (4.5) follows from (4.4), using (1.3).

The differentials (4.2) follow from the result of [2] or [3, Proposition 1.3.5] that $H^{4i+1}(K_2;\mathbb{Z}) \approx \mathbb{Z}/2^{\nu(i)+2} \oplus \bigoplus \mathbb{Z}_2$. The ASS converging to $H^*(K_2;\mathbb{Z})$ has $E_2 = \operatorname{Ext}_{A_0}(\mathbb{Z}_2, H^*K_2)$, where $A_0 = \langle 1, Q_0\rangle$. We depict this E_2 similarly to our ASS for $ku^*(K_2)$. It has an h_0-tower for each element of $H_*(H^*K_2, Q_0)$, which was described in Lemma 3.4. These come in pairs in grading $4i$ and $4i+1$ corresponding to u_2^{2i} and $u_2^{2i-2}u_5$. There must be a $d_{\nu(i)+2}$-differential, as pictured on the right hand side of Figure 4.6.

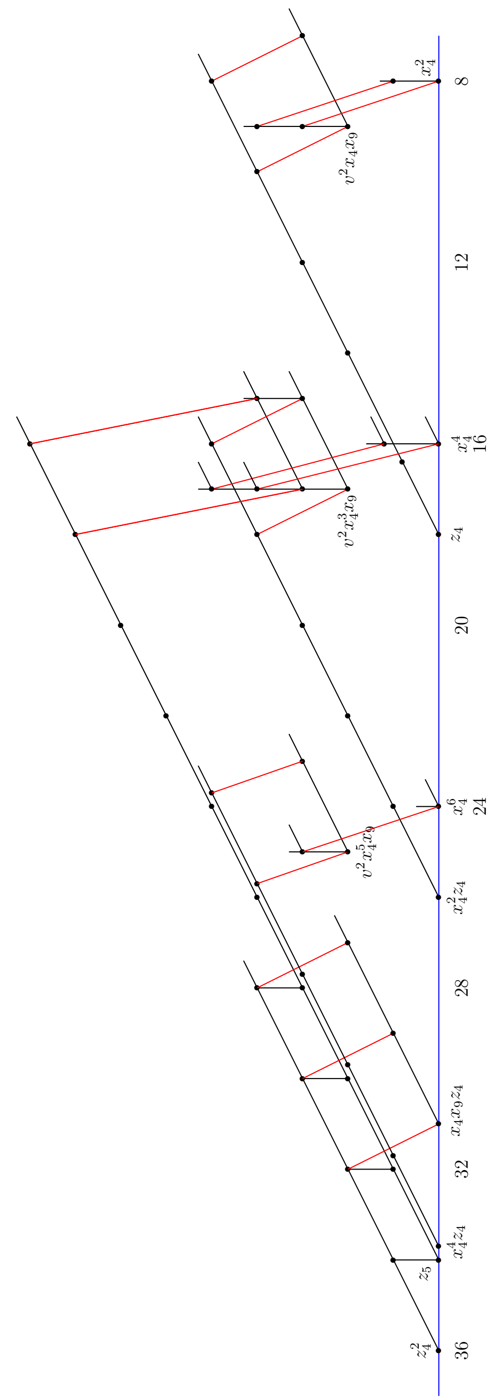
Similarly to Figures 3.14 and 3.15, we have, for $i \geq 1$, an h_0-tower in the ASS for $ku^*(K_2)$ arising from $G^{4i+1,2}$, called either $h_0^2 x_4^{i-1} x_5$ or $v^2 x_4^{i-2} x_9$. There is also an h_0-tower arising from $x_4^i \in G^{4i,0}$. The classes x_4 and x_5 correspond to cohomology classes u_2^2 and $u_5 + u_2u_3$. Under the morphism $ku^*(K_2) \to H^*(K_2;\mathbb{Z})$, these towers map across, as suggested in Figure 4.6. We deduce the $d_{\nu(i)+2}$-differential claimed in (4.2), promulgated by the action of v.

Figure 4.6. $ku^*(K_2) \to H^*(K_2;\mathbb{Z})$



In Figure 4.7, we depict many of the differentials asserted in Theorem 4.1 in grading ≤ 36. Not included in this is the $P[x_4] \otimes \langle x_8, x_{10}, h_0x_8 = vx_{10}\rangle$ portion of Theorem 3.2. (The classes called x_{10} here are sometimes called z_3, because that fits nicely in (1.6).) Also not included are the portions involving (4.2) and (4.3) when i is odd, as this portion self-annihilates. What is shown is (4.2) for $i = 2$, 4, and 6, (4.4) for $(t, k) = (1, 0)$, $(1, 1)$, $(1, 2)$, and $(2, 0)$, and (4.5) with $t = 1$, $k = 0$, and $j = 4$.

Figure 4.7. Some differentials.



In order to establish some of the differentials, we will need the following description of $k(1)^*(K_2)$, which is proved in [6, Theorem 9.3]. It involves classes x_4, x_8, and z_j

for $j \geq 3$, which are reductions of the corresponding classes in $ku^*(K_2)$, an element p_3 which is the reduction of x_9, and an additional class p_4 with $|p_4| = 17$. There are composite elements p_e for $e > 4$ defined recursively by $p_{e+2} = x_4^{2^{e-3}} p_e z_{e+1}$. For $5 \leq e \leq 8$, $|p_e|$ is 31, 59, 113, 221.

We introduce functions h and h' whose first few values are given in Table 1. Successive values can be obtained using $h(e+2) - h(e) = 2^e + 1$ and $h'(e+2) - h'(e) = 2^{e+1} - 1$.

e	1	2	3	4	5	6	7	8	9
$h(e)$	0	2	4	7	13	24	46	89	175
$h'(e)$	1	2	4	9	19	40	82	167	337

Table 1: The functions h and h'

Our description of $k(1)^*(K_2)$ is given in the following theorem.

Theorem 4.8. *$k(1)^*(K_2)$ consists of the following three types of elements.*

a *For each split $\mathbb{Z}_2$ in $ku^*(K_2)$ in grading d, there are split $\mathbb{Z}_2$'s in $k(1)^*(K_2)$ in gradings d and $d - 1$.*

b *Additionally, there are split $\mathbb{Z}_2$'s, also of v-height 1, corresponding to a basis of $\mathbb{Z}_2[x_4] \otimes E[p_3] \otimes \bigoplus_{j \geq 4} z_j^2 \Lambda_{j+1}$, and also $\{x_8, z_3\} \otimes \mathbb{Z}_2[x_4]$.*

c *For $e \geq 2$, there are summands $\overline{E}[p_{e+1}] \otimes E[p_{e+2}] \otimes \mathbb{Z}_2[x_4^{2^{e-1}}] \otimes \Lambda_{e+2}$ and $\overline{E}[z_{e+2}] \otimes E[p_{e+2}] \otimes \mathbb{Z}_2[x_4^{2^{e-1}}] \otimes \Lambda_{e+3}$, consisting of classes of v-height $h(e)$ and $h'(e)$, respectively.*

Proof. Part (c) was proved in [6, Theorem 9.3], with the following correspondence of notation. Our z_j is their z_{j-1}, our p_j is their w_{j-1}, our $h(j)$ is their $r(j-1)$, and our $x_4^{2^j}$ is their y_{j+1}. Part (a) is true since a copy of E_1 with top class in grading d is the sum of copies of $E[Q_1]$ with top classes in grading d and $d - 1$. The classes in part (b) play an important role in Sections 6 and 7. The E_1-module N in Figure 3.7 has free $E[Q_1]$-summands with top classes in gradings 8 and 10, and so the N-part of Proposition 3.8 yields the second part of (b) in the theorem. In Remark 3.16, we illustrate how M_7 has free $E[Q_1]$-summands with top classes corresponding to z_6^2, $z_5^2 z_6$, and $z_4^2 z_5 z_6$. Thus the $j = 7$ summand in (3.13) contributes to the R-part of Proposition 3.8 all monomials in $\bigoplus_{j \geq 4} z_j^2 \Lambda_{j+1}$ whose first omitted factor is z_7, and so consideration of all $j \geq 4$ in (3.13) yields all of $\bigoplus_{j\geq 4} z_j^2\Lambda_{j+1}$. For the S-part of Proposition 3.8, this is just tensored with $x_9 = p_3$. ■

Elements of the first few v-heights in $k(1)^*(K_2)$ are listed in Table 2.

v-height	elements
$h(2) = 2$	$\overline{E}[p_3] \otimes E[p_4] \otimes \mathbb{Z}_2[x_4^2] \otimes \Lambda_4$
$h'(2) = 2$	$\overline{E}[z_4] \otimes E[p_4] \otimes \mathbb{Z}_2[x_4^2] \otimes \Lambda_5$
$h(3) = 4$	$\overline{E}[p_4] \otimes E[p_5] \otimes \mathbb{Z}_2[x_4^4] \otimes \Lambda_5$
$h'(3) = 4$	$\overline{E}[z_5] \otimes E[p_5] \otimes \mathbb{Z}_2[x_4^4] \otimes \Lambda_6$
$h(4) = 7$	$\overline{E}[p_5] \otimes E[p_6] \otimes \mathbb{Z}_2[x_4^8] \otimes \Lambda_6$
$h'(4) = 9$	$\overline{E}[z_6] \otimes E[p_6] \otimes \mathbb{Z}_2[x_4^8] \otimes \Lambda_7$
$h(5) = 13$	$\overline{E}[p_6] \otimes E[p_7] \otimes \mathbb{Z}_2[x_4^{16}] \otimes \Lambda_7$
$h'(5) = 19$	$\overline{E}[z_7] \otimes E[p_7] \otimes \mathbb{Z}_2[x_4^{16}] \otimes \Lambda_8$
$h(6) = 24$	$\overline{E}[p_7] \otimes E[p_8] \otimes \mathbb{Z}_2[x_4^{32}] \otimes \Lambda_8$

Table 2: Elements of $k(1)^*(K_2)$

Two things from Theorem 4.8 that will be important in proving the differentials in the ASS of $ku^*(K_2)$ are summarized in the following corollary.

Corollary 4.9. (1) *In the morphism of ASSs induced by* $ku^*(K_2) \xrightarrow{\rho} k(1)^*(K_2)$, *the v-towers on* $x_4^{2^{e-3}j} z_e$ *map across. The target tower is truncated at height* $h'(e-2)$, *and so* $\rho(v^s x_4^{2^{e-3}j} z_e) = 0$ *for* $s \geq h'(e-2)$, *as there are no higher-filtration elements for it to hit.*

(2) *In* $k(1)^{*-1}(K_2) \to ku^*(K_2)$, $|v^{h(e-1)}p_e| = |v^{2^{e-2}}z_e| - 1$, *which will be important in deducing that* $v^{2^{e-2}}z_e$ *is hit by a differential.*

Now we continue the proof of Theorem 4.1. We have already proved (4.2) and (4.3). As noted earlier, the z_j's are infinite cycles by [9], and so the differentials in (4.5) are implied as soon as the corresponding differential in (4.4) is proved. We start with the case $t = 1$ of (4.4). In even gradings ≤ 14, $k(1)^*(K_2) = 0$ in positive filtration, using Table 2. Thus the map $ku^*(K_2) \to k(1)^*(K_2)$ implies that in $ku^*(K_2)$, $v^s z_4$ is either hit by a differential or divisible by 2 for $s \geq 2$. In grading < 8, there is nothing that can divide it, and the only odd-grading v-tower in that range is on $v^2x_4x_9$. Thus $d_2(v^2x_4x_9) = v^4z_4$, the case $t = 1$, $k = 0$ of (4.4). Since $d_2(x_4^{2k}) = 0$ by (4.2), the case $t = 1$ of (4.4) follows for any k by the derivation property.

