

2026 LEHIGH UNIVERSITY HIGH SCHOOL MATH CONTEST

1. PROBLEMS,

annotated with (a,b), where

a is the number of the top 23 students answering correctly, and

b is the number of the other 93 students answering correctly.

1. [2 pts] (23, 85) a_0, a_1, a_2, a_3, a_4 is an arithmetic sequence whose sum is 30. What number **must** appear in the sequence?
2. [2 pts] (23, 77) What is the largest integer k such that $n^3 - n$ is divisible by k for all integers $n \geq 2$?
3. [2 pts] (22,49) How many 5-digit numbers $ABCDE$ satisfy $A > 0, C = A + B, D = B + C,$ and $E = C + D$?
4. [2 pts] (22, 70) Evaluate $\sqrt{11112^2 - 44444}$ as an expression with no $\sqrt{\quad}$.
5. [2 pts] (21, 62) How many 0's appear at the end of $2026!$? (That's 2026 factorial.)
6. [3 pts] (23, 68) List all primes p for which there is an integer n such that both $n + p$ and $n + 7p$ are perfect squares.
7. [3 pts] (22, 58) Compute the sum of the reciprocals of the roots of the polynomial $x^3 - 7x^2 + 13x - 7$.
8. [3 pts] (21, 25) Define a sequence by $a_0 = 1$ and $a_k = 7a_{k-1} + 4$ for $k \geq 1$. Find the limiting value of $\frac{a_k}{7^k}$ as k approaches infinity.
9. [3 pts] (23, 70) Find the sum of the areas of all rectangles with integer sides for which the perimeter equals the area. (Congruent rectangles only count once.)
10. [3 pts] (17, 27) The Constitution of the Kingdom of Nepal stipulates that the flag be constructed as follows: Draw a line AB of the required length from left to right. From A draw a line AC perpendicular to AB making AC equal to AB plus one-third

AB. On AC mark off D making line AD equal to line AB. Join BD. On BD mark off E making BE equal to AB. Let F be on AC with FE parallel to AB. Extend FE beyond E to G with FG equal to AB. Join CG. If the length of AB is 1, what is the area of the flag; i.e., the area enclosed by the lines?

11. [4 pts] (23, 75) Alex has a bag containing more than 400 candies. He wants to split it equally among his 20 friends and himself. After doing so, there are 5 left over. Alethea arrives, and they redistribute the same batch of candies evenly again among all 22 kids. This time there are 3 left over. What is the smallest number of candies that the bag might have contained?
12. [4 pts] (21, 37) You have two fair 6-sided dice. One has the numbers 1 through 6, while the other has 3 through 8. Twice you randomly select one of the dice and roll it. Given that the sum of the two rolls is 9, what is the probability that you rolled the same die both times?
13. [4 pts] (10, 3) For every integer n from 0 to 6, we have three indistinguishable weights with weight 2^n . How many ways are there to form a total weight of 263 using a subset of these weights?
14. [4 pts] (20, 30) How many ordered triples (a, b, c) of integers satisfy $0 < c < b < a$ and $a + b + c = 100$?
15. [4 pts] (22, 32) Square $ABCD$ has side length 4. Points P and Q lie on sides BC and CD , respectively, with $BP = DQ = 1$. Segments AQ and DP intersect at X . What is the area of triangle PQX ?
16. [5 pts] (17, 7) Find the value of x which has the maximum value of $|\sqrt{x^2 + 4x + 8} - \sqrt{x^2 + 2x + 10}|$.

