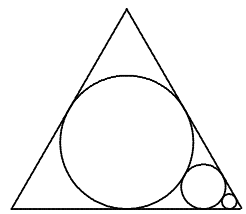
If a question asks you to "find all" or "list all" and you think there are none, write "None."

1. Simplify $1/(\frac{1}{3} - \frac{1}{4})$.
2. The price of an item increases by 10% and then by another 10%. What is the overall price increase of the item, in percent?
3. What is the area of a triangle with sides 5, 5, and 6?
4. A student has scores of 71, 81, 74, and 86 on her first four exams. What range of scores on the fifth exam will leave her in the C-range? The C-range is the interval $[70, 80)$, i.e., all averages satisfying $70 \le \text{ave} < 80$. You should express your answer either in interval notation or else as an inequality.
5. What is the volume of a cube inscribed in a sphere of radius 1?
6. You randomly select two distinct integers from 1 to 10. What is the probability that they are consecutive numbers; i.e., that they differ by 1?
7. Which of the following numbers is the largest? (Leave your answer in the given form.) $\sin(20^o)$, $\cos(20^o)$, $\tan(20^o)$, $1/\sin(20^o)$, $1/\cos(20^o)$, $1/\tan(20^o)$.
8. How many real solutions are there to the equation $x^4 + |x| = 5$?
9. Write 104060401 as a product of factors which are primes or powers of primes.
10. Simplify $\sqrt{19 + \sqrt{297}} - \sqrt{19 - \sqrt{297}}$.
11. Let $S_n = 1 - 2 + 3 - 4 + \cdots + (-1)^{n-1}n$. What is the value of $\sum_{n=1}^{2007} S_n$?
12. Rectangle $ABCD$ is divided into four subrectangles by two lines, one parallel to AB and the other parallel to BC. Three of the subrectangles have areas 1, 2, and 3. What are the possible values for the area of the rectangle $ABCD$?
13. Find all ordered pairs (x, y) of positive real numbers such that $3, x, y$ is a geometric progression, while $x, y, 9$ is an arithmetic progression.
14. Let $f(x) = 6/(1 + 7e^x)$ with domain all real numbers x. How many integers are in the range of f? (Here $e = 2.78128\ldots$. You may replace e^x by 2^x if you wish; the answer will be the same.)
15. Inside a regular hexagon $ABCDEF$ is drawn triangle ACE. What is the ratio of the area of triangle ACE to that of the hexagon?
16. In triangle ABC, $\angle A = 30^o$ and B is a right angle. The point D lies on AB so that $AD = 2$ and $\angle CDB = 45^o$. What is the length of BC?
17. John randomly selects an integer from 1 to 9, and so does Mary, independently of John's choice. What is/are the most likely value(s) for the units digit of the sum of their numbers?

18. Out of all positive real numbers x and y, what is the smallest value of $x+y$ for which $x+y \leq xy$?
19. Write a simple value for the sum $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{99}{100!}$. Your answer may involve the factorial symbol.
20. Two squares of sidelength 1 have a vertex of one at the center of the other. What is the smallest possible value for the area of their intersection?
21. Triangle ABC has $\tan(A) = 3/4$ and $\tan(B) = 21/20$. What is AC/BC?
22. If n is a positive integer, let $r(n)$ denote the number obtained by reversing the order of the digits of n. For example, $r(16) = 61$. For how many 2-digit positive integers n is $n + r(n)$ a square of a positive integer?
23. Find all ordered pairs (x, y) of positive integers such that $x^4 = y^2 + 71$.
24. How many diagonals are there in a regular 19-gon? (A diagonal is a line connecting two non-adjacent vertices.)
25. What is the sum of all positive integers n for which there exists a positive integer m satisfying $6(n! + 3) = m^2 + 5$?
26. Three circles of radius 1 are tangent to one another. What is the area of the small region bounded by the three points of tangency and minor arcs of the three circles?
27. A square train car is moving forward at 60 miles per hour. An ant in a front corner of the train crawls toward the diagonally opposite corner on the floor of the car at 1 mile per hour. At what speed in miles per hour is the ant moving relative to the ground?
28. A triangular number is one which is the sum of the first n positive integers for some n. An equidigit number is one all of whose digits are equal to one another. List all 3-digit equidigit triangular numbers.
29. How many integers between 1 and 1000 cannot be expressed as the difference of squares of integers?
30. A silo-shaped figure is formed by placing a semicircle of diameter 1 on top of a unit square, with the diameter coinciding with the top of the square. What is the radius of the smallest circle which contains this figure?
31. Let $p(x)$ be a polynomial of degree 4 satisfying $p(2) = p(-2) = p(-3) = 1$ and $p(1) = p(-1) = 2$. What is $p(0)$?
32. A cube of sidelength 4 has top square $ABCD$ directly above bottom square $EFGH$. Thus A is directly above E, etc. Point X lies on AD at distance 1 from A. Point Y lies on HG at distance 1 from H. Point Z lies on BF at distance 1 from F. What is the area of triangle XYZ?
33. A Fibonacci-like sequence of numbers is defined by $a_1 = 1$, $a_2 = 3$, and for $n \geq 3$, $a_n = a_{n-1} + a_{n-2}$. One can compute that $a_{29} = 1149851$ and $a_{30} = 1860498$. What is the value of $\sum_{n=1}^{28} a_n$?