Similarly v^sz_5 must be hit or divisible for $s \geq 4$, and examination of options in Figure 4.7 shows that we must have $d_5(h_0v^2x_4^3x_9) = v^8z_5$, preceded by extensions. Since $d_5(x_4^8) = h_0^3v^2x_4^7x_9$, we deduce the case $t = 2$, k even of (4.4) using the derivation property, (3.3), and $h_0z_4 = 0$. We do not have *a priori* knowledge that $x_4^4z_5$ is a permanent cycle in the ASS of $ku^*(K_2)$. However, if it supported a nonzero differential, then the tower of v-height 4 on $x_4^4z_5$ in the ASS of $k(1)^*(K_2)$ would have to map to v^tC for $0 \leq t \leq 3$ for some C in positive filtration in grading 51 in the ASS of $ku^*(K_2)$. Then v^4C must be $d_r(B)$ with $r \geq 5$ and B in filtration 0 in grading 42. (B cannot have higher filtration since everything is v-towers, and v^3C cannot be hit.) But the only possible B is $x_4^6z_4$, and we already know that $v^4x_4^6z_4 \in \operatorname{im}(d_4)$. (Ordinarily this would not preclude the possibility of B supporting a differential, but it does since everything is v-towers.) Thus $x_4^4z_5$ is a permanent cycle, and consideration of its image in $k(1)^*(K_2)$ implies that $v^sx_4^4z_5$ is hit by a differential for some $s \geq 4$. The only element in odd grading < 42 not yet accounted for is $h_0v^2x_4^7x_9$ in grading 33. This is the case $t = 2$, $k = 1$ of (4.4). The validity for all odd k (and $t = 2$) now follows similarly to what we did for even k at the beginning of this paragraph.

The proof of (4.4) for $t \geq 3$ is much more delicate. For all non-2-powers n, write $n = 2^p(2k+1)$ and let $T(n) = v^2h_0^{p+2}x_4^{2^{p+3}k-1}x_9$ and $M(n) = x_4^{2^{p+3}(k-1)}z_{p+6}$. We will prove $d_{2^{p+4}-p-4}(T(n)) = v^{2^{p+4}}M(n)$, which is (4.4), with a new k. From now on, we will denote such a differential as $T(n) \to M(n)$. If we write $T(n) \to M(m)$, then the exponent of v accompanying $M(m)$ will be $\frac{1}{2}(|M(m)| - |T(n)| - 1)$. In Table 3, we consider the range $33 \leq n \leq 63$. We also include $n = 96$ for future reference. We omit writing the v^2x_9 factors of $T(n)$, and write x instead of x_4. The values $M'(n) = |M(n)| - 2h'(p+4)$ will be important, as we shall explain later.

There are two main constraints. Constraint (1) says that if $T(n) \to M(m)$, then $|T(n)| < M'(m)$. This is true since the image of $M(m)$ in $k(1)^*(K_2)$ has v-height $h'(p+4)$, with $p = \nu(m)$. Thus the v-tower on $M(m)$ cannot be hit by a differential in grading $> |M(m)| - 2h'(p+4) = M'(m)$. This also requires that we know, as in

n	p	k	$\|T(n)\|$	$\|M(n)\|$	$M'(n)$	$T(n)$	$v^{2^{p+4}}M(n)$
33	0	16	513	546	528	$h_0^2x^{127}$	$v^{16}x^{120}z_6$
34	1	8	513	578	540	$h_0^3x^{127}$	$v^{32}x^{112}z_7$
35	0	17	545	578	560	$h_0^2x^{135}$	$v^{16}x^{128}z_6$
36	2	4	513	642	562	$h_0^4x^{127}$	$v^{64}x^{96}z_8$
37	0	18	577	610	592	$h_0^2x^{143}$	$v^{16}x^{136}z_6$
38	1	9	577	642	604	$h_0^3x^{143}$	$v^{32}x^{128}z_7$
39	0	19	609	642	624	$h_0^2x^{151}$	$v^{16}x^{144}z_6$
40	3	2	513	770	606	$h_0^5x^{127}$	$v^{128}x^{64}z_9$
41	0	20	641	674	656	$h_0^2x^{159}$	$v^{16}x^{152}z_6$
42	1	10	641	706	668	$h_0^3x^{159}$	$v^{32}x^{144}z_7$
43	0	21	673	706	688	$h_0^2x^{167}$	$v^{16}x^{160}z_6$
44	2	5	641	770	690	$h_0^4x^{159}$	$v^{64}x^{128}z_8$
45	0	22	705	738	720	$h_0^2x^{175}$	$v^{16}x^{168}z_6$
46	1	11	705	770	732	$h_0^3x^{175}$	$v^{32}x^{160}z_7$
47	0	23	737	770	752	$h_0^2x^{183}$	$v^{16}x^{176}z_6$
48	4	1	513	1026	692	$h_0^6x^{127}$	$v^{256}z_{10}$
49	0	24	769	802	784	$h_0^2x^{191}$	$v^{16}x^{184}z_6$
50	1	12	769	834	796	$h_0^3x^{191}$	$v^{32}x^{176}z_7$
51	0	25	801	834	816	$h_0^2x^{199}$	$v^{16}x^{192}z_6$
52	2	6	769	898	818	$h_0^4x^{191}$	$v^{64}x^{160}z_8$
53	0	26	833	866	848	$h_0^2x^{207}$	$v^{16}x^{200}z_6$
54	1	13	833	898	860	$h_0^3x^{207}$	$v^{32}x^{192}z_7$
55	0	27	865	898	880	$h_0^2x^{215}$	$v^{16}x^{208}z_6$
56	3	3	769	1026	862	$h_0^5x^{191}$	$v^{128}x^{128}z_9$
57	0	28	897	930	912	$h_0^2x^{223}$	$v^{16}x^{216}z_6$
58	1	14	897	962	924	$h_0^3x^{223}$	$v^{32}x^{208}z_7$
59	0	29	929	962	944	$h_0^2x^{231}$	$v^{16}x^{224}z_6$
60	2	7	897	1026	946	$h_0^4x^{223}$	$v^{64}x^{192}z_8$
61	0	30	961	994	976	$h_0^2x^{239}$	$v^{16}x^{232}z_6$
62	1	15	961	1026	988	$h_0^3x^{239}$	$v^{32}x^{224}z_7$
63	0	31	993	1026	1008	$h_0^2x^{247}$	$v^{16}x^{240}z_6$
96	5	1	1025	2050	1376	$h_0^7x^{255}$	$v^{512}z_{11}$

Table 3: Differentials

the case of $x_4^4 z_5$ discussed earlier, that each $M(n)$ is a permanent cycle. We prove this in Lemma 4.10. Constraint (2) says that if $n_1 < n_2$ and $|T(n_1)| = |T(n_2)|$ and $T(n_1) \to M(m_1)$ and $T(n_2) \to M(m_2)$, then $|M(m_1)| < |M(m_2)|$. This is true since moving up an h_0-tower requires higher differentials.

Lemma 4.10. *In the algorithm described in this section, $M(n)$ is a permanent cycle.*

Proof. Recall that $M(n) = x_4^{2^{p+3}(k-1)} z_{p+6}$. We present the proof when $p = 1$, and then explain how it generalizes. The algorithm illustrated in Table 3 purports to prove that $d_{27}(v^2 h_0^3 x_4^{16k-1} x_9) = v^{32} x_4^{16(k-1)} z_7$, and an important part of the argument is that, by consideration of the image of $x_4^{16(k-1)} z_7$ in the ASS for $k(1)^*(K_2)$, the v-tower on $x_4^{16(k-1)} z_7$ is hit by a d_r-differential with $r \geq 19$. This argument would go awry if $x_4^{16(k-1)} z_7$ supported a differential in the ASS of $ku^*(K_2)$. If it did support a differential, then in the ASS morphism of $k(1)^*(K_2) \to ku^{*+1}(K_2)$, the height-19 v-tower on $x_4^{16(k-1)} z_7$ will map nontrivially, increasing filtration by at least 1. The target v-tower must be truncated by a d_r-differential with $r \geq 20$ emanating from filtration 0 in grading $64(k-1) + 130 - 38 = 64k + 28$. We seek to show that no such differential is possible.

The class supporting such a differential cannot be an $M(m)$ with $m < n$, since they have already been shown to be targets of differentials, nor can it be a product of z_j's times such $M(m)$, for the same reason. It can't be an $M(m)$ with $m \geq n$ because their grading is too large.

We must also rule out the possibility that this unwanted differential is one of the (4.3) differentials. If so, the i in (4.3) must satisfy $\nu(i) \geq 18$, and the class supporting the differential is $x_4^i Z$, where Z is a product of z_j's with $j \geq 22$ and all j's distinct, except that the smallest one might occur twice. Since $|z_j| = 2^j + 2$, $|x_4^i Z| = 64k + 28$ implies that there must be 14 z_j's, with the largest j being ≥ 34. Hence $64k > 2^{34}$.

If k is minimal such that $d_{27}(v^2 h_0^3 x_4^{16k-1} x_9) = v^{32} x_4^{16(k-1)} z_7$ does not hold due to the problem we have been describing, then we have just seen that $16k > 2^{32}$. By the minimality assumption, the d_{27} formula is valid if $16k$ is replaced by $16k - 2^{32}$. By (4.2), $d_{27}(x_4^{2^{32}}) = 0$. Hence by the derivation property, the formula holds as stated.

For arbitrary p, the above argument goes through with

$$(16k, 7, 27, 19 \pm 1, 64k, 28, 22, 14, 34, 2^{32})$$

replaced by

$$(2^{p+3}k, 6+p, 2^{p+4}-(p+4), h'(p+4)\pm 1, 2^{p+5}k, 2^{p+5}+2-2h'(p+4),$$
$$h'(p+4)+3, 2^{p+4}+1-h'(p+4), 2^{p+4}+2, 2^{2^{p+4}}).$$

The final step follows from $d_{2^{p+4}-(p+4)}(x_4^{2^{2^{p+4}}})=0$. ■

Now we can explain how the description of ku^{od} in (1.11) is obtained from (4.3) and Lemma 4.10. We illustrate with the case $k=7$ in (1.11), so we want $x_4^{15}x_9S_{7,\ell}$ for $\ell\ge 8$. It is formed from $P[v]x_4^{15}x_9W_\ell$ (with W_ℓ as in Theorem 3.2) by truncating the first (leftmost) $\ell-7$ v-towers at height 6, while the last four support differentials. The differentials from (4.3) are

$$\begin{aligned} d_6(x_4^{16}z_j\cdot z_j\cdots z_{\ell-1}) &= v^6x_4^{15}x_9z_{j-4,j}z_j\cdots z_{\ell-1}\\ &= v^6x_4^{15}x_9z_{j-4,\ell},\quad 8\le j\le \ell-1 \\ d_6(x_4^{16}z_\ell) &= v^6x_4^{15}z_{\ell-4,\ell}. \end{aligned}\tag{4.11}$$

After tensoring with $P[x_4^{2^{k-2}}]\otimes\Lambda_{\ell+1}$, all of (1.11) is obtained in this way.