17. [5 pts] (19, 20) A bag has 10 balls of 10 different colors. Four times you draw a ball, **with replacement**. What is the expected number of colors among the balls you picked?
18. [5 pts] (19, 4) Triangle ABC has $AB = 5$, $BC = 7$, and $AC = 8$. Points E and F lie on AC , G and H on CB , and D and J on AB with each of DE , FG , and HJ tangent to the incircle of ABC and each parallel to the opposite side of ABC . Compute the area of hexagon $DEFGHJ$.
19. [5 pts] (10, 2) What is the minimum value of $x^3 + 4y^2 + 9z$ under the constraints $xyz = 1$ and $x, y, z \geq 0$?
20. [5 pts] (8, 6) How many paths are there from $(0, 0)$ to $(14, 14)$ in which each step increments by either $(1, 0)$, $(0, 1)$, or $(1, 1)$, and the path never passes through a lattice point (m, n) with $m + n$ a multiple of 3 (except the initial point)?
21. [6 pts] (11, 0) Let C_1 be the set of points in the complex plane at distance 1 from the point 0, C_2 the points at distance 2 from the point $4 - 2i$, and C_3 the points at distance 4 from $2 + 8i$. What is the area of the set of points of the form $z_1 + z_2 + z_3$ with $z_1 \in C_1$, $z_2 \in C_2$, and $z_3 \in C_3$?
22. [6 pts] (11, 4) In a regular pentagon of area 1, connect each vertex to the two vertices farthest from it, forming a smaller regular pentagon inside. What is the area of the smaller pentagon? Express your answer without any trig functions.
23. [6 pts] (10, 1) What is the smallest number which can be written as $a^2 - b^2$ for integers $0 \leq b \leq a$ in exactly 13 ways?
24. [6 pts] (6, 0) Let X_1, X_2, \dots be a sequence of strings of 0's and 1's defined by $X_1 = 1$, and X_{n+1} is formed from X_n by replacing each 0 by a 1, and replacing each 1 by 11000. Thus, $X_2 = 11000$ and $X_3 = 1100011000111$. How many times does 01 occur in X_{2026} ?

25. [6 pts] (2, 0) The 503 ordered pairs

$(1007, 2), (1006, 3), (1005, 4), \dots, (506, 503), (505, 504)$

are written on the board in this order. Every minute, you choose a pair of adjacent pairs $(x, y), (x', y')$ (in this order), erase them and replace them by the single ordered pair $\left(\frac{xyx'}{y'}, \frac{xyy'}{x'}\right)$. After 502 minutes, there will be only one ordered pair left. How many possibilities are there for this final ordered pair?

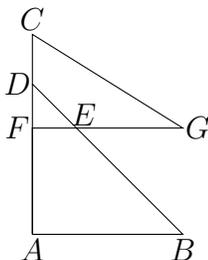
2. SOLUTIONS

6. The middle term is the average, so $30/5$.
6. $n^3 - n = (n - 1)n(n + 1)$. Three consecutive integers must contain at least one which is divisible by 2, and one which is divisible by 3. Therefore $n^3 - n$ is always divisible by 6. Since $2^3 - 2 = 6$, $n^3 - n$ is not always divisible by any number > 6 .
8. The number is determined by A and B . The only ones that make $E \leq 9$ are 10112, 11235, 12358, 20224, 21347, 30336, 31459, and 40448.
11110. $\sqrt{(11111 + 1)^2 - 44444} = \sqrt{11111^2 + 1 - 22222} = \sqrt{(11111 - 1)^2}$.
505. Each such 0 is formed from a factor of 2 and a factor of 5. There are more factors of 2 than of 5, so we only need to count the number of factors of 5. This will equal $\lfloor \frac{2026}{5} \rfloor + \lfloor \frac{2026}{25} \rfloor + \lfloor \frac{2026}{125} \rfloor + \lfloor \frac{2026}{625} \rfloor = 405 + 81 + 16 + 3 = 505$.
2. If $n + 7p = a^2$ and $n + p = b^2$, then $6p = (a + b)(a - b)$. At least one of $a + b$ and $a - b$ must be even, but since they are congruent mod 2, both must be even. Therefore p must be even, so $p = 2$. ($n = 2, a = 4, b = 2$)
- 13/7. If the roots of the polynomial are r, s , and t , then

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = \frac{st + rt + rs}{rst} = \frac{13}{7}.$$