34. A circle is inscribed in an equilateral triangle of side length 2. Then another circle is inscribed tangent to two sides of the triangle and to the first circle. An infinite sequence of such circles is constructed, each tangent to two sides of the triangle and to the previous circle. The figure below depicts the first three circles. What is the sum of the areas of all the infinitely many circles?



35. Let a_n denote the n-digit positive integer all of whose digits are 1. For example, $a_3 = 111$. What is the greatest common divisor of a_{45} and a_{140}?

36. Find all ordered triples (x, y, z) of integers satisfying $x^2 + y^2 + z^2 + 3 < xy + 3y + 2z$.

37. If n is a positive integer, let $s(n)$ denote the integer obtained by removing the last digit of n and placing it in front. For example, $s(731) = 173$. What is the smallest positive integer n ending in 6 satisfying $s(n) = 4n$?

38. Find all ordered triples (a, b, c) of prime numbers satisfying $a(b+c) = 234$ and $b(a+c) = 220$.

39. If $n = \underline{a}\,\underline{b}\,\underline{c}$ is a 3-digit number with digits a, b, and c, let

$$f(n) = a + b + c + ab + ac + bc + abc.$$

Here the things being summed are products of digits. For example, $f(234) = 2 + 3 + 4 + 6 + 8 + 12 + 24 = 59$. How many 3-digit positive integers n satisfy $f(n) = n$?

40. Let $ABCD$ be a square of sidelength 2, and let E and F be midpoints of AD and CD, respectively. Let G and H be the points where AF intersects EB and BD, respectively. What is the area of quadrilateral $DEGH$?

Solutions to 2008 contest

The first number in brackets is the total number of students (out of 257) who answered it correctly, and the second number is the number of correct responses from the 32 people who scored at least 20.

1. 12. [234,32] This follows since $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

2. 21 or 21%. [198,29] The price is multiplied by $1.1 \cdot 1.1 = 1.21$.

3. 12. [198,32] The isosceles triangle is composed of two 3-4-5 right triangles.

4. $[38, 88)$. [158,32] Her total so far is 312. Out of 500 points, the C-range starts at a total of 350 and ends just before 400.

5. $8\sqrt{3}/9$. [53,25] Place the sphere as $x^2 + y^2 + z^2 = 1$. One vertex of the cube is (x, x, x) satisfying $3x^2 = 1$, so $x = 1/\sqrt{3}$. Thus one side goes from $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ to $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, and hence has $s = 2/\sqrt{3}$. The volume is thus $8/(3\sqrt{3})$.

6. 1/5. [105,28] There are $\binom{10}{2} = 10 \cdot 9/2$ possible pairs of numbers, and of these, 9 pairs (from {1,2} to {9,10}) are consecutive. The probability is 9/45.

7. $1/\sin(20^o)$. [144,30] We have $\frac{1}{\tan(20^o)} = \frac{\cos(20^o)}{\sin(20^o)} < \frac{1}{\sin(20^o)}$ and $\frac{1}{\cos(20^o)} < \frac{1}{\sin(20^o)}$.

8. 2. [111,23] For $x > 0$, $x^4 + x$ is an increasing function going from 0 to ∞. Thus it equals 5 for exactly one positive value of x. The values of $x^4 + |x|$ for negative x are exactly the same as those of $x^4 + x$ for positive x, and so $x^4 + |x| = 5$ for exactly one negative value of x.

9. 101^4. [38,22] It is $1 + 4 \cdot 100 + 6 \cdot 100^2 + 4 \cdot 100^3 + 100^4 = (1 + 100)^4$, and 101 is prime.