The last $\nu(e+1)$ v-towers in $x_4^eW_\ell$ support differentials. To see this, first note that, similarly to (4.11), the image of (4.3) hits v-towers on all $x_4^ex_9z_{s,j}\Lambda_{j+1}$ with $j-s\ge\nu(e+1)$. In $P[v,x_4]x_9\bigoplus_{j\ge4}W_j\otimes\Lambda_{j+1}$ of Theorem 3.2, this is all but the last $\nu(e+1)$ v-towers in the W_j's. By Lemma 4.10 and the fact that z_j's are permanent cycles, all the v-towers on the right-hand side of (4.4) and (4.5) are permanent cycles. Thus there is nothing which can hit these last $\nu(e+1)$ odd-graded v-towers, and since no infinite v-towers are present in E_∞ by [1], we deduce the claim of this paragraph. Thus the elements of (1.11), which were obtained in the preceding paragraph, are the totality of $ku^{\mathrm{od}}(K_2)$.

Now we proceed with the proof of (4.4) for $t\ge3$. We begin by showing that if we have proved $T(n)\to M(n)$ for all non-2-power $n\le 8a$, then $T(8a+b)\to M(8a+b)$ for $1\le b\le 3$. We show this for $a=4$, and then note that the same argument works for any a since $n\not\equiv 0$ mod 8 implies that increasing n by 8 increases each of $|T(n)|$, $|M(n)|$, and $M'(n)$ by 128. Refer to Table 3. Constraint (1) implies that $M(33)$ and $M(34)$ must be hit by some $T(n)$ with $|T(n)|<540$ so $|T(n)|=513$, and by Constraint (2) this must be $T(33)\to M(33)$ and $T(34)\to M(34)$. Constraint (1) says that $M(35)$ must be hit by some $T(n)$ with $|T(n)|=513$ or 545, and Constraint (2) says it cannot be hit by one with $|T(n)| = 513$ since $|M(35)| = |M(34)|$. Therefore $T(35) \to M(35)$.

Constraints (1) and (2) allow a possibility of $T(16i+4) \to M(16i+5)$, $T(16i+5) \to M(16i+6)$, $T(16i+6) \to M(16i+8)$, and $T(16i+8) \to M(16i+4)$ for $i \geq 1$. Since this alternative involves an aberration of a d_{12}-differential, and $x_4^{2^{10}}$ survives to E_{12}, multiplicativity implies that the first time that this alternative might occur must be in grading $< 2^{12}$. If $i = 2j+1$ is odd, this alternative would say that $v^{96}x_4^{128j}z_9$ is hit by a differential. Theorem 4.8 says that $k(1)^*(K_2)$ has classes $x_4^{128j}p_9$ with v-height 89. We have $|v^{89}x_4^{128j}p_9| = 257 + 512j = |v^{128}x_4^{128j}z_9| - 1$, and the expectation is that in the $k(1)^{*-1}(K_2) \to ku^*(K_2)$ portion of the exact sequence, $v^{89-s}x_4^{128j}p_9$ maps to $v^{128-s}x_4^{128j}z_9$ for $1 \leq s \leq 32$. In the alternative scenario, with $v^{96}x_4^{128j}z_9 = 0$, there is nothing for $v^{89-s}x_4^{128j}p_9$ to hit for $1 \leq s \leq 32$. (This is easy to check because of our order of listing the classes. For example, letting $j = 1$, all subsequent $|T(n)|$'s are > 833, so all the higher v-towers are truncated before they get to grading 833.) So these classes must be in the image from $ku^{*-1}(K_2)/2$. In odd gradings, these are just the $S_{k,\ell}$ classes,[3] which have v-heights $k-1$ arising from filtration 0 in gradings $> 2^{k+1}$, roughly. In grading $< 2^{12}$, which is where we noted the first case of the alternative scenario must occur, the maximum v-height in $S_{k,\ell}$'s is 10, which is not nearly large enough to map onto the portion of the p_9-tower that needs to be hit. This shows that this alternative scenario cannot occur when i is odd.

Combining this with the previous observation about the first few values of n yields the desired $T(n) \to M(n)$ for $32j + 17 \leq n \leq 32j + 27$, and the result for $32j + 28 \leq n \leq 32j + 31$ follows easily from Constraints (1) and (2), as can be seen in lines 60 to 62 of Table 3.

When i is even, a different argument must be used because $x_4^{64}p_9$ does not exist in $k(1)^*(K_2)$. For $i = 2$, we will be considering values of n in Table 3 from 36 to 48, and a similar argument applies for any $i = 4j + 2$. There are various scenarios consistent with Constraints (1) and (2) for which it is not the case that $T(n) \to M(n)$ for all n in this range.

[3]We introduce the term "$S_{k,\ell}$ classes" to refer to the classes of (1.11), so they are accompanied by $x_4^{2^{k-3}-1}x_9$ and perhaps by powers of $x_4^{2^{k-2}}$ and monomials in $\Lambda_{\ell+1}$.

Assume first that there is an odd number n in this range for which it is not the case that $T(n) \to M(n)$. Then there is a deviation from a d_{12}-differential, and so, as above, we can assert that the first such deviation occurs in grading $< 2^{12}$. (For $i = 2$, we are clearly in grading $< 2^{12}$, but this argument is applying to all $i = 4j+2$.) If it is not the case that $T(48) \to M(48)$, then $v^s z_{10}$ is hit by a differential for some $s \le 224$, since the only $|T(n)|$'s not yet handled are ≥ 577. The $v^{175-t} p_{10}$ which wanted to map to $v^{256-t} z_{10}$ will be mapping to 0 for $t \le 32$. It must be hit by a v-tower of height ≥ 32 in some $S_{k,\ell}$, but these have v-height < 12 in grading $< 2^{12}$. Thus we conclude that $T(48) \to M(48)$, and $v^{255} z_{10} \ne 0$ in $ku^*(K_2)$.

However, the image of $v^s z_{10}$ in $k(1)^*(K_2)$ is 0 for $s \ge 167$, as there is nothing for it to hit. Thus these elements must be divisible by 2, and so there is an element C in $ku^{692}(K_2)$ (with $2C = v^{167} z_{10}$) such that $v^{88} C \ne 0$. The only possible C is $v^{39} x_4^{64} z_9$, and so $v^{127} x_4^{64} z_9 \ne 0$. Therefore $M(40)$ must be hit by $T(40)$. It is easy to check that this, together with Constraints (1) and (2), implies that $T(n) \to M(n)$ for $33 \le n \le 48$, and similarly for any $33 + 64j \le n \le 48 + 64j$, contradicting the assumption that $T(n) \not\to M(n)$ for some odd n in this range.

Now we may assume that $T(n) \to M(n)$ for all odd n in the range under consideration. One easily checks that Constraints (1) and (2) then imply that either $T(n) \to M(n)$ for all n in $[33, 48]$ or else there is a deviation from a d_{27}-differential. Hence the first such deviation must occur in grading $< 2^{27}$ (since $x_4^{2^{25}} \in E_{27}$). Since $27 < 32$, the same argument as above applies. But for subsequent continuation of the argument, we strengthen it. Under this assumption about $T(n) \to M(n)$ for all odd n, some ranges in the previous argument can be doubled. If $T(48) \not\to M(48)$, then $v^s z_{10}$ is hit for some $s \le 256 - 64$. Then part of the v-tower on p_{10} must be hit by a v-tower of height ≥ 64 in an $S_{k,\ell}$, but, for the first occurrence, these heights are ≤ 27. Hence $v^{255} z_{10} \ne 0$ in $ku^*(K_2)$. The second part of the argument, involving $M(40)$, goes through exactly as above, and so we have proved $T(n) \to M(n)$ for $33 + 64j \le n \le 48 + 64j$.

Next we consider the cases where $n \in [65, 80] \cup \{96\}$, the only remaining cases less than 128. For $n \in [65, 80]$, the values of $|T(n)|$, $|M(n)|$, and $M'(n)$ are 512 greater than those for $n - 32$ tabulated in Table 3, and the $v^{2^{p+4}} M(n)$ column has an extra factor of x^{128}. The entries for $n = 96$ are in the last line of Table 3.

A direct adaptation of the argument used for $n\in[33,48]$ breaks down where it said "the $v^{175-t}p_{10}$ which wanted to map to $v^{256-t}z_{10}$" because the z_{10} is now multiplied by x_4^{128}, and there is not a corresponding class $x_4^{128}p_{10}$ in $k(1)^*(K_2)$.

If it is not the case that $T(96)\to M(96)$, then v^sz_{11} is hit by a differential for some $s\le 512-2^\rho$, where $\rho=5$ if $T(n)\not\to M(n)$ for some odd n, else $\rho=6$ if $T(n)\not\to M(n)$ for some $n\equiv 2$ mod 4, else $\rho=7$. Similarly to the earlier argument, the last 2^ρ classes on the v-tower on p_{11} will have to be hit by a v-tower from some $S_{k,\ell}$, but, for the first such occurrence, the maximum v-heights in any $S_{k,\ell}$ are $\le 2^{\rho-1}-(\rho-1)$. (Here we are again using the derivation property and (4.2).) Thus $T(96)\to M(96)$, and the v-tower on z_{11} in $ku^*(K_2)$ has height 512.

The image of z_{11} in $k(1)^*(K_2)$ has v-height 337, and there is nothing else for the end of the v-tower on z_{11} in $ku^*(K_2)$ to hit. Thus there is a class C in $ku^*(K_2)$ with $2C=v^{337}z_{11}$ and $v^{174}C\ne 0$. The only possible C is $v^{81}x_4^{128}z_{10}$, and so $v^{255}x_4^{128}z_{10}\ne 0$, and hence $T(80)\to M(80)$. (Constraint (2) implies that $M(80)$ could not be hit by $T(72)$, since there would be nothing with larger $|M(m)|$ for $T(80)$ to hit.)

Now we do a similar step to show that $T(72)\to M(72)$. Indeed, the image of $x_4^{128}z_{10}$ in $k(1)^*(K_2)$ has v-height 167, and so $v^{167}x_4^{128}z_{10}$ must be $2C'$ with $v^{88}C'\ne 0$, and the only possibility is $v^{39}x_4^{192}z_9$. Hence $v^{127}x_4^{192}z_9\ne 0$, and $T(72)\to M(72)$. We now easily deduce using Constraints (1) and (2) that $T(n)\to M(n)$ for all n in $[65,80]\cup\{96\}$, and similarly for shifts of this by multiples of 128.