8. $5/3$ or $1 + \frac{2}{3}$. Let $b_k = a_k/7^k$. Then $b_0 = 1$ and $b_k = b_{k-1} + \frac{4}{7^k}$.
 The limit is $1 + \sum_{k=1}^{\infty} \frac{4}{7^k} = 1 + \frac{4}{7(1 - \frac{1}{7})} = 1 + \frac{2}{3}$.
9. 34. $LW = 2(L + W)$ implies $L = \frac{2W}{W-2} = 2 + \frac{4}{W-2}$. This is a positive integer if and only if $W = 3, 4,$ or 6 , giving $(L, W) = (6, 3), (4, 4),$ or $(3, 6)$. We only count one of $(6, 3)$ and $(3, 6)$.
10. $(5+3\sqrt{2})/12$ or $\frac{5}{12} + \frac{1}{4}\sqrt{2}$. We have $AB = AD = BE = FG = 1$, and $AC = \frac{4}{3}$. By Pythagoras, $BD = \sqrt{2}$. By similar triangles, $\frac{AF}{AD} = \frac{BE}{BD}$, so $AF = \frac{1}{\sqrt{2}}$. Therefore $CF = \frac{4}{3} - \frac{1}{\sqrt{2}}$ and $DF = 1 - \frac{1}{\sqrt{2}}$. By similar triangles, $\frac{EF}{AB} = \frac{DF}{AD}$, so $EF = 1 - \frac{1}{\sqrt{2}}$. The desired area equals $|CFG| + |ABEF| = (\frac{2}{3} - \frac{1}{2\sqrt{2}}) + (\frac{1}{\sqrt{2}} - \frac{1}{4})$, since $|ABEF| = \frac{1}{\sqrt{2}} \cdot \frac{1}{2}(1 + 1 - \frac{1}{\sqrt{2}})$.

Figure 2.1. Diagram for problem 10.



11. 509. If the bag contained x candies, then $x \equiv 5 \pmod{21}$, and $x \equiv 3 \pmod{22}$. This is a Chinese Remainder Theorem problem. Since $22x \equiv 110 \pmod{462}$ and $21x \equiv 63 \pmod{462}$, we obtain $x \equiv 47 \pmod{462}$. The smallest such number greater than 400 is 509.
12. $2/5$. If you rolled the same die twice, the probability that you rolled 9 is $4/36$. ($(3,6), (4,5), (5,4),$ or $(6,3)$ with either die.) If you rolled each die once, the probability that you rolled 9 is $6/36$. ($(1,8), (2,7), (3,6), (4,5), (5,4), (6,3)$). Without regard for the outcome, the probability that you rolled the same die twice is $1/2$. Thus the probability that you chose the same die

twice and rolled 9 is $1/18$, while the probability that you chose different dice and rolled 9 is $1/12$. Given that you rolled 9, the probability that you had rolled the same die twice is

$$\frac{\frac{1}{18}}{\frac{1}{18} + \frac{1}{12}} = \frac{2}{5}.$$

13. 60. We seek the coefficient of x^{263} in

$$\begin{aligned} & (1 + x + x^2 + x^3)(1 + x^2 + x^4 + x^6) \cdots (1 + x^{64} + x^{128} + x^{192}) \\ &= \frac{x^4 - 1}{x - 1} \cdot \frac{x^8 - 1}{x^2 - 1} \cdot \frac{x^{16} - 1}{x^4 - 1} \cdot \frac{x^{32} - 1}{x^8 - 1} \cdot \frac{x^{64} - 1}{x^{16} - 1} \cdot \frac{x^{128} - 1}{x^{32} - 1} \cdot \frac{x^{256} - 1}{x^{64} - 1} \\ &= \frac{x^{128} - 1}{x - 1} \cdot \frac{x^{256} - 1}{x^2 - 1} \\ &= (1 + x + x^2 + \cdots + x^{127})(1 + x^2 + x^4 + \cdots + x^{254}). \end{aligned}$$

The x^{263} term is $\sum_{i=68}^{127} x^{263-2i} x^{2i} = 60x^{263}$.

14. 784. We first enumerate the (a, b, c) with all numbers positive and sum equal to 100, but allowing any order and possible equality. Start with one in each slot and then do “stars and bars” on the others, yielding $\binom{97+2}{2} = 99 \cdot 49$ ways. (You are arranging 97 stars and 2 bars separating them into three (possibly empty) groups.) Of these, there are $3 \cdot 49$ in which two are equal. (The 3 represents the position of the unequal number, and the 49 counts the possibilities for the two equal integers. They can’t all be equal.) So there are $96 \cdot 49$ in which $a, b,$ and c are distinct, but in any order. Divide by 6 to obtain the desired amount, $16 \cdot 49$.