10. $\sqrt{22}$. [32,17] Squaring the given expression yields $38 - 2\sqrt{361 - 297} = 38 - 16$ using the Binomial Theorem.

11. 1004. [120,31] Compute the first few values. $S_1 = 1$, $S_2 = -1$, $S_3 = 2$, and $S_4 = -2$. This pattern will continue, and so $S_1 + \cdots + S_{2006} = 0$, and the desired sum equals $S_{2007} = (2007 + 1)/2$.

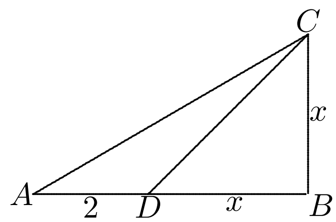
12. $6\frac{2}{3}$, $7\frac{1}{2}$, and 12. [48,15] Arrange the subrectangles so that the unknown one is in the lower right. If the upper left rectangle has area, respectively, 1, 2, or 3, then the lower right one has area 6, 3/2, or 2/3, each of which must be added to 6 to obtain the total area.

13. $(\frac{9}{2}, \frac{27}{4})$. [27,14] Since $\frac{x}{3} = \frac{y}{x}$, we have $x^2 = 3y$. Since $y - x = 9 - y$, we have $2y = 9 + x$. This implies $2x^2 = 3x + 27$, and so $x = (3 \pm 15)/4$. Only the positive solution $x = 9/2$ can be used.

14. 5. [64,23] The denominator can equal any real number greater than 1. Thus the range of f is $(0, 6)$, all positive numbers less than 6. The integers in the range are 1, 2, 3, 4, and 5.

15. 1/2. [148.5,32] Let O denote the center of the hexagon. Then triangles ABC and AOC are congruent, and similarly for two other pairs of triangles. Triangle ABC is composed of three of these, while the hexagon is composed of six of them.

16. $1+\sqrt{3}$. [77.5,26.5] In the diagram below, we have $\frac{x+2}{x} = \sqrt{3}$. Thus $x = \frac{2}{\sqrt{3}-1} = \sqrt{3}+1$.



17. 0. [106,29] Out of the 81 equally likely pairs, we tabulate the number of ways each sum can occur:

2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	2	3	4	5	6	7	8	9	8	7	6	5	4	3	2	1

We see that a last digit 0 occurs in nine ways, while any other last digit occurs in eight ways.

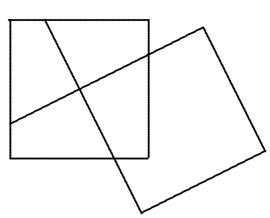
18. 4. [131,30] Adding $4xy$ to both sides of $(x-y)^2 \geq 0$ yields $(x+y)^2 \geq 4xy$. Thus, if the asserted inequality holds, then $4(x+y) \leq 4xy \leq (x+y)^2$, and so $x+y \geq 4$. As the inequality holds for $x = y = 2$, we find that 4 is the smallest value of $x+y$ for which the inequality holds.

19. $1 - \frac{1}{100!}$. [31,19] The trick here is to notice that $\frac{n}{(n+1)!} = \frac{n+1}{(n+1)!} - \frac{1}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}$. Thus the desired sum is

$$\left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \cdots + \left(\frac{1}{99!} - \frac{1}{100!}\right),$$

and all the intermediate terms cancel.

20. 1/4. [165,31] It is always 1/4. Two perpendicular lines passing through the center of a square divide it into four congruent parts.



21. 35/29. [23,13.5] By the Law of Sines, or a comparison of right triangles, we have $\frac{AC}{BC} = \frac{\sin(B)}{\sin(A)}$. Looking at 3-4-5 and 20-21-29 right triangles shows that $\sin(A) = 3/5$ and $\sin(B) = 21/29$. Thus

$$\frac{AC}{BC} = \frac{21}{29} \cdot \frac{5}{3} = \frac{35}{29}.$$

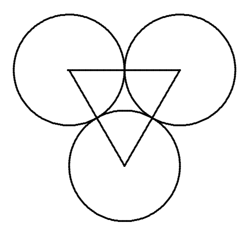
22. 8. [88,30] The numbers n are 29, 38, 47, 56, 65, 74, 83, and 92. To see this, note that if $n = 10a + b$ with $1 \le a \le 9$ and $0 \le b \le 9$, then $n + r(n) = 11(a + b)$. For this to be a square, we must have $a + b = 11$, and the listed ways are the only ways to accomplish this.