We have now shown that $T(n)\to M(n)$ for all non-2-power $n\le 127$, and in the range $[129,255]$ all are done except for $[129,144]\cup\{160,192\}$. These can be handled by the same method as used above, with one extra step. If these values are increased by multiples of 256, the same argument applies. This procedure can be continued for all n.

We discuss briefly how Theorems 3.2 and 4.1 lead to (1.1), modulo exotic extensions. We have already seen, in the discussion surrounding (4.11), how the description of $ku^{\mathrm{od}}(K_2)$ in (1.11) follows from Theorems 3.2 and 4.1.

The part of Theorem 3.2 called $\langle x_8,x_{10},h_0x_8=vx_{10}\rangle$ is A_3. (Recall that $x_{10}=z_3$.) Then $x_4^{2^i-1}x_4^{c2^{i+1}}A_3$ is a subset of $x_4^{c2^{i+1}}A_{3+i}$. Thus the second half of the second displayed line of Theorem 3.2 exactly yields the A_3-portion of (1.5) tensored with $\mathbb{Z}_2[x_4^{2^{k-2}}]$.

All elements in the part of Theorem 3.2 called $P[h_0, v, x_4]v^2x_9$ are either targets in (4.2) or support differentials in (4.4), while the $P[h_0, v, x_4]$ part of Theorem 3.2 supports differentials in (4.2).

This leaves the v-towers on monomials $x_4^t z_{i,j}\Lambda_{j+1}$ with $4 \le i \le j$. Those with $i \ge 4+\nu(t)$ support differentials (4.3). Those with $\nu(t) \ge i-3$ are hit by differentials (4.4) and (4.5), and the v-heights are as in our definitions of T_k^A and T_k^B in Section 1. It remains to see how these monomials occur in the summands of (1.1).

It is convenient to let $y_i = x_4^{2^{i-3}}$ and $E_i = E[y_j, z_j : j \ge i]$. The monomials in question are all those of the form $z_i M$ with $M \in E_i$, $i \ge 4$. Let k be the smallest integer $\ge i$ such that either both or neither of y_k and z_k are factors of M. If we divide (1.1) into its three parts, including the $\mathbb{Z}_2[x_4^{2^{k-2}}]$ in each, then the first (resp. third) part has those monomials containing neither y_k nor z_k in M, and no (resp. some) factors z_p with $p > k$, while the second part is those with both y_k and z_k. Moreover, the k in (1.1) agrees with the k in this paragraph.

For example, we consider the second part of (1.1) with $k = 7$. All terms have factors y_7z_7, and possibly some factors y_j and z_j with $j > 7$. The z_4E_4 terms have, in addition to these and the z_4, the following factors corresponding to the successive summands in (1.9).

$$z_4z_5z_6,\ z_4z_5y_6,\ z_4y_5z_6,\ z_4y_5y_6,\ y_4z_5z_6,\ y_4y_5z_6,\ y_4z_5y_6,\ y_4y_5y_6.$$

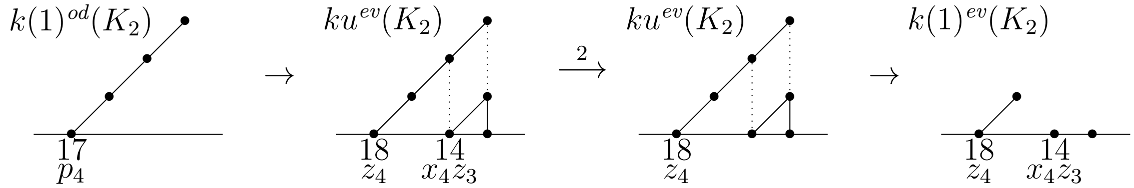
These can be seen in Figure 1.10 in gradings 126, 102, 118, 84, 126, 108, 92, and 74, respectively. There are also monomials in z_5E_5, z_6E_6, and z_7.

5. The exotic extensions

The extensions in (1.6) are established in various A_k. They are then promulgated under multiplication by products of one or more z_j's. Parts of the formula are implied by h_0 in Ext. The rest are deduced using the exact sequence (1.19).

The first exotic extension, $2x_4z_3 = v_1^2 z_4$, can be seen in the lower right corner of Figure 1.10, after dividing by x_4^{14}. To prove it, first note that the v-tower on $z_4 \in ku^{18}(K_2)$ has height 4. The elements v^2z_4 and v^3z_4 map to 0 in $k(1)^*(K_2)$, since it contains no elements in even grading ≤ 18 in filtration > 1. Table 2 is useful in seeing this. Thus v^2z_4 and v^3z_4 must be in the image of $\xrightarrow{2}$, hence the extension. Figure 5.1 shows the relevant elements in this portion of the exact sequence (1.19).

Figure 5.1. Portion of exact sequence.



A similar argument works to prove

$$2 \cdot x_4^{2^{j-3}} z_j = v x_4^{2^{j-3}} z_{j-1}^2 + v^{2^{j-2}} z_{j+1}, \tag{5.2}$$

which was the last equation in (1.6). The first term is seen in Ext. To see the second term, we consider $j = 6$ as a typical example. It has the advantage that we can refer to Figure 1.10. The v-heights of z_7 in $ku^*(K_2)$ and $k(1)^*(K_2)$ are 32 and 19, respectively. The elements $v^m z_7$ for $20 \le m \le 31$ are in filtration ≥ 20 in gradings ≤ 90. It is easy to check that $k(1)^*(K_2)$ is 0 in this range. Thus these $v^m z_7$ must all be divisible by 2 in $ku^*(K_2)$. The elements $v^{m-16} x_4^8 z_6$ are the only possible classes that can do this. [[If $2 \cdot C = v^{20} z_7$, then $2 \cdot v^{11} C = v^{31} z_7 \ne 0$. But $v^4 x_4^8 z_6$ is the only class C with $|C| = 90$ and $v^{11} C \ne 0$. Other multiples of z_6 are not in this range, and the v-height of z_5 is 8.]] Knowing that $v^{20} z_7 = 2 v^4 x_4^8 z_6$ implies (5.2) for $j = 6$, as is easily seen in Figure 1.10. Essentially the same argument works for all z_j.

A similar argument applies to deduce that (5.2) is valid after multiplication by $x_4^{c2^{j-2}}$. The same comparison of v-heights applies as when $c = 0$. This was discussed in part (1) of Corollary 4.9. Thus $v^m x_4^{c2^{j-2}} z_{j+1}$ is divisible by 2 for $m \ge h'(j-1)$. It is convenient to also be in the range where $h_0 x_4^{c2^{j-2}} x_4^{2^{j-3}} z_j = 0$. This will occur for $v^m x_4^{c2^{j-2}} z_{j+1}$ with $m \ge 2^{j-2} + 2^{j-3} - j + 2$. This requires slightly larger values of m than did the $h'(j-1)$ condition. For example, the values are 19 and 20 when $j = 6$, and are 40 and 43 when $j = 7$. For $m = 2^{j-2} + 2^{j-3} - j + 2$, there must be an element Y in $ku^*(K_2)$ with $2Y = v^m x_4^{c2^{j-2}} z_{j+1}$ and $v^{2^{j-1}-1-m} Y \ne 0$ (since $v^{2^{j-1}-1} x_4^{c2^{j-2}} z_{j+1} \ne 0$). The only possible Y is $v^{m-2^{j-2}} x_4^{c2^{j-2}} x_4^{2^{j-3}} z_j$.

Table 3 can help us see this. We consider a specific case, $j = 7$, $c = 6$, but it should be clear that it generalizes. The relevant lines of Table 3 are $57 \le n \le 60$. The nice thing is that the table shows all[4] classes that are not products[5] of more than one z,

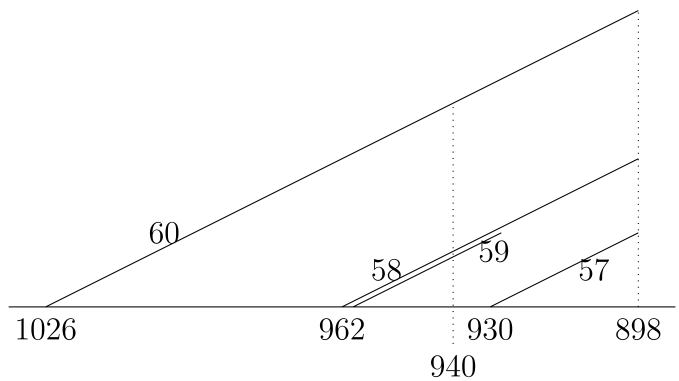
[4]The table does not include the short v-towers on z_4 and z_5. These could be filled in, at the expense of greatly lengthening the table.

[5]Regarding other classes, see Remark 5.4.

and it lists the towers roughly in order of grading. Figure 5.3 depicts the only four relevant v-towers in this range, labeled by their n-value. The class Y has $|Y| = 940$.

The key thing is that tower 57 lies outside grading 940, and tower 59 does not extend far enough back to support the extension all the way back, as must occur. It must be the class in tower 58 which supports the extension. In general, ignoring the $x_4^{c2^{j-2}}$, the extension occurs into $v^{2^{j-2}+2^{j-3}-j+2}z_{j+1}$, and the next lower v-tower (after the one that works) is $x_4^{2^{j-3}+2^{j-4}}z_{j-1}$, whose grading is lower than that of the extension.

Figure 5.3. Depiction of some v-towers.



Remark 5.4. Because z_i's are elements of $ku^*(K_2)$, multiplication by z_i preserves extension formulas. This explains why the class which extends into $v^m x_4^{c2^{j-2}} z_{j+1}$ cannot be divisible by more than one z_i. This is because the first such occurrence would be on a class z_iC for which $2 \cdot C$ has already been seen to be compatible with our extension formulas.

6. Proposed formulas for the exact sequence (1.19)

In this section, we propose what we feel must be correct complete formulas for the exact sequence (1.19). Some homomorphisms are forced by naturality, but many others involve significant filtration jumps. However, they all occur in several families with nice properties. The 10-term exact sequence (6.2) shows how the $S_{k,\ell}$ portions and the exotic extensions yield compatibility of the differing v-tower heights in $ku^*(K_2)$ and $k(1)^*(K_2)$. In Section 7, we show that all elements of $k(1)^*(K_2)$ are accounted for exactly once in these homomorphisms, which implies that there can be no more exotic extensions. This does not require us to prove that our formulas are actually correct, as discussed at the end of Section 1.