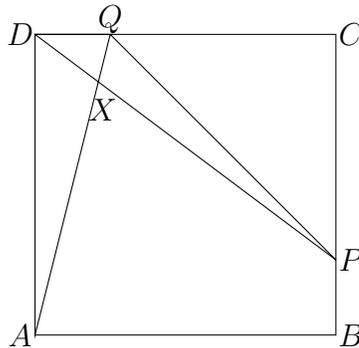
15. $45/38$. The desired area equals $|PQD| - |QDX|$, and $|PQD| = \frac{3}{2}$. Let $\theta = \angle QDX = \angle CDP$. We have $\sin \theta = \frac{3}{5}$ and $\cos \theta = \frac{4}{5}$.

Now

$$\begin{aligned}
 2 &= |QDA| \\
 &= |QDX| + |XDA| \\
 &= \frac{1}{2} \cdot DX \cdot (\sin \theta + 4 \sin(\frac{\pi}{2} - \theta)) \\
 &= \frac{1}{2} \cdot DX \cdot (\sin \theta + 4 \cos \theta),
 \end{aligned}$$

so $DX = \frac{20}{19}$. Thus $|QDX| = \frac{1}{2} \cdot \frac{20}{19} \cdot \frac{3}{5} = \frac{6}{19}$, and $|PQX| = \frac{3}{2} - \frac{6}{19} = \frac{45}{38}$.

Figure 2.2. Diagram for problem 15.



16. -4. Write the expression as

$$|\sqrt{(x+2)^2 + (0-2)^2} - \sqrt{(x+1)^2 + (0-3)^2}|.$$

If $P = (x, 0)$, $A = (-2, 2)$, and $B = (-1, 3)$, this is the absolute difference between distances PA and PB . By the triangle inequality, this is $\leq AB$. The maximum value occurs when A , B , and P are collinear, and so $P = (-4, 0)$.

17. $3439/1000$ or 3.439 . For each color, the probability that it will be drawn at least once is $1 - (\frac{9}{10})^4 = \frac{3439}{10000}$. This is its contribution to the expected number of colors. By linearity of expectation, the desired expected value is $10 \cdot \frac{3439}{10000}$.

18. $\frac{31}{5}\sqrt{3}$. By Heron's formula $|ABC| = \sqrt{10 \cdot 5 \cdot 2 \cdot 3} = 10\sqrt{3}$.

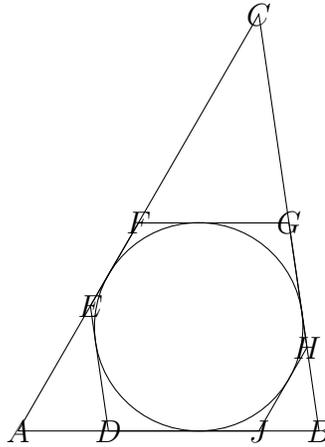
The radius of the incircle is

$$r = \frac{2|ABC|}{AB + BC + AC} = \frac{20\sqrt{3}}{20} = \sqrt{3}.$$

The altitude from A to BC is $\frac{2|ABC|}{BC} = \frac{20\sqrt{3}}{7}$, so the altitude from A to DE is $\frac{20\sqrt{3}}{7} - 2r = \frac{6\sqrt{3}}{7}$. By similar triangles, $|ADE| = \left(\frac{6\sqrt{3}/7}{20\sqrt{3}/7}\right)^2 |ABC| = \frac{9}{100}|ABC|$. Similarly, we get the altitude from B to AC is $\frac{5}{2}\sqrt{3}$, then the altitude from B to HJ is $\frac{1}{2}\sqrt{3}$, and $|BHJ| = \frac{1}{25}|ABC|$. Again, the altitude from C to AB is $4\sqrt{3}$, the altitude from C to FG is $2\sqrt{3}$, and $|CFG| = \frac{1}{4}|ABC|$. Finally,

$$\begin{aligned} |DEFGHJ| &= |ABC| - |ADE| - |BHJ| - |CFG| \\ &= |ABC|\left(1 - \frac{9}{100} - \frac{1}{25} - \frac{1}{4}\right) = 10\sqrt{3} \cdot \frac{31}{50}. \end{aligned}$$

Figure 2.3. Diagram for problem 18.



19. $11\sqrt[11]{27/4}$. By the arithmetic-geometric mean inequality

$$\frac{2 \cdot \frac{x^3}{2} + 3 \cdot \frac{4y^2}{3} + 6 \cdot \frac{9z}{6}}{11} \geq \sqrt[11]{\left(\frac{x^3}{2}\right)^2 \cdot \left(\frac{4y^2}{3}\right)^3 \cdot \left(\frac{9z}{6}\right)^6}.$$

Setting $xyz = 1$ and multiplying by 11 yields the result. Note that the 2, 3, and 6 were chosen to make the exponents of x , y , and z equal.