23. $(6, 35)$. [35,21] We have $(x^2 - y)(x^2 + y) = 71$. Thus $x^2 + y = 71$ and $x^2 - y = 1$. Adding and subtracting these equations yields the only solution.

24. 152. [106,29] For an n-gon, there are $n(n-3)/2$ diagonals. This is obtained by noting that each vertex can be connected to $n-3$ vertices by diagonals, but each diagonal is counted twice in the product $n(n-3)$.

25. 5. [68,24] For $1 \le n \le 4$, we have that $n = 1$ and $n = 4$ do not yield a corresponding integer m, but $n = 2$ and $n = 3$ yield $m = 5$ and $m = 7$, respectively. If $n \ge 5$, then $n!$ ends in 0, so $6(n! + 3)$ ends in 8, but no square ends in 3, so there are no more m's.

26. $\sqrt{3} - \frac{\pi}{2}$. [81,29] The centers of the circles are vertices of an equilateral triangle of area $\sqrt{3}$. Inside the intersection of the triangle with the circles, and hence outside our region, are three wedges of area $\pi/6$ each.

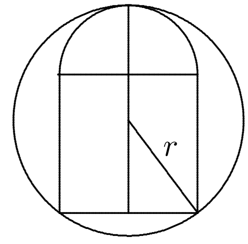


27. $\sqrt{3601 - 60\sqrt{2}}$. [30,14] Its velocity vector is $(60 - \frac{1}{\sqrt{2}})\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$, and so its speed is given by $\sqrt{(60 - \frac{1}{\sqrt{2}})^2 + \frac{1}{2}} = \sqrt{3600 - 60\sqrt{2} + \frac{1}{2} + \frac{1}{2}}$.

28. 666. [45,24] Such a number must be of the form $111a$ for $1 \le a \le 9$ and of the form $n(n+1)/2$ for some n. Thus $n(n+1) = 2 \cdot 3 \cdot 37 \cdot a$. Either n or $n+1$ must be divisible by 37. But $n = 37$ requires a to be divisible by 19, which it isn't. The only solution is obtained when $n + 1 = 37$, yielding $a = 6$. If $n + 1 \ge 74$, then a is too large.

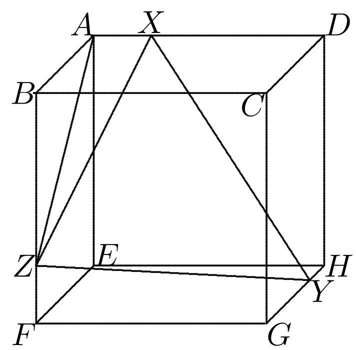
29. 250. [20,14] It is all numbers of the form $4k + 2$ from 2 to 998. To see this, first note that if $n = x^2 - y^2 = (x - y)(x + y)$, then n is the product of two even numbers or of two odd numbers, and hence n cannot be of the form $4k + 2$. On the other hand, $2k + 1 = (k + 1)^2 - k^2$ and $4(m + 1) = (m + 2)^2 - m^2$.

30. 5/6. [25,17] See the figure below. The radius r satisfies $r + \sqrt{r^2 - 1/4} = 3/2$. Thus $r^2 - 1/4 = (3/2 - r)^2$, and so $3r = 10/4$.



31. 5/2. [19,12] We have $p(x) = (x-2)(x+2)(x+3)(ax+b)+1$, and then $2 = (-1)\cdot 3\cdot 4(a+b)+1$ and $2 = (-3)\cdot 1\cdot 2(-a+b)+1$. Thus $1 = -12a - 12b$ and $1 = 6a - 6b$. We obtain $3 = -24b$, and so $p(0) = (-2)\cdot 2\cdot 3/(-8)+1 = 5/2$.

32. $13\sqrt{3}/2$. [26.5,20] See the figure below. Each side of triangle XYZ is the hypotenuse of a right triangle with legs 1 and $\sqrt{3^2+4^2}$. Thus the side lengths are $\sqrt{26}$, and the desired area is $26\sqrt{3}/4$.



33. 1860495. [18,12] We have that for all n, $\sum_{i=1}^{n} a_i = a_{n+2} - 3$. (A similar, but not identical, relation holds for the ordinary Fibonacci numbers.) The claimed relation is easily proved by induction on n. (Verify it for $n=1$; then assume it is true for n and add a_{n+1} to both sides to see that it is true for $n+1$.) Thus the desired sum is $a_{30} - 3$.