We propose that (1.19) can be split into exact sequences of length 4 and 10 (not including 0's at the end). There are subgroups of $k(1)^*(K_2)$ called G_k^1 and G_k^2 for $k \geq 3$ and $G_{k,\ell}^i$ for $3 \leq i \leq 6$ and $3 \leq k < \ell$ such that there are exact sequences

$$0 \to G_k^1 \to A_k \xrightarrow{2} A_k \to G_k^2 \to 0 \tag{6.1}$$

for $k \geq 3$, and, for $3 \leq k < \ell$,

$$\begin{aligned} 0 \quad \to \quad & G_{k,\ell}^3 \to x_4^{2^{k-3}} B_k \prod_k^{\ell-1} z_i \xrightarrow{2} x_4^{2^{k-3}} B_k \prod_k^{\ell-1} z_i \to G_{k,\ell}^4 \to x_4^{2^{k-3}-1} x_9 S_{k,\ell} \\ \xrightarrow{2} \quad & x_4^{2^{k-3}-1} x_9 S_{k,\ell} \to G_{k,\ell}^5 \to B_k z_\ell \xrightarrow{2} B_k z_\ell \to G_{k,\ell}^6 \to 0. \end{aligned} \tag{6.2}$$

The sequence (6.1) can be tensored with $\mathbb{Z}_2[x_4^{2^{k-2}}]$, while (6.2) can be tensored with $\mathbb{Z}_2[x_4^{2^{k-2}}] \otimes \Lambda_{\ell+1}$. Note that $B_3 = 0$, so that (6.2) only has four nontrivial terms when $k = 3$. We will study these exact sequences by breaking them up into short exact sequences and isomorphisms involving kernels and cokernels of $\cdot 2$.

In studying these exact sequences, $K_k^A := \ker(2|A_k)$ and $K_k^B := \ker(2|B_k)$ are very important. Important elements of each are given in Table 4.

k	g_k
3	z_3
4	z_4
5	vz_5
6	$x_4^2z_4z_5+v^3z_6$
7	$vx_4^4z_5z_6+v^8z_7$
8	$x_4^{10}z_4z_5z_7+v^3x_4^8z_6z_7+v^{18}z_8$
9	$vx_4^{20}z_5z_6z_8+v^8x_4^{16}z_7z_8+v^{39}z_9$
10	$x_4^{42}z_4z_5z_7z_9+v^3x_4^{40}z_6z_7z_9+v^{18}x_4^{32}z_8z_9+v^{81}z_{10}$

Table 4: Elements g_k in K_k

For example g_7 can be seen in Figure 1.10 in grading 114, and can be verified using (1.6) to see that $2g_7=v^9z_6^2+v^9z_6^2=0$. A recursive formula is

$$g_{k+2}=x_4^{2^{k-3}}g_kz_{k+1}+v^{h'(k-1)-1}z_{k+2}. \tag{6.3}$$

Note that the first part of this formula is analogous to the recursive formula for p_e. The occurrence of $h'(k-1)$ here is a bridge between $ku^*(K_2)$ and $k(1)^*(K_2)$.

The isomorphisms $G_k^1\to K_k^A$ and $G_{k,\ell}^3\to x_4^{2^{k-3}}K_k^B\prod_k^{\ell-1}z_i$ are determined, on elements of v-height >1, by $p_i\mapsto g_i$, multiplied by various things. The main place where the A- and B-versions differ is in the element of largest v-height. This is g_k for each. However, its v-height in K_k^A (resp. K_k^B) is $2^{k-2}-(h'(k-3)-1)$ (resp. $2^{k-2}-(k-2)-(h'(k-3)-1)$). In $k(1)^*(K_2)$, the v-height of p_k is $h(k-1)$ if it is not accompanied by z_k, as will be the case when mapping to K_k^A, while its v-height is $h'(k-2)$ if it is accompanied by z_k, as will be the case for the map out of $G_{k,\ell}^3$. One can verify that these v-heights match, i.e., $2^{k-2}-h'(k-3)+1=h(k-1)$ and $2^{k-2}-(k-2)-h'(k-3)+1=h'(k-2)$.

Other elements of v-height >1 will have the same v-height in the A- and B-versions. We just list it when $k=7$, where we have Figure 1.10 to look at. These elements are hit by v-towers in $k(1)^*(K_2)$ of the same height as follows:

$$\begin{aligned} &\text{ht } h'(2) = 2 && (p_4 \mapsto g_4) \cdot z_4\{z_5, x_4^4\}\{z_6, x_4^8\} \\ &\text{ht } h'(3) = 4 && (p_5 \mapsto g_5) \cdot z_5\{z_6, x_4^8\} \\ &\text{ht } h'(4) = 9 && (p_6 \mapsto g_6) \cdot z_6. \end{aligned} \tag{6.4}$$

The notation such as $\{z_5, x_4^4\}$ means that the homomorphism is multiplied by either z_5 or x_4^4. For example, (6.4) means that $p_5 z_5 z_6 \mapsto g_5 z_5 z_6$ and also $p_5 z_5 x_4^8 \mapsto g_5 z_5 x_4^8$. You can see all of the target elements in Figure 1.10, and can verify that the preimage elements occur in Table 2 with the prescribed v-height. This generalizes to arbitrary k in an obvious way. In the B case, these formulas must be multiplied by $x_4^{2^{k-3}}$ and by $\prod_k^{\ell-1} z_i$, or by z_ℓ with $\ell > k$. In both the A- and B-cases, they can also be multiplied by the things which we said the exact sequences can be multiplied by. None of this changes any of the v-heights.

There are elements of v-height 1 in K_k^A and K_k^B. When $k = 7$, you can see these in Figure 1.10 in gradings 124, 108, 106, 90, and (for B but not A) 76, 74, and 72. The basic formulas for the morphisms from G_k^1 and $G_{k,\ell}^3$ follow a pattern which should be clear from the first three:

$$x_4^3 p_3 z_{4,5} \mapsto x_4^2 v z_4 z_5 \tag{6.5}$$

$$x_4^7 p_3 \big(\!\big(z_{4,6}, z_{5,6}\big)\!\big) \mapsto x_4^4 \big(\!\big(v^4 z_5, v x_4^2 z_4\big)\!\big) z_6 \tag{6.6}$$

$$x_4^{15} p_3 \big(\!\big(z_{4,7}, z_{5,7}, z_{6,7}\big)\!\big) \mapsto x_4^8 \big(\!\big(v^{11} z_6, v^4 x_4^4 z_5, v x_4^6 z_4\big)\!\big) z_7. \tag{6.7}$$

We use $\big(\!\big(-\big)\!\big)$ notation to indicate an ordered list. For example, (6.6) means that $x_4^7 p_3 z_{4,6} \mapsto x_4^4 v^4 z_5 z_6$ and $x_4^7 p_3 z_{5,6} \mapsto v x_4^6 z_4 z_6$. It is different than the set symbols that we used to mean “choose one.” The v-exponents in the targets are various $2^t - t - 1$. The preimage elements are of the second type in Theorem 4.8. Note that this morphism involves large filtration jumps.

The formula (6.5) occurs in G^1 and G^3 in many ways. Later we will list additional ways that it occurs in G^5.

- as stated in $G_6^1 \to A_6$;
- multiplied by x_4^4 in $G_{5,6}^3 \to x_4^4 B_5 z_5$;
- multiplied by $x_4^8 z_6$ in $G_{6,7}^3 \to x_4^8 B_6 z_6$;
- multiplied by $\{z_6, x_4^8\}$ in $G_7^1 \to A_7$;

- multiplied by $x_4^{16}\{z_6,x_4^8\}z_7$ in $G^3_{7,8}\to x_4^{16}B_7z_7$;
- multiplied by $\{z_6,x_4^8\}\{z_7,x_4^{16}\}$ in $G^1_8\to A_8$;
- etc.

For $G^3_{k,\ell}\to x_4^{2^{k-3}}B_k\prod_k^{\ell-1}z_i$ with $\ell>k+1$, multiply the formula by an additional $z_{k+1}\cdots z_{\ell-1}$. Of course, formulas (6.6) and (6.7) and subsequent formulas have similar manifestations. For the subsequent formulas after (6.7), increase subscripts of A, B, G, and z and i in $x_4^{2^i}$ by appropriate amounts, and extend the vectors. In Figure 1.10, multiples of (6.5) apply to the elements in 124 and 90, while (6.6) applies to elements in 108 and 106, and (6.7) to z_7 times elements in 76, 74, and 72.[6]

Next we describe the isomorphisms $C_k\to G_k^2$ and $C_kz_\ell\to G^6_{k,\ell}$, where $C_k:=\operatorname{coker}(2|A_k)=\operatorname{coker}(2|B_k)$[7] and $\ell\ge k+1$. These isomorphisms are defined simply by sending an element to one with the same name. Perusal of Figure 1.10 makes it quite clear that the elements of C_7 with v-height >1 are as listed below with their v-heights, in a pattern whose generalization to any k should be clear.

$$\begin{array}{rl} \text{ht } 19=h'(5) & z_7\\ \text{ht } 9=h'(4) & x_4^8z_6\\ \text{ht } 4=h'(3) & x_4^4z_5\{z_6,x_4^8\}\\ \text{ht } 2=h'(2) & x_4^2z_4\{z_5,x_4^4\}\{z_6,x_4^8\}.\end{array}\tag{6.8}$$

We explain the v-height of z_7, again referring to Figure 1.10. In grading 92, we have

$$2(x_4^{10}z_4z_5+v^3x_4^8z_6)=v^{19}z_7,\tag{6.9}$$

so $v^{19}z_7=0$ in C_7, corresponding to the v-height of z_7 in $k(1)^*(K_2)$. Note that $v^{18}z_7\ne0$ in C_7, since $2\cdot v^2x_4^8z_6=v^{18}z_7+v^3x_4^8z_5^2$. The relation (6.9) is closely related to the formula for g_8 in Table 4: if (6.9) is multiplied by z_7, then $vz_7^2=2z_8$ implies the relation $2g_8=0$.

The elements of v-height 1 in C_7 are

$$\begin{array}{rl} z_{j,7} & 4\le j\le 6\\ x_4^8z_{j,6} & 4\le j\le 5\\ x_4^4z_{j,5}\{z_6,x_4^8\} & j=4.\end{array}\tag{6.10}$$

[6] These elements do not exist as kernel elements without being multiplied by sone z_k with $k\ge7$.
[7] except for the elements $x_4^{2^{k-3}-1}z_3$ and $x_4^{2^{k-3}-1}x_8$ in $\operatorname{coker}(2|A_k)$.

Note that these have v-height 1 in C_7 because v times them is divisible by 2 in $ku^*(K_2)$. This generalizes to any k.