20. 322. Let D denote a diagonal step, and S a horizontal or vertical step. If you are at a point with $m + n \equiv 1 \pmod{3}$, then you must move S, while if you are where $m + n \equiv 2 \pmod{3}$, you must move D. If the first step is D, then the path must be $D(DS)^8D$, with four of the S's horizontal and four vertical. There are $\binom{8}{4}$ such paths. If the first step is S, the path must be $SS(DS)^8D$ with five S's horizontal and five vertical. There are $\binom{10}{5}$ of these. Thus the total is $70 + 252$.
21. 48π . Points in C_1 are of the form w_1 with $|w_1| = 1$, those in C_2 are $4 - 2i + w_2$ with $|w_2| = 2$, and those in C_3 are $2 + 8i + w_3$ with $|w_3| = 4$. Since the sum of the three centers is $6 + 6i$, the desired set consists of points of the form $6 + 6i + w_1 + w_2 + w_3$. This has the same area as the set S of points $w_1 + w_2 + w_3$. We have $\max |w_1 + w_2 + w_3| = 1 + 2 + 4 = 7$, and

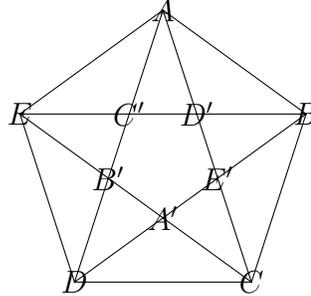
$$\begin{aligned} & \min |w_1 + w_2 + w_3| \\ &= \min |w_3 - w_2 - w_1| \\ &\geq \min(|w_3| - |w_2| - |w_1|) \\ &= 1, \end{aligned}$$

with the minimum value achieved when $w_1 = -1$, $w_2 = -2$, and $w_3 = 4$. Thus $|w_1 + w_2 + w_3|$ can have any value between 1 and 7, and the angles of w_1 , w_2 , and w_3 can be chosen to all have the same value from 0 to 2π . Hence S is a disk of radius 7 with a hole of radius 1. The area is 48π .

22. $\frac{1}{2}(7 - 3\sqrt{5})$. Label the original pentagon $ABCDE$, and the smaller one $A'B'C'D'E'$, with each primed vertex farthest away from the corresponding unprimed one. Let $AB = s$, $A'B' = x$ and $A'C = y$. By similar triangles and symmetry, $\frac{AB}{A'B'} = \frac{AD}{B'D} =$

$\frac{AC}{E'C} = \frac{AC}{A'C}$, so $\frac{s}{x} = \frac{2y+x}{y}$. Since $AD \parallel BC$, triangles ABC and $AB'C$ are congruent, so $x + y = s$. Let $r = \frac{x}{s}$. Then $\frac{1}{r} = \frac{2(1-r)+r}{1-r}$, so $r^2 - 3r + 1 = 0$ and $r = \frac{1}{2}(3 - \sqrt{5})$. Finally the area of the smaller pentagon is $r^2 = \frac{1}{4}(14 - 6\sqrt{5})$.

Figure 2.4. Diagram for problem 22.



23. 5184. Let $\#(n)$ denote the number of ways of writing $n = a^2 - b^2$ as requested. This equals $\lceil \frac{P(n)}{2} \rceil$, where $P(n)$ is the number of pairs $(a - b, a + b)$ or $(a + b, a - b)$ with $0 \leq b < a$ and $(a - b)(a + b) = n$. (Note that $P(n)$ can be odd if n is a perfect square.) If $n = 2^{e_2}3^{e_3}\cdots$ with $e_2 > 0$, then $P(n) = (e_2 - 1)(e_3 + 1)(e_5 + 1)\cdots$. This is because the exponents of p in a factorization in $P(n)$ can be split $(i, e_p - i)$ for $0 \leq i \leq e_p$ if p is odd, but only for $1 \leq i \leq e_2 - 1$ if $p = 2$ since $a - b$ and $a + b$ must have the same parity. For example, if $n = 2^45^2 = 400$, the 9 products of $\{(2, 2^3), (2^2, 2^2), (2^3, 2)\}$ times $\{(5^2, 1), (5, 5), (1, 5^2)\}$ yield the 5 pairs $(a - b, a + b) = (2 \cdot 5, 2 \cdot 5^3), (2^3, 2 \cdot 5^2), (2^2, 2^25^2), (2, 2^35^2)$, and $(2^25, 2^25)$. For $\#(n) = 13$, we must have $P(n) = 25$ or 26 . The smallest n with $P(n) = 26$ is $2^{14}3$, while the smallest n with $P(n) = 25$ is 2^63^4 . Since $2^8 > 3^3$, the smallest n is $64 \cdot 81 = 5184$. If $e_2 = 0$, then $P(n) = (e_3 + 1)\cdots$, and n would have to be much larger, since it would involve powers of 3 and 5.