34. $3\pi/8$. [15,13] Since the altitudes of an equilateral triangle meet at a point one third of the way along each, the radius of the first circle is $\frac{1}{3}\sqrt{3}$ and so its area is $\pi/3$. All remaining circles form a shrunken version of the entire set of circles, but shrunk by a linear factor of $1/3$. Thus the area of all the circles except the first is $1/9$ of the total desired area. If the desired area is A, then we have $A = \pi/3 + A/9$, and so $A = 3\pi/8$.

35. 11111 or a_5. [35,15] Every common divisor of a_{140} and a_{45} is a divisor of

$$a_{140} - 10^{95}a_{45} - 10^{50}a_{45} - 10^{5}a_{45} = a_5.$$

On the other hand, a_5 is a divisor of both a_{140} and a_{45}. For example, $a_{45} = a_5(10^{40} + 10^{35} + \cdots + 10^5 + 1)$. Since a_5 is a common divisor and is a multiple of every common divisor, it is the gcd.

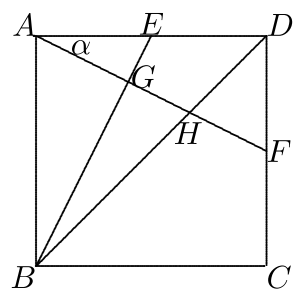
36. $(1,2,1)$. [10,6] First express the inequality as $(x - \frac{y}{2})^2 + \frac{3}{4}(y-2)^2 + (z-1)^2 < 1$. Since x, y, and z are integers, this implies $z = 1$ and $y = 1$, 2, or 3. If $y = 1$, we obtain $x^2 - x < 0$, which has no integer solutions. If $y = 2$, we obtain $x^2 - 2x < 0$, whose only integer solution is $x = 1$. If $y = 3$, we obtain $x^2 - 3x + 2 < 0$, which has no integer solutions.

37. 153846. [22,15] Since $4\cdot 6 = 24$, n ends 46. Since $4\cdot 46 = 184$, n ends 846. Since $4\cdot 846 = 3384$, n ends 3846. Since $4\cdot 3846 = 15384$, n ends 53846. Since $4\cdot 53846 = 215384$, n ends 153846. Since $4\cdot 153846 = 615384$, $n = 153846$.

38. $(13, 11, 7)$. [24,17] Since $a(b+c) = 2 \cdot 3^2 \cdot 13$, $b(a+c) = 2^2 \cdot 5 \cdot 11$, and $c(a-b) = 2 \cdot 7$, then $b+c$ must equal 117, 78, or 18, b must equal 2, 5, or 11, and c must equal 2 or 7. Thus we must have $b = 11$ and $c = 7$. Then $a = 13$ and all equations are satisfied.

39. 9. [13,7] Note that $f(n) = (a+1)(b+1)(c+1) - 1$. Thus we must have $(a+1)(b+1)(c+1) = 100a+10b+c+1$. This simplifies to $a(b+1)(c+1)+b(c+1) = 100a+10b$. Since $b, c \leq 9$, we have $(b+1)(c+1) \leq 100$ and $c+1 \leq 10$, and so equality is obtained if and only if $b = c = 9$. However, a can equal $1, \dots, 9$.

40. 7/15. [5,5] We use parentheses to denote area of a figure, and refer to the diagram below. We have $(HDF) = \frac{1}{2} HD \cdot DF \sin(45) = HD/(2\sqrt{2})$, and similarly $(AHD) = HD/\sqrt{2}$. Since $(HDF) + (AHD) = 1$, we obtain $HD = 2\sqrt{2}/3$ and $(HDF) = 1/3$. If $\alpha = \angle FAD$, then $\sin(\alpha) = 1/\sqrt{5}$ and $\sin(90-\alpha) = 2/\sqrt{5}$. Then $(AGE) = AG/(2\sqrt{5})$ and $(AGB) = 2AG/\sqrt{5}$. Since $(AGE) + (AGB) = 1$, we obtain $AG = 2\sqrt{5}/5$ and $(AGE) = 1/5$. Finally $(DEGH) = (ADF) - (HDF) - (AGE) = 1 - \frac{1}{3} - \frac{1}{5}$.



Alternatively, one can write equations of the three diagonal lines, and use these to find the coordinates of the points G and H, obtaining that the altitudes from G to AE and from H to DF are 2/5 and 2/3, respectively. Thus the areas of triangles AGE and HDF are 1/5 and 1/3, respectively.