Let $S^K_{k,\ell} = \ker(2|S_{k,\ell})$ and $S^C_{k,\ell} = \operatorname{coker}(2|S_{k,\ell})$. We study the short exact sequence

$$0 \to x_4^{2^{k-3}} C_k z_k P^\ell_{k+1} \to G^4_{k,\ell} \to x_4^{2^{k-3}-1} x_9 S^K_{k,\ell} \to 0, \tag{6.11}$$

where $P^\ell_{k+1} := \prod_{k+1}^{\ell-1} z_i$. Note that $S^K_{k,\ell}$ contains just the v-tower of height $k-1$ on $z_{4,\ell}$, and classes of v-height 1 for each $v^{k-2} z_{i,\ell}$ with $5 \le i \le \ell - k + 3$. We deal with the latter elements first. The map from $G^4_{k,\ell}$ sends

$$x_4^{2^{k-3}} z_{i+k-4,\ell} \mapsto v^{k-2} x_4^{2^{k-3}-1} x_9 z_{i,\ell},\ 5 \le i \le \ell - k + 3. \tag{6.12}$$

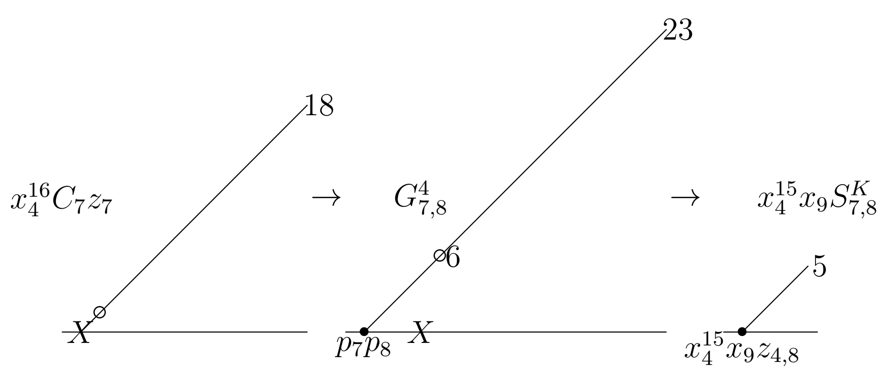
The classes of v-height 1 in C_k, described in the preceding paragraph, when multiplied by $x_4^{2^{k-3}} z_k P^\ell_{k+1}$, map to elements with the same name in $G^4_{k,\ell} \subset k(1)^*(K_2)$.

Of the v-towers in C_k of v-height > 1, after multiplication by $x_4^{2^{k-3}} z_k P^\ell_{k+1}$, all except the one on z_k map to v-towers with the same name. The only tower of v-height > 1 in $x_4^{2^{k-3}-1} x_9 S^K_{k,\ell}$ is $x_4^{2^{k-3}-1} x_9 z_{4,\ell}$, with v-height $k-1$. It is hit by $p_k p_{k+1} P^\ell_{k+1}$, which has v-height $h(k-1)$. The class which hits $v^{k-1} p_k p_{k+1} P^\ell_{k+1}$ is $v x_4^{2^{k-3}} z_{k,\ell}$, which has v-height $h'(k-2)-1$ in $x_4^{2^{k-3}} C_k z_k P^\ell_{k+1}$, as it corresponds to $z_k \in C_k$. (See (6.8) for the v-height.) These match since

$$h(k-1) - (k-1) = h'(k-2) - 1.$$

The generator of the v-tower $x_4^{2^{k-3}} z_{k,\ell}$ in $x_4^{2^{k-3}} C_k z_k P^\ell_{k+1}$ maps to the class with the same name in $k(1)^*(K_2)$. A schematic when $k = 7$ and $\ell = 8$ appears in Figure 6.13. Elements with $\circ$, $\bullet$, or $X = x_4^{16} z_7^2$ map to elements with the same symbol, and numbers indicate filtration.

Figure 6.13. Towers in exact sequence.



Finally, we study the short exact sequence

$$0 \to x_4^{2^{k-3}-1} x_9 S^C_{k,\ell} \to G^5_{k,\ell} \to K^B_k z_\ell \to 0 \tag{6.14}$$

with $\ell \geq k+1$. First, $S^C_{k,\ell}$ has classes $z_{i,\ell}$ for $4 \leq i \leq \ell - k + 3$, which, after multiplying by $x_4^{2^{k-3}-1} x_9$, map to classes with the same name (except that x_9 is replaced by p_3) in $G^5_{k,\ell} \subset k(1)^*(K_2)$. The target classes have v-height 1, as do the domain classes in $x_4^{2^{k-3}-1} x_9 S^C_{k,\ell}$, except the one with $i = \ell - k + 3$, which has v-height $k-1$. Similarly to the discussion following (6.3)[8], $K^B_k z_\ell$ has summands of v-height $h'(e-2)$ for $4 \leq e \leq k-1$ with generators $g_e P$, with

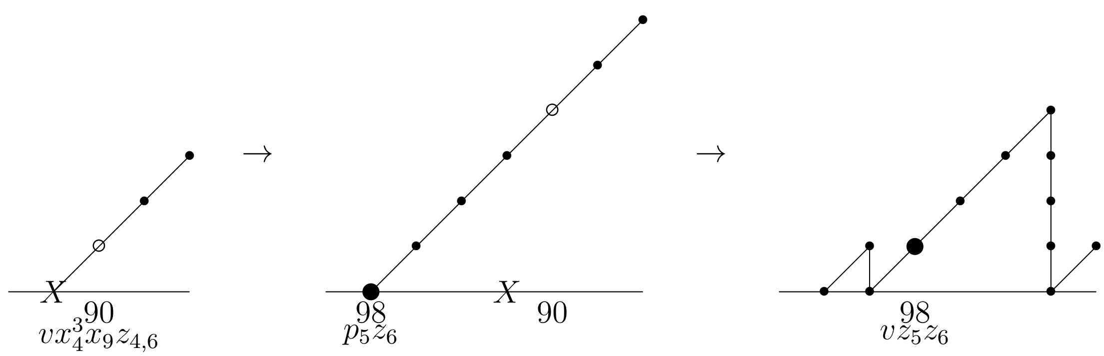
$$P := z_e \prod_{j=e+1}^{k-1} \{z_j, x_4^{2^{j-3}}\} z_\ell,$$

and $g_k z_\ell$ of v-height $h'(k-2)$. The classes $g_e P$ are mapped to by $p_e P$ in $G^5_{k,\ell}$ with the same v-height. However, $g_k z_\ell$ is hit by $p_k z_\ell$ of v-height $h(k-1) = h'(k-2) + k - 2$. To compensate, $v^{h'(k-2)} p_k z_\ell$ is hit by $v x_4^{2^{k-3}-1} x_9 z_{3+\ell-k,\ell}$, which is v times the generator of the only part of $x_4^{2^{k-3}-1} x_9 S^C_{k,\ell}$ of v-height > 1. One can check that

$$|v x_4^{2^{k-3}-1} x_9 z_{3+\ell-k,\ell}| + 1 = |v^{h'(k-2)} p_{k-1} z_\ell|.$$

We illustrate this key phenomenon in Figure 6.15, which shows all of $x_4^3 x_9 S_{5,6}$ and $B_5 z_6$, and part of $G^5_{5,6}$.

Figure 6.15. $x_4^3 x_9 S_{5,6} \to G^5_{5,6} \to B_5 z_6$



There are also two families of elements of v-height 1 in $K^B_k z_\ell$ which are hit from $G^5_{k,\ell}$ similar to those described in (6.5)-(6.7). First, in G^5, (6.5) occurs

as follows (6.16)

[8] See especially (6.4).

- in $G^5_{6,\ell}\to B_6z_\ell$ for $\ell\ge 7$, multiplied by z_ℓ,
- in $G^5_{7,\ell}\to B_7z_\ell$ for $\ell\ge 8$, multiplied by $\{z_6,x_4^8\}z_\ell$,
- in $G^5_{8,\ell}\to B_8z_\ell$ for $\ell\ge 9$, multiplied by $\{z_6,x_4^8\}\{z_7,x_4^{16}\}z_\ell$,
- etc.

These can also be tensored with $\Lambda_{\ell+1}$ and, for $G^5_{k,\ell}$, by $\mathbb{Z}_2[x_4^{2^{k-2}}]$. There are analogous occurrences of (6.6), (6.7), and their successors.

In G^5, there are also generalizations of (6.5)-(6.7) as follows.

$$
\begin{aligned}
x_4^3p_3z_{\ell-1,\ell} &\mapsto x_4^2vz_4z_\ell,\ \ell\ge 6 &\quad(6.17)\\
x_4^7p_3\big((z_{\ell-2,\ell},z_{\ell-1,\ell})\big) &\mapsto x_4^4\big((v^4z_5,vx_4^2z_4)\big)z_\ell,\ \ell\ge 7 &\quad(6.18)\\
&\quad \text{etc.}
\end{aligned}
$$

Formula (6.17) occurs in $G^5_{5,\ell}\to B_5z_\ell$ and can be tensored with $\mathbb{Z}_2[x_4^8]\Lambda_{\ell+1}$, and (6.18) occurs in $G^5_{6,\ell}\to B_6z_\ell$ and can be tensored with $\mathbb{Z}_2[x_4^{16}]\Lambda_{\ell+1}$.

7. All accounted for

In this section, we show that all elements of $k(1)^*(K_2)$ are involved in exactly one of the homomorphisms involving some G-group described in the preceding section. As discussed earlier, this implies that there can be no exotic extensions in $ku^*(K_2)$ other than those in (1.6), because such an extension would decrease the number of elements in $\ker(2|ku^*(K_2))$ and $\operatorname{coker}(2|ku^*(K_2))$, and these must correspond to the elements of G-groups.

Let

$$
G^i=\begin{cases}\displaystyle\bigoplus_{k\ge 3}\mathbb{Z}_2[x_4^{2^{k-2}}]\otimes G^i_k & 1\le i\le 2\\ \displaystyle\bigoplus_{3\le k<\ell}\mathbb{Z}_2[x_4^{2^{k-2}}]\otimes G^i_{k,\ell}\otimes\Lambda_{\ell+1} & 3\le i\le 6.\end{cases}
$$

This section is devoted to the proof of the following theorem.

Theorem 7.1. *$G^1\oplus\cdots\oplus G^6$ consists precisely of classes of the following four types.*

i. $\{x_8,z_3\}\otimes\mathbb{Z}_2[x_4]$.

ii. *For $e\ge 2$, v-towers of height $h(e)$ on $\overline{E}[p_{e+1}]\otimes E[p_{e+2}]\otimes\mathbb{Z}_2[x_4^{2^{e-1}}]\otimes\Lambda_{e+2}$;*

iii. *For $e\ge 3$, v-towers of height $h'(e-1)$ on $\overline{E}[z_{e+1}]\otimes E[p_{e+1}]\otimes\mathbb{Z}_2[x_4^{2^{e-2}}]\otimes\Lambda_{e+2}$;*

iv. *For $e\ge 4$, v-towers of height 1 on $\mathbb{Z}_2[x_4]\otimes E[p_3]\otimes\overline{E}[z_e^2]\otimes\Lambda_{e+1}$.*

This and Theorem 4.8 immediately imply the following result.

Corollary 7.2. *$G^1 \oplus \cdots \oplus G^6$ exactly gives all of $k(1)^*(K_2)$ except for the split $\mathbb{Z}_2$'s (of the first type in Theorem 4.8) coming from free E_1-summands in $H^*(K_2)$.*

Proof of Theorem 7.1. **Case** i. The mod-2 reduction of A_3 is $\{x_8, z_3\}$, and, as noted near the end of Section 4, $x_4^{2^i-1}x_4^{c2^{i+1}}A_3 \subset x_4^{c2^{i+1}}A_{3+i}$. These map to classes with the same name in G^2.