24. $\frac{3^{2025}-3}{4}$. Let u_n , v_n , and w_n denote the number of 0's, 1's, and 01's, respectively, in X_n . The only way to get 01 in X_n is from a 1 in X_{n-1} , and every 1 in X_{n-1} leads to a 01 in X_n unless it occurs at the end of X_{n-1} . Since X_n ends with 0 if n even and 1 if n odd,

$$w_n = \begin{cases} v_{n-1} & n \text{ odd} \\ v_{n-1} - 1 & n \text{ even.} \end{cases}$$

Clearly $u_{n+1} = 3v_n$ and $v_{n+1} = 2v_n + u_n = 2v_n + 3v_{n-1}$. Such a recurrence implies that $v_n = C_1r_1^n + C_2r_2^n$, where r_1 and r_2 are solutions of $r^2 - 2r - 3 = 0$. Thus $v_n = C_13^n + C_2(-1)^n$, where C_1 and C_2 are determined from $v_1 = 1$ and $v_2 = 2$. We obtain $C_1 = \frac{1}{4}$ and $C_2 = -\frac{1}{4}$, so $v_n = (3^n - (-1)^n)/4$. Since 2026 is even, $w_n = \frac{3^{2025} + 1}{4} - 1$.

25. 126,252. It is convenient to replace each ordered pair (x, y) by the alternate form $[xy, \frac{x}{y}]$. If $[a, b]$ is an alternate form, then the original can be retrieved as $(\sqrt{ab}, \sqrt{\frac{a}{b}})$. If $\#$ denotes the replacement operation, we have

$$\begin{aligned} & [a_1, b_1] \# [a_2, b_2] \\ &= \left(\sqrt{a_1 b_1}, \sqrt{\frac{a_1}{b_1}} \right) \# \left(\sqrt{a_2 b_2}, \sqrt{\frac{a_2}{b_2}} \right) \\ &= \left(a_1 b_2, \frac{a_1}{b_2} \right) = [a_1^2, b_2^2]. \end{aligned}$$

We may replace all of the original pairs by their alternate forms, and use the $[a_1^2, b_2^2]$ replacement rule. At the outset, the pair on the far left will be $[2014, \frac{1007}{2}]$, and the rightmost pair will be $[254520, \frac{505}{504}]$. Let ℓ be the number of times the leftmost pair is erased, and r the number of times the rightmost pair is erased, during the entire sequence of steps. Then the final pair is

$$\left[2014^{2^\ell}, \left(\frac{505}{504} \right)^{2^r} \right].$$

Every step except the last cannot involve both the leftmost and rightmost pairs, so $\ell + r \leq 503$. Since every pair must be erased at least once, $\ell \geq 1$ and $r \geq 1$. However, we cannot have $\ell = r = 1$. (Letting $q = 505/504$, if the last pair is $[2014^{2^\ell}, q^{2^r}]$, then it must be preceded by $[2014^{2^{\ell-1}}, y_1], [y_2, q^{2^{r-1}}]$, and the step prior to this must involve either $2014^{2^{\ell-2}}$ or $q^{2^{r-2}}$.) Every pair of integers ℓ and r satisfying these inequalities can occur, by making $503 - \ell - r$ moves involving only pairs in the middle, then $\ell - 1$ moves involving the left pair, and finally r moves involving the right pair. Since the final answer is determined uniquely by ℓ and r , the number of possible pairs at the end is 1 less than

$$\sum_{\ell=1}^{502} (503 - \ell) = \sum_{\ell=1}^{502} \ell = \frac{(502)(503)}{2} = 126253.$$