Case ii. Our work in Section 6 showed that the v-towers of height $h(e)$ in the G's are

- $p_{e+1}\mathbb{Z}_2[x_4^{2^{e-1}}]$ in G^1,
- $p_{e+1}p_{e+2}\mathbb{Z}_2[x_4^{2^{e-1}}]\Lambda_{e+2}$ in G^4, and
- $p_{e+1}\mathbb{Z}_2[x_4^{2^{e-1}}]\overline{\Lambda}_{e+2}$ in G^5.

The first and third combine to give the portion of Theorem 7.1(ii.) which does not contain the p_{e+2} in $E[p_{e+2}]$, while the second part contains the portion which does.

Case iii. The work in Section 6 showed that the v-towers of height $h'(e-1)$ in the G's are

- $p_{e+1}z_{e+1}\bigoplus_{i\ge e+1}\mathbb{Z}_2[x_4^{2^{i-1}}]\otimes\prod_{j=e+2}^{i}\{z_j, x_4^{2^{j-3}}\}$ in G^1,
- $\mathbb{Z}_2[x_4^{2^{e-1}}]z_{e+1}\oplus x_4^{2^{e-2}}z_{e+1}\bigoplus_{i\ge e+1}\mathbb{Z}_2[x_4^{2^{i-1}}]\otimes\prod_{j=e+2}^{i}\{z_j, x_4^{2^{j-3}}\}$ in G^2,
- $p_{e+1}z_{e+1}\big(x_4^{2^{e-2}}\mathbb{Z}_2[x_4^{2^{e-1}}]\Lambda_{e+2}\oplus\bigoplus_{i\ge e+1}x_4^{2^{i-2}}\mathbb{Z}_2[x_4^{2^{i-1}}]\otimes\prod_{j=e+2}^{i}\{z_j, x_4^{2^{j-3}}\}\cdot z_{i+1}\Lambda_{i+2}\big)$ in G^3,
- $x_4^{2^{e-2}}z_{e+1}\bigoplus_{i\ge e+1}x_4^{2^{i-2}}\mathbb{Z}_2[x_4^{2^{i-1}}]\otimes\prod_{j=e+2}^{i}\{z_j, x_4^{2^{j-3}}\}\cdot z_{i+1}\Lambda_{i+2}$ in G^4,
- $p_{e+1}z_{e+1}\bigoplus_{i\ge e+1}\mathbb{Z}_2[x_4^{2^{i-1}}]\otimes\prod_{j=e+2}^{i}\{z_j, x_4^{2^{j-3}}\}\cdot\overline{\Lambda}_{i+2}$ in G^5, and
- $\mathbb{Z}_2[x_4^{2^{e-1}}]z_{e+1}\overline{\Lambda}_{e+2}\oplus x_4^{2^{e-2}}z_{e+1}\bigoplus_{i\ge e+1}\mathbb{Z}_2[x_4^{2^{i-1}}]\otimes\prod_{j=e+2}^{i}\{z_j, x_4^{2^{j-3}}\}\cdot\overline{\Lambda}_{i+2}$ in G^6.

The $G^1\oplus G^3\oplus G^5$ part is all divisible by $p_{e+1}z_{e+1}$. We remove those factors, and combine G^1 into G^5 to remove the bar over Λ. This combines with the G^3-part to give

$$x_4^{2^{e-2}}\mathbb{Z}_2[x_4^{2^{e-1}}]\Lambda_{e+2}\oplus\bigoplus_{i\ge e+1}\mathbb{Z}_2[x_4^{2^{i-1}}]\otimes\{1,z_{i+1}x_4^{2^{i-2}}\}\prod_{j=e+2}^{i}\{z_j,x_4^{2^{j-3}}\}\cdot\Lambda_{i+2}. \tag{7.3}$$

We will show that the $\bigoplus$ part equals $\mathbb{Z}_2[x_4^{2^{e-1}}]\Lambda_{e+2}$. Thus the entire expression equals $\mathbb{Z}_2[x_4^{2^{e-2}}]\Lambda_{e+2}$, and so this $G^1\oplus G^3\oplus G^5$ part gives the portion of Theorem 7.1(ii) which includes the p_{e+1} in $E[p_{e+1}]$. A very similar argument shows that the $G^2\oplus G^4\oplus G^6$ part gives the portion which includes just the 1 in $E[p_{e+1}]$, concluding the proof of Case iii, modulo the claim.

To prove the claim, it is convenient to think of $\mathbb{Z}_2[x_4^{2^{i-1}}]$ as an exterior algebra of $\{x_4^{2^t}: t\ge i-1\}$. Any monomial in $\mathbb{Z}_2[x_4^{2^{e-1}}]\Lambda_{e+2}$ can be described by a sequence of choices: $\big(\big((z_{e+2},x_4^{2^{e-1}}),(z_{e+3},x_4^{2^e}),\ldots\big)\big)$. In each pair, which was included: neither, both, or which one? Note that $\mathbb{Z}_2[x_4^{2^{i-1}}]\Lambda_{i+2}$ allows all possible choices beginning with $(z_{i+2},x_4^{2^{i-1}})$. A monomial corresponding to the i-term in the $\bigoplus$ in (7.3) chooses exactly one of z_j and $x_4^{2^{j-3}}$ in each position for $j<i+1$, then chooses neither or both of z_{i+1} and $x_4^{2^{i-2}}$, and then makes all possible choices after that. Thus all monomials in $\mathbb{Z}_2[x_4^{2^{e-1}}]\Lambda_{e+2}$ are chosen exactly once.

Case iv. Now we study the classes of v-height 1. We begin with those not divisible by p_3. These are exactly those coming from G^2, G^4, and G^6, except that Case i handled a few from G_2. Now we list the terms in each which contain the factor z_e^2, for some $e\ge 4$. The desired answer is $z_e^2\mathbb{Z}_2[x_4]\Lambda_{e+1}$.

From G^2, we have

$$z_e^2\bigoplus_{i\ge e+1}\mathbb{Z}_2[x_4^{2^{i-2}}]\prod_{j=e+2}^{i}\{z_{j-1},x_4^{2^{j-4}}\},$$

and from G^6 the same thing with $\overline{\Lambda}_{i+1}$ appended, so that these combine to give

$$z_e^2\bigoplus_{i\ge e+1}\mathbb{Z}_2[x_4^{2^{i-2}}]\prod_{j=e+2}^{i}\{z_{j-1},x_4^{2^{j-4}}\}\cdot\Lambda_{i+1}. \tag{7.4}$$

From G^4, there are three types. One, from (6.10), is

$$z_e^2\bigoplus_{i\ge e+1}\mathbb{Z}_2[x_4^{2^{i-2}}]z_ix_4^{2^{i-3}}\prod_{j=e+2}^{i}\{z_{j-1},x_4^{2^{j-4}}\}\cdot\Lambda_{i+1}. \tag{7.5}$$

(The seven cases of (6.10) multiplied by $x_4^{16}z_7$ give the seven cases of (7.5) with $i=7$, prior to tensoring either with $\mathbb{Z}_2[x_4^{32}]\Lambda_8$.) This combines with (7.4) to give $z_e^2\mathbb{Z}_2[x_4^{2^{e-2}}]\Lambda_{e+1}$ in exactly the same way as was done two paragraphs above. The element X of Figure 6.13 and its generalizations give $z_e^2x_4^{2^{e-3}}\mathbb{Z}_2[x_4^{2^{e-2}}]\Lambda_{e+1}$, so now we have all $z_e^2x_4^t\Lambda_{e+1}$ with $\nu(t)\ge e-3$. The classes $z_e^2x_4^t\Lambda_{e+1}$ with $\nu(t)\le e-4$ are exactly those in (6.12) since $\nu(t)=k-3$, $e=i+k-4$, and $i\ge 5$.

The terms divisible by p_3 are a bit harder. Those with x_4^{2*} and x_4^{4*+1} are easily handled, as they all come from (6.14) with $k=3$ and 4, since $S_{k,\ell}$ can be producted with $\mathbb{Z}_2[x_4^{2^{k-2}}]\Lambda_{\ell+1}$. Note all $z_{i,\ell}\Lambda_{\ell+1}$ with $4\le i\le \ell-1$ gives all of $\bigoplus_{e\ge4}z_e^2\Lambda_{e+1}$.

The domain classes in G^1 obtained from (6.5)-(6.7) and those in G^5 related to the group (6.16) combine to give, for $e\ge 4$,

$$z_e^2p_3\bigoplus_{i\ge e+1}x_4^{2^{i-3}-1}\prod_{t=e+1}^{i-1}z_t\cdot\bigoplus_{j\ge i}\mathbb{Z}_2[x_4^{2^{j-1}}]\Lambda_{j+2}\prod_{s=i+1}^{j}\{z_s,x_4^{2^{s-3}}\}. \tag{7.6}$$

We first consider the terms in G^3 of v-height 1 which are divisible by $p_3z_e^2$ with $e=5$. It may be helpful to refer to the paragraph following (6.5)-(6.7). From (6.6), we obtain

$$x_4^7p_3z_5^2\big(x_4^8\Lambda_7\mathbb{Z}_2[x_4^{16}]\oplus x_4^{16}z_7\Lambda_8\mathbb{Z}_2[x_4^{32}]\oplus x_4^{32}z_8\{z_7,x_4^{16}\}\Lambda_9\mathbb{Z}_2[x_4^{64}]\oplus\cdots\big). \tag{7.7}$$

From (6.7), we obtain

$$x_4^{15}p_3z_5^2z_6\big(x_4^{16}\Lambda_8\mathbb{Z}_2[x_4^{32}]\oplus x_4^{32}z_8\Lambda_9\mathbb{Z}_2[x_4^{64}]\oplus x_4^{64}z_9\{z_8,x_4^{32}\}\Lambda_{10}\mathbb{Z}_2[x_4^{128}]\oplus\cdots\big). \tag{7.8}$$

These extend in an obvious way, and the pattern for arbitrary $e\ge 4$ should be apparent, with all subscripts and 2-power exponents modified appropriately.

In addition, the generalization of (6.17) and (6.18) contribute to G^5, for $e\ge 5$,

$$z_e^2p_3\bigoplus_{i\ge e-2}x_4^{2^{i-1}-1}\mathbb{Z}_2[x_4^{2^i}]\Lambda_{i+4}\prod_{j=e+1}^{i+2}z_j. \tag{7.9}$$

Finally, G^5 contains image terms from the $S_{k,\ell}$ part of (6.14). We have already discussed how the part for $k=3$ and 4 gives all desired terms with factors x_4^{2*} and

x_4^{4*+1}. The remaining terms combine to yield

$$\begin{aligned} & \bigoplus_{e\ge 4} z_e^2 p_3 \bigoplus_{k\ge 5} x_4^{2^{k-3}-1}\mathbb{Z}_2[x_4^{2^{k-2}}] \bigoplus_{\ell\ge k+e-3} \prod_{j=e+1}^{\ell-1} z_j \cdot \Lambda_{\ell+1} \\ = & \bigoplus_{e\ge 4} z_e^2 p_3 \bigoplus_{k\ge 5} x_4^{2^{k-3}-1}\mathbb{Z}_2[x_4^{2^{k-2}}] \prod_{j=e+1}^{k+e-4} z_j \cdot \Lambda_{k+e-3}. \end{aligned} \tag{7.10}$$

Now we prove Case iv of Theorem 7.1 for classes divisible by p_3. To simplify exposition, we restrict our attention to the case $e = 5$. We wish to show that all monomials in $x_4^s z_5^2 p_3 \Lambda_6$ are obtained exactly once in $G^1 \oplus G^3 \oplus G^5$, whose classes have been described in the previous several paragraphs. We let $\nu = \nu(s+1)$ and $Z(t) = z_6 \cdots z_t$ for $t \ge 6$, and $Z(5) = 1$. The cases $\nu < 2$ have already been handled.

From (7.9), we obtain all $z_5^2 p_3 x_4^s Z(\nu+3)\Lambda_{\nu+5}$. From (7.10), we obtain all $z_5^2 p_3 x_4^s Z(\nu+4)\Lambda_{\nu+5}$. Combining these gives $z_5^2 p_3 x_4^s Z(\nu+3)\Lambda_{\nu+4}$. If $\nu = 2$, this is as desired.

Now restrict to $\nu \ge 3$. We consider the family beginning with (7.7) and (7.8) but omit the first term of each sum. When these are combined with (7.6), we obtain expressions which can be simplified using exactly the same method that was used to simplify (7.3), and we obtain $x_5^2 p_3 x_4^s Z(\nu+2)\Lambda_{\nu+4}$. When this is combined with the previous combined expression, we obtain $z_5^2 p_3 x_4^s Z(\nu+2)\Lambda_{\nu+3}$. Finally, the first terms of the (7.7)-(7.8) family give all monomials in $z_5^2 p_3 x_4^s \Lambda_6$ not divisible by $Z(\nu+2)$. This and $z_5^2 p_3 x_4^s Z(\nu+2)\Lambda_{\nu+3}$ exactly fill out $x_4^s z_5^2 p_3 \Lambda_6$. To justify the claim about the "first terms," note that (7.7) and (7.8) are the first two of a succession of similar expressions, of which we are considering the first terms of each. Terms with a certain value of $\nu \ge 4$ will appear among the first $\nu - 3$ of these. For example, with $\nu = 6$, the first of these contains all terms with no z_6, the second those with z_6 but no z_7, and the third those with $z_6 z_7$ but no z_8. These comprise all terms not divisible by $Z(8)$.

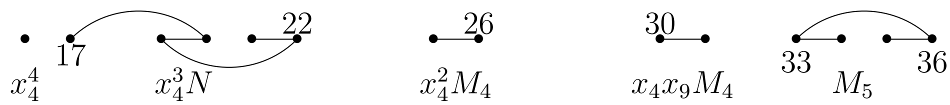
The argument that we have illustrated when $e = 5$ generalizes to arbitrary $e \ge 4$ in an obvious way. ■

8. An explanation of self-duality of B_k

In this optional section, we discuss some observations about the ASS of $ku^*(K_2)$ and $ku_*(K_2)$ which, among other things, provide an explanation of the self-dual nature of the B_k charts which occur in both $ku^*(K_2)$ and $ku_*(K_2)$.

We first observe that, for $k \geq 3$, there is an E_1-submodule, $\mathcal{M}_k$, of $H^*(K_2)$ such that $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, \mathcal{M}_k)$ (resp. $\mathrm{Ext}_{E_1}(\mathcal{M}_k, \mathbb{Z}_2)$) is closed under the differentials in the ASS converging to $ku^*(K_2)$ (resp. $ku_*(K_2)$), yielding the chart A_k (resp. the homology analogue of A_k discussed in Theorem 2.4). For example, with M_j as in (3.10) and N as in Figure 3.7, $\mathcal{M}_5$ is as depicted in Figure 8.1.

Figure 8.1. The E_1-module $\mathcal{M}_5$.



The two ASSs for $\mathcal{M}_5$ will yield the charts for A_5 and its homology analogue pictured in [4].

The situation for B_k is slightly more complicated. There is no E_1-submodule of $H^*(K_2)$ which, by itself, can give a chart B_k or B_kz_ℓ. Some of the differentials that truncate v-towers in B_kz_ℓ come from classes that are part of a summand that includes $x_4^{2^{k-3}-1}x_9S_{k,\ell}$. We find that, for $4 \leq k < \ell$, there is an E_1-submodule $\mathcal{M}_{k,\ell}$ of H^*K_2 such that $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, \mathcal{M}_{k,\ell})$ is closed under the differentials in the ASS converging to $ku^*(K_2)$ and yields the chart

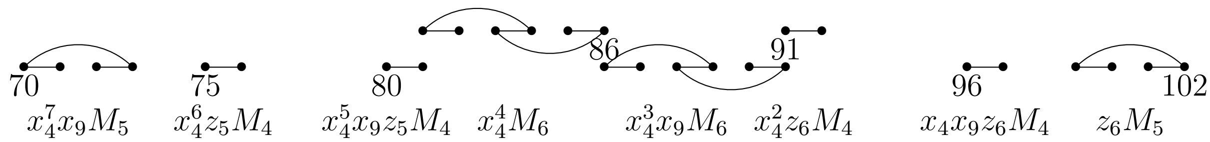
$$B_kz_\ell \oplus x_4^{2^{k-3}-1}x_9S_{k,\ell} \oplus x_4^{2^{k-3}}B_kZ_k^{\ell-1},$$

where $Z_k^{\ell-1} = z_k \cdots z_{\ell-1}$. Note that these three subsets of $ku^*(K_2)$ appeared together in the 10-term exact sequence (6.2).

This $\mathcal{M}_{k,\ell}$ is symmetric; i.e., there is an integer D such that $\mathcal{M}_{k,\ell}^*$ and $\mathcal{M}_{k,\ell}$ are isomorphic E_1-modules, where $\mathcal{M}_{k,\ell}^*$ is obtained from $\mathcal{M}_{k,\ell}$ by negating gradings and reversing direction of Q_0 and Q_1. This implies that the v-towers in $\mathrm{Ext}_{E_1}(\mathbb{Z}_2, \mathcal{M}_{k,\ell})$ and $\mathrm{Ext}_{E_1}(\mathcal{M}_{k,\ell}, \mathbb{Z}_2)$ correspond nicely. Moreover, the differentials in the two ASSs correspond, too, obtaining isomorphic charts, although the gradings in one decrease from left to right, while in the other they increase.

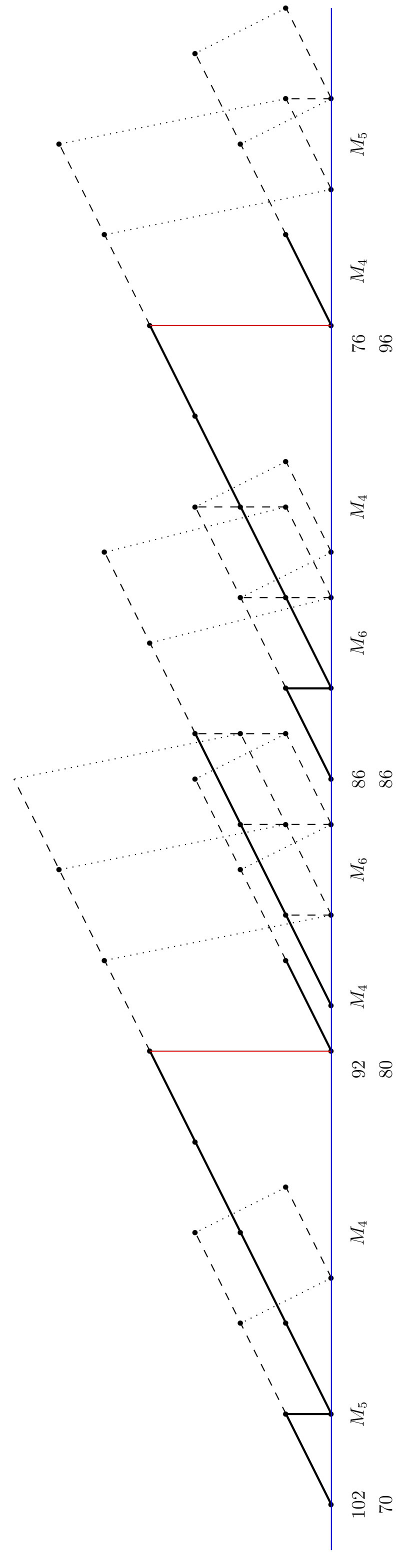
We illustrate with an example, $\mathcal{M}_{5,6}$, and then discuss the implication for self-duality of B_k, and finally discuss briefly the general case. In Figure 8.2, we depict $\mathcal{M}_{5,6}$.

Figure 8.2. The E_1-module $\mathcal{M}_{5,6}$.



In Figure 8.3, we depict the ASS chart for both $\text{Ext}_{E_1}(\mathbb{Z}_2, \mathcal{M}_{5,6})$ and $\text{Ext}_{E_1}(\mathcal{M}_{5,6}, \mathbb{Z}_2)$. They are isomorphic except that, from left to right, the gradings start with 102 for the first and 70 for the second. We label the portions of the chart corresponding to the eight summands of $\mathcal{M}_{5,6}$ just by the M-factor, since accompanying factors differ for the two versions. For example, the M_5 on the left-hand side is z_6M_5 for the first spectral sequence, and is $x_4^7x_9M_5$ for the second.

Figure 8.3. Two ASSs for $\mathcal{M}_{5,6}$.



For the $ku^*(K_2)$ version, $B_5 z_6$ is on the left hand side of Figure 8.3, and $x_4^4 B_5 z_5$ on the right hand side, with $x_4^3 x_9 S_{5,6}$ separating them. The duality isomorphism in Theorem 2.1 says that the Pontryagin dual of $B_5 z_6$ is isomorphic as a ku_*-module to Σ^4 of the right hand side of the $ku_*(K_2)$ version of Figure 8.3, and we see that this is isomorphic to a shifted version of B_5 with indices negated. This is the self-duality statement, that the Pontryagin dual of B_k is isomorphic as a ku_*-module to a shifted version of B_k with indices negated.

Finally, we explain how the eight summands in $\mathcal{M}_{5,6}$ in Figure 8.2 generalize. Note that (1.8) is the generalization of (1.9). We explain the general case using $k = 7$ and (1.9). Let U_i be the coefficient of x^{2i} in (1.9) with T_j^B replaced by M_j. Then, for $\ell \geq 8$, $\mathcal{M}_{7,\ell}$ in backwards order is

$$\begin{aligned} & z_\ell M_7 \oplus \bigoplus_{i=1}^{7} \big(x_4^{2i-1} x_9 U_i z_\ell \oplus x_4^{2i} U_i z_\ell \big) \oplus x_4^{15} x_9 M_\ell \\ \oplus \quad & x_4^{16} M_\ell \oplus \bigoplus_{i=1}^{7} \big(x_4^{2i+15} x_9 U_i Z_7^{\ell-1} \oplus x_4^{2i+16} U_i Z_7^{\ell-1} \big) \oplus x_4^{31} x_9 Z_7^{\ell-1} M_7. \end{aligned}$$